Singular perturbation approach to rattleback reversals

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Lagrangian systems with constraints are common models of fundamental or idealized physical systems. Holonomic constraints, typified in the example of a freely moving rigid body, give rise to systems with certain special properties, such as symplectic Hamiltonian systems. Nonholonomic constraints arise in such systems as a disk which rolls without slipping.

Holonomic and nonholonomic systems have differing mathematical structures, and they have different behaviors [1, 3, 4, 9, 21, 22, 24]. For example, when a vehicle has good contact with a road, then it is behaving as a nonholonomic system. Angular momentum is not conserved; otherwise, the vehicle could not be steered into a turn. Under icy conditions, the vehicle is essentially a holonomic system; then steering cannot change its angular momentum, the vehicle cannot be turned, and whatever spin it has will persist. Conservation of energy, however, is a dominant feature of both systems.

A rattleback is a toy top in the shape of long, narrow boat, with a slight, usually imperceptible asymmetry, either in its shape, or in its mass distribution. Many people anticipate that, when spun on a table, the rattleback will behave as other tops do i.e. *holonomically*. And, when spun in one direction, the rattleback will behave like this. When spun in the opposite direction, rattlebacks will spontaneously reverse direction, exhibiting *nonholonomic* non-conservation of angular momentum. As it turns out, because of the asymmetry, the table and the rattleback are coupled nonholonomically. Angular momentum is not conserved: some time after spinning in the unstable direction, the rattleback is observed to be spinning, at nearly the same rate, in the *opposite* direction. The transition between the two spins is dynamically complicated: it occurs through a non-spinning longitudinal wobbling motion. This is the rattleback's *spin reversal*.

Rattlebacks have been observed, and the basic mathematical model obtained, for over a century [26, 27], and the have been researched off-an-on since then [2, 5, 6, 7, 8, 15, 16, 17, 19, 20, 23, 25, 28]. But the spin reversal is a global dynamical feature, and its understanding is incomplete. Part of the problem is the sheer complexity of the system. The (reduced) rattleback equations of motion, for a body with surface $\mathcal{M}$ rolling on the plane, are

$$\begin{aligned} &\boldsymbol{L}_{\mathcal{M}}(s)\,\frac{ds}{dt} = \Omega\times\boldsymbol{n}_{\mathcal{M}}(s),\\ &\bigl(I - m(s^{\wedge})^2\bigr)\,\frac{d\Omega}{dt} = -U, \quad U \equiv \Omega\times I\,\Omega + ms\times\left(\Omega\times\frac{ds}{dt} + (s\cdot\Omega)\Omega + g\,\boldsymbol{n}_{\mathcal{M}}\right), \end{aligned}$$

where Ω is the body-reference angular velocity, as in the Euler equation for a rigid body; $s\in\mathcal{M}$ is the body-reference contact point; $\boldsymbol{n}_{\mathcal{M}}$ is the unit normal of $\mathcal{M}$; and $\boldsymbol{L}_{\mathcal{M}}$ is the Weingarten map of $\mathcal{M}$. Putting the equations in the usual first-order form requires inverting $\boldsymbol{L}_{\mathcal{M}}(s)$, with result a complicated nonlinear differential equation, on the five dimensional manifold $\mathcal{M}\times\mathbb{R}^3$.

In the workshop, we explored a new idea about rattleback spin reversals, with objective a better understanding, and especially a proof, that it occurs in some dynamical regimes. Taking advantage of the slender geometry, we view the rattleback as a perturbation of a cylindrical reference system i.e. in the reference system, $\mathcal{M} = \mathcal{C} \times [-l/2, l/2]$, $l > 0$, where $\mathcal{C}$ is a curve. Since cylinders are not curved, they cannot wobble, so spin reversal cannot occur in the reference system, only in small perturbations of it. The reference system has, however, a robust feature: for a cylinder, $\boldsymbol{L}_{\mathcal{M}}(s)$ is non-invertible, because the principle curvature of a cylinder along its axis is zero. As a result the perturbations we are considering are singular, such as, for example, the differential equation

$$\epsilon \frac{dx}{dt} = (v + \alpha x)\cos x, \qquad \frac{dv}{dt} = -\sin x \tag{1}$$

as ϵ is perturbed from zero. Such singular perturbations are central in Applied Mathematics. Singularly perturbed systems display fast transitions between slow invariant manifolds, of which there is a related sophisticated dynamical and geometric theory [10, 11, 12, 13, 14, 18].

At the BIRS workshop, we began our research into singular perturbations and rattleback spin-reversals. We managed to see that the rapid transitions between relatively stable spin and wobbling modes could be explained as fast transitions of a singularly perturbed system. We computed what the invariant manifolds might be, and how the fast vector field meets the codimension 1 invariant manifolds, where transversally or not. And we analyzed possible of the behaviors of the simpler systems (1).

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