On the relationship between Semi–Lagrangian and Lagrange–Galerkin schemes

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- Some history
- Setting of the model problem
- The Semi–Lagrangian approach
- The Lagrange–Galerkin approach
- Recasting SL as LG schemes – constant speed
- Recasting SL as LG schemes – general linear case

Some history

• **Semi–Lagrangian schemes**: introduced as first–order schemes by Courant, Isaacson and Rees (CPAM, '52), then improved by Wiin-Nielsen (Tellus, '59), Robert (Atmosphere-Ocean, '81), Staniforth, Côté, Smolarkiewicz...

• **Lagrange–Galerkin schemes**: introduced independenly by Douglas and Russell (SINUM, '82) and Pironneau (Num. Math, '82), improved by Russell, Bercovier, Pironneau, Süli, Lesaint,...

Setting of the model problem

For simplicity, we will discuss SL and LG schemes focusing on the model problem

$$\begin{cases} v_t(x,t) + f(x,t) \cdot \nabla v(x,t) = 0, & \text{in } \mathbb{R}^N \times \mathbb{R} \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

posed on the whole of $\mathbb{R}^N$.

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posed on the whole of $\mathbb{R}^N$.

- We avoid the treatment of boundary conditions
- We treat separately and more explicitly the case of constant speed

Any large time–step technique (in particular, both Semi–Lagrangian and Lagrange–Galerkin approximations) stem from the method of characteristics. Let a system of characteristic trajectories $X(x, s; t)$ for the model equation be defined by:

$$\begin{cases} X(x, s; s) &= x, \\ \dfrac{d}{dt} X(x, s; t) &= f(X(x, s; t), t). \end{cases}$$

Then, the solution is constant along such trajectories, which means that the following representation formula

$$v(X(x, t; t + \tau), t + \tau) = v(x, t).$$

holds for the solution v.

Writing the representation formula with $\tau = -\Delta t$, we have the time-discrete version

$$v(x,t) = v(X(x,t;t-\Delta t), t-\Delta t).$$

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Its numerical discretization is obtained by combining:

• A numerical technique to integrate backwards the ODE of characteristics

• A reconstruction to approximate the value $v(X(x_j,t;t-\Delta t), t-\Delta t)$, since in general the foot of the characteristic $X(x_j,t;t-\Delta t)$ does not coincide with any grid point.

- The approximation of characteristics will be neglected in the sequel

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- On the other hand, the reconstruction of the value $v(X(x_j, t; t - \Delta t), t - \Delta t)$ is the crucial point in the theoretical analysis, and it is also what makes the difference between SL and LG schemes

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• In SL schemes, the reconstruction is performed by an interpolation, whereas LG schemes perform this step as a Galerkin projection

• We assume in both cases that reconstruction is invariant for $\Delta x \cdot \mathbb{Z}$ translations (this rules out high–order finite element bases)

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The Semi–Lagrangian approach

In the SL scheme, the representation formula is discretized as

$$v_i^{n+1} = I[V^n](X(x_i, t^{n+1}; t^n)) = \sum_j v_j^n \psi_j(X(x_i, t^{n+1}; t^n))$$

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(for $i \in \mathbb{Z}$), $\{\psi_j\}$ being the basis for the interpolation operator $I[\cdot]$. In particular, this form holds for Lagrange interpolation, for which

$$\psi_j(\xi) = \psi\left(\frac{\xi_1}{\Delta x_1} - j_1\right) \cdots \psi\left(\frac{\xi_N}{\Delta x_N} - j_N\right) = \prod_{k=1}^{N} \psi\left(\frac{\xi_k}{\Delta x_k} - j_k\right).$$

where $j = (j_1, \ldots, j_N)$ is a multiindex and we have denoted by ψ the (one-dimensional) reference basis function and by Δx_k the space step along the k–th direction (clearly, $\psi(0) = 1$, $\psi(i) = 0$ $(i \neq 0)$).

The case of Lagrange reconstruction of order n still allows to single out a proper SL basis. For n odd, the form of $\psi^{(n)}$ is

$$\psi^{(n)}(t) = \begin{cases} \displaystyle\prod_{k\neq 0, k=-[n/2]}^{[n/2]+1} \frac{t-k}{-k} & \text{if } 0 \leq t \leq 1 \\ \vdots & \vdots \\ \displaystyle\prod_{k=1}^{n} \frac{t-k}{-k} & \text{if } [n/2] \leq t \leq [n/2]+1 \\ 0 & \text{if } t > [n/2]+1 \end{cases}$$

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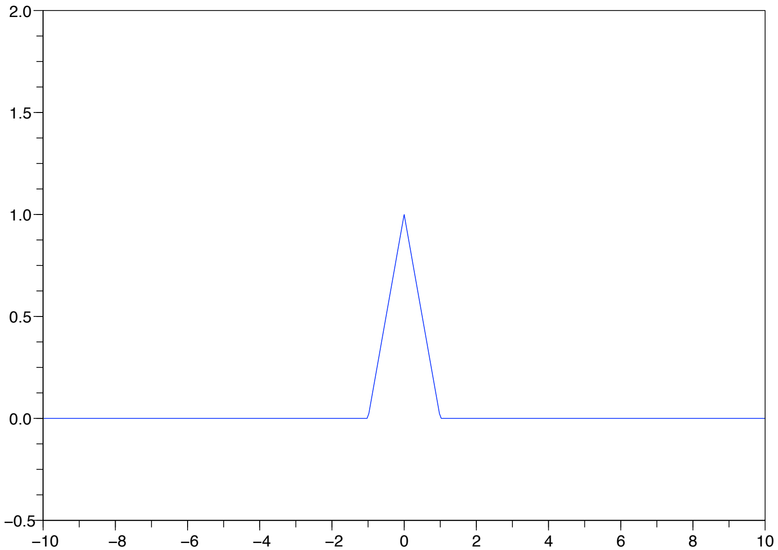
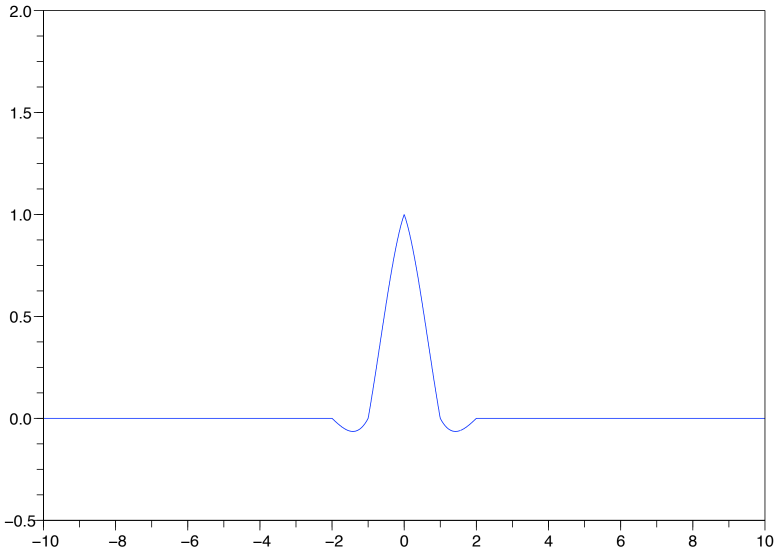
• This general structure is obtained by setting $\Delta x = 1$ and reconstructing the vector e_0)



The reference function ψ for $\mathbb{P}_1$ interpolation



The reference function ψ for cubic interpolation

Existing results and known facts:

- Stability of SL schemes has not yet a complete theoretical analysis

- Older results: Von Neumann stability analysis, with no closed form solution (e.g. Falcone - F., SINUM '98)

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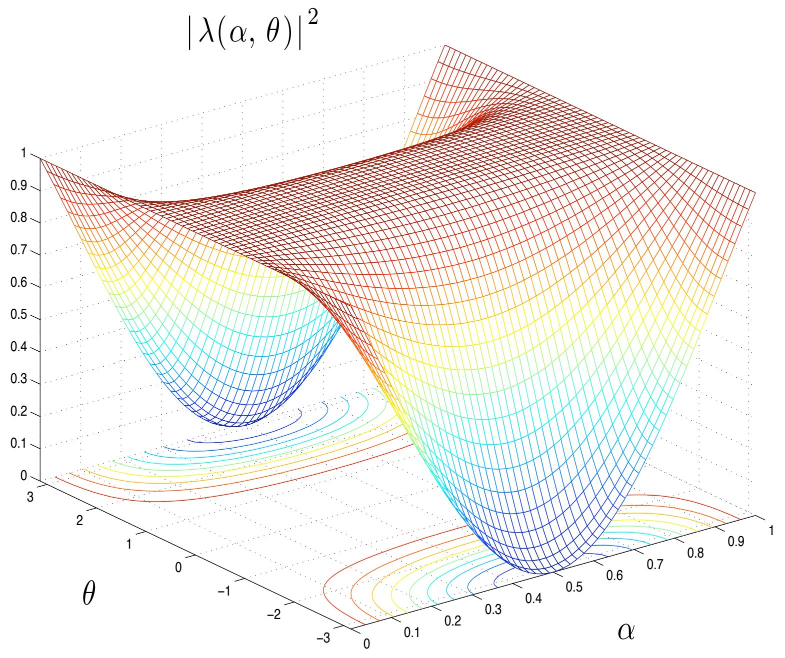
- Older results: Von Neumann stability analysis, with no closed form solution (e.g. Falcone - F., SINUM '98)

- A recent result of Von Neumann analysis, with a closed form solution (Besse - Mehrenberger, Math. Comp. '07)



Amplitude of the amplification factors λ for cubic interpolation

The Lagrange–Galerkin approach

In the LG scheme, the representation formula is discretized as

$$\int_{\mathbb{R}^N} \sum_j v_j^{n+1} \phi_j(\xi)\phi_i(\xi) d\xi = \int_{\mathbb{R}^N} \sum_j v_j^n \phi_j(X(\xi, t^{n+1}; t^n))\phi_i(\xi) d\xi$$

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- $\phi_i(\xi)$ is the test function

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(for $i \in \mathbb{Z}$), that is,

$$\sum_j v_j^{n+1} \int_{\mathbb{R}^N} \phi_j(\xi)\phi_i(\xi) d\xi = \sum_j v_j^n \int_{\mathbb{R}^N} \phi_j(X(\xi, t^{n+1}; t^n))\phi_i(\xi) d\xi.$$

$\{\phi_j\}$ being the basis for the Galerkin projection.

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \phi_j(\xi)\phi_i(\xi) d\xi & \longrightarrow \text{mass matrix} \\[2ex] \displaystyle\int_{\mathbb{R}^N} \phi_j(X(\xi, t^{n+1}; t^n))\phi_i(\xi) d\xi & \longrightarrow \text{"upwinded" mass matrix} \end{cases}$$

The Galerkin basis is supposed to have the same tensorized structure as the SL basis:

$$\phi_j(\xi) = \frac{1}{\sqrt{\Delta x_1 \cdots \Delta x_N}} \, \phi\left(\frac{\xi_1}{\Delta x_1} - j_1\right) \cdots \phi\left(\frac{\xi_N}{\Delta x_N} - j_N\right) =$$

$$= \frac{1}{\sqrt{\Delta x_1 \cdots \Delta x_N}} \prod_{k=1}^{N} \phi\left(\frac{\xi_k}{\Delta x_k} - j_k\right).$$

where ϕ is the reference LG basis function.

- The factor $\frac{1}{\sqrt{\Delta x_1 \cdots \Delta x_N}}$ gives the correct scaling in the integration

Existing results and known facts:

• LG present different techniques to account for the deformation of the basis functions which does not allow for an exact computation of the right–hand side integrals

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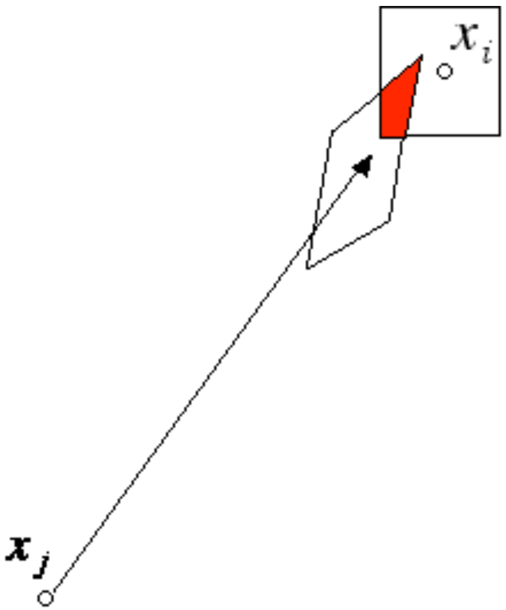
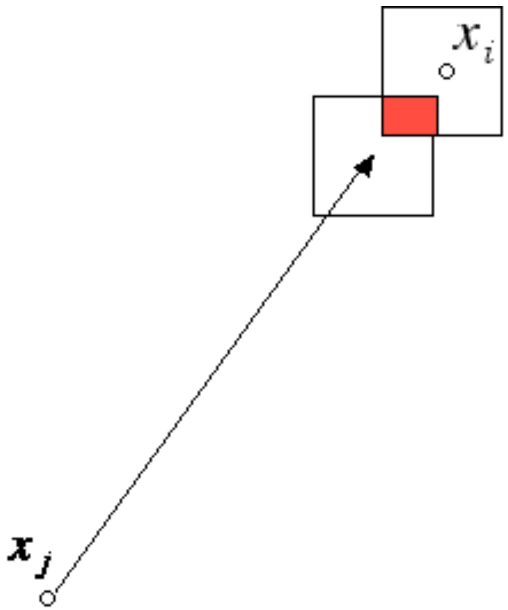
• LG present different techniques to account for the deformation of the basis functions which does not allow for an exact computation of the right–hand side integrals

• Stability results: rigorous L^2 stability analysis for exact schemes in a variety of cases, Von Neumann stability analysis for the inexact integration case

• A rigorous stability analysis is also possible if basis functions are assumed not to be deformed by advection



Pure LG scheme: the basis functions are deformed by advection



Rigid advection: the basis functions are translated without deformations

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Recasting SL as LG schemes – constant speed

In this case:

- The advection does not deform the basis functions so that the LG scheme could be exactly implemented

- The condition of equivalence between SL and LG schemes expresses the reference function ϕ as a function of ψ and takes the form of an integral equation:

$$\int_{\mathbb{R}} \phi(\eta + t)\phi(\eta)d\eta = \psi(t)$$

that is, ϕ must have ψ as its autocorrelation

This problem has a solution (in general, nonunique) if and only if:

- The function ψ is positive definite, that is

$$\sum_{k=1}^{n}\sum_{j=1}^{n} a_k \psi(t_k - t_j)\bar{a}_j \geq 0$$

for any $t_k \in \mathbb{R}$, $a_k \in \mathbb{C}$ $(k = 1, \ldots, n)$ and for all $n \in \mathbb{N}$

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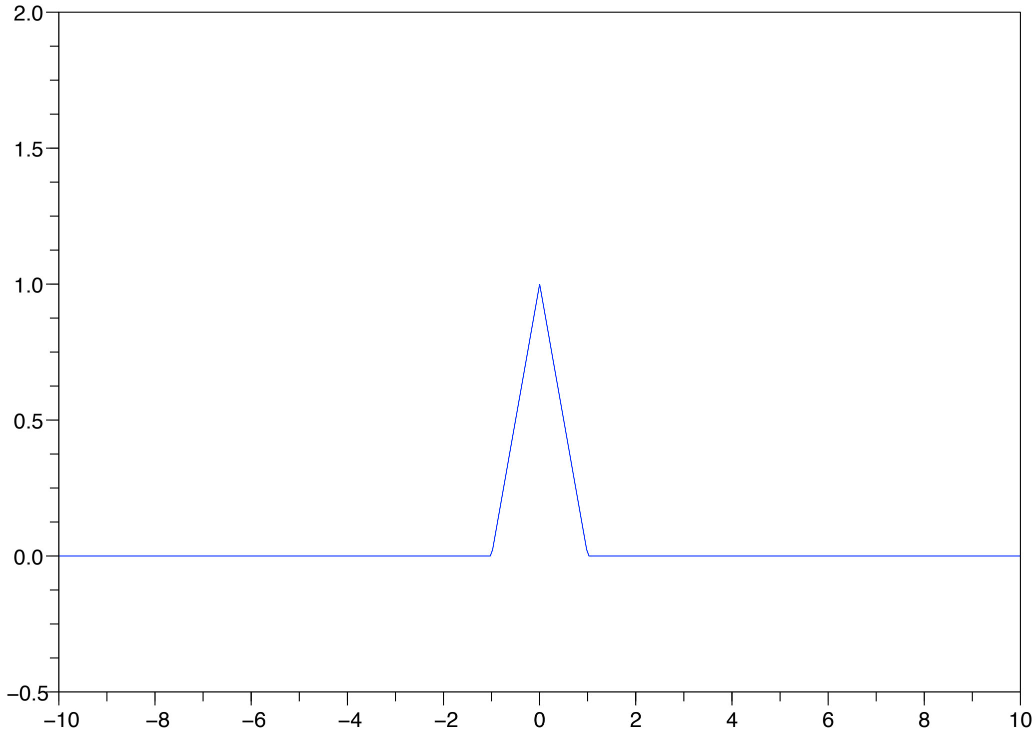
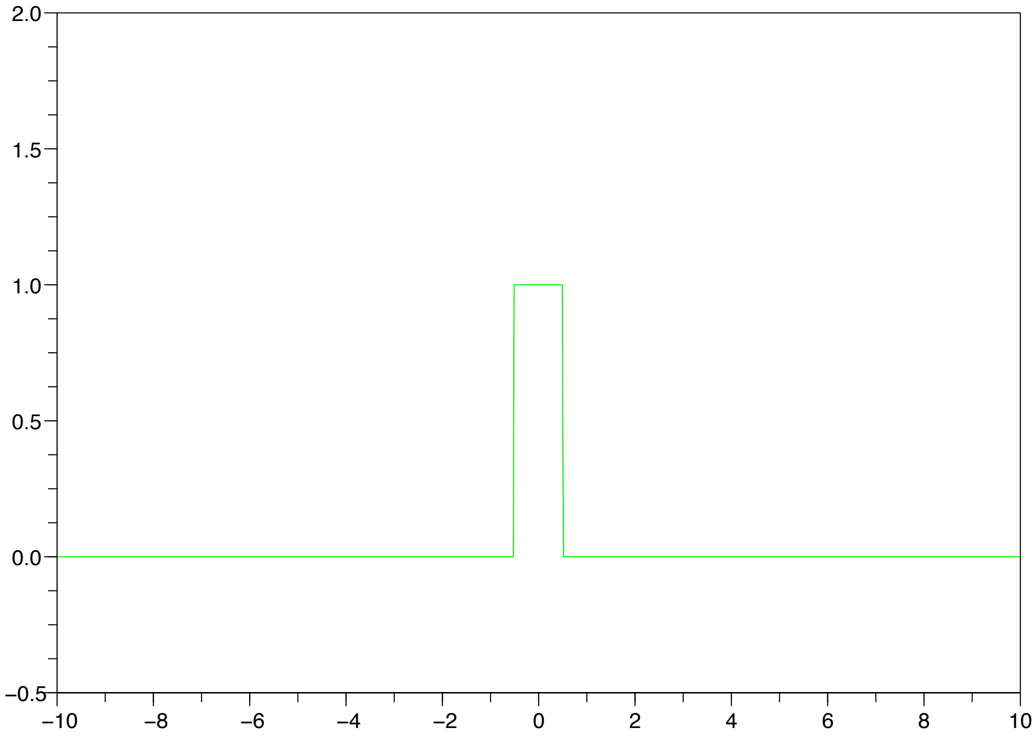
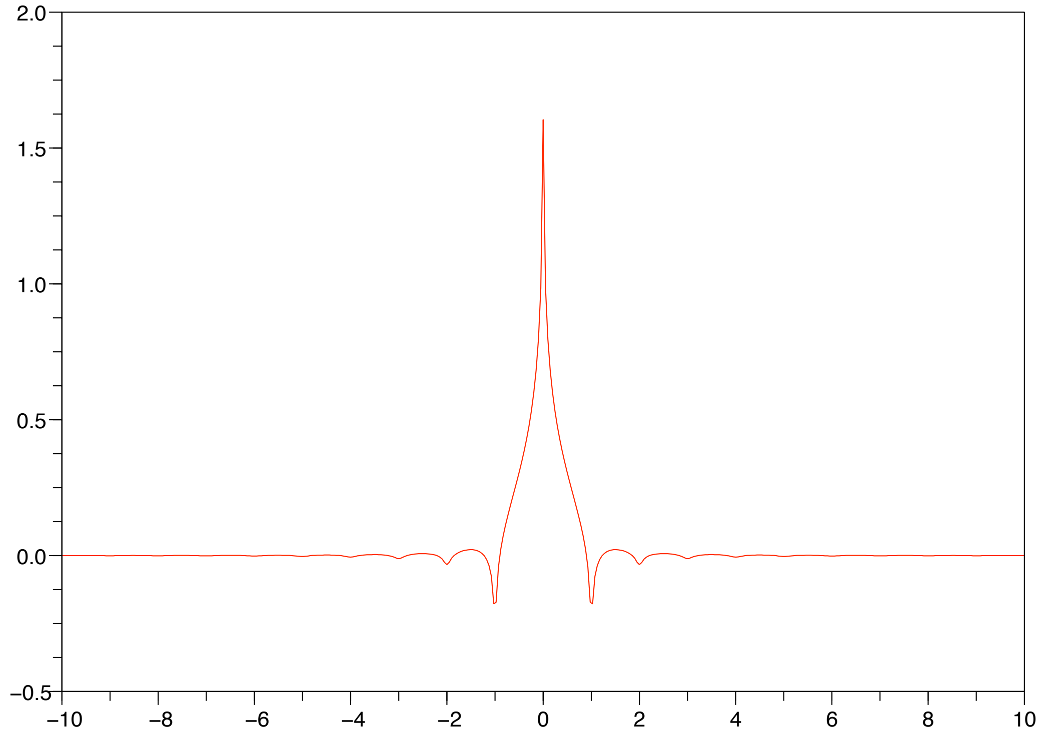
- Existence of a solution implies L^2 stability of SL schemes



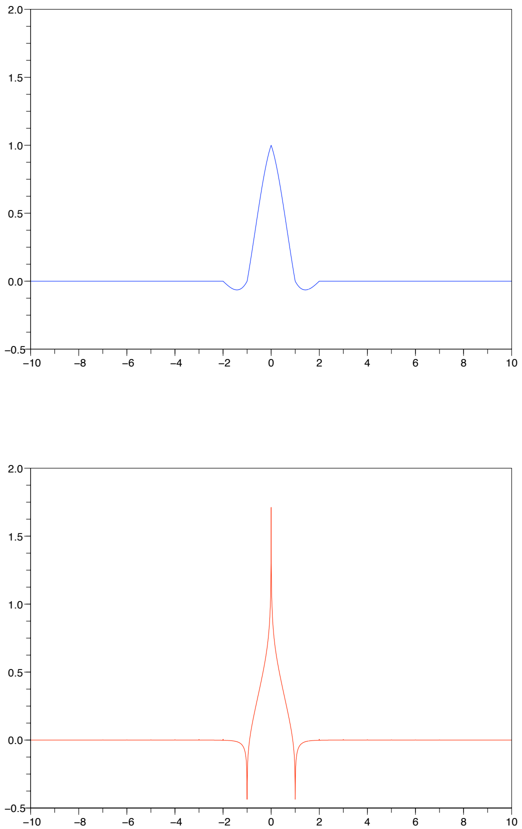
The reference function ψ for $\mathbb{P}_1$ interpolation



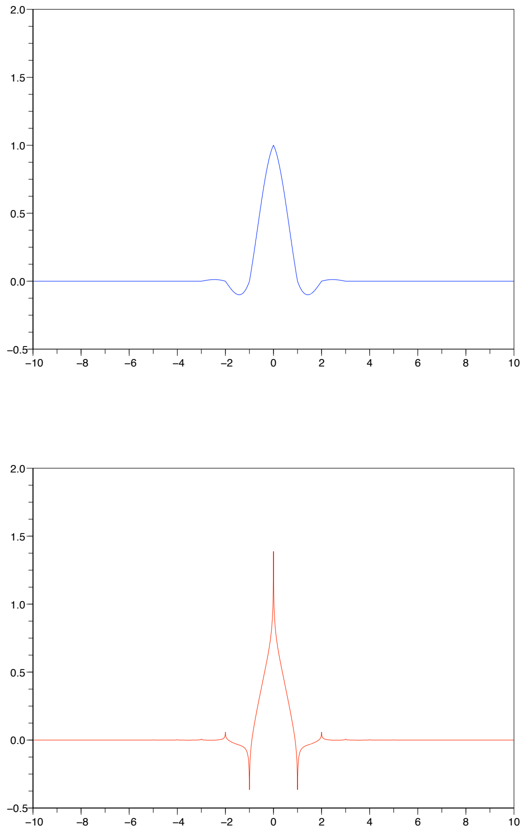
The "obvious" LG counterpart ϕ for $\mathbb{P}_1$ interpolation



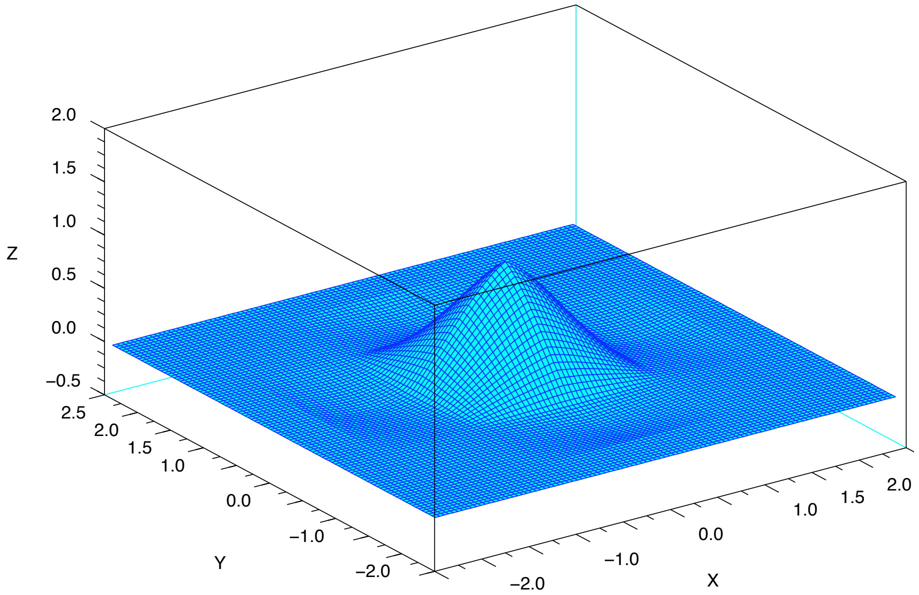
The minimal phase LG counterpart ϕ for $\mathbb{P}_1$ interpolation



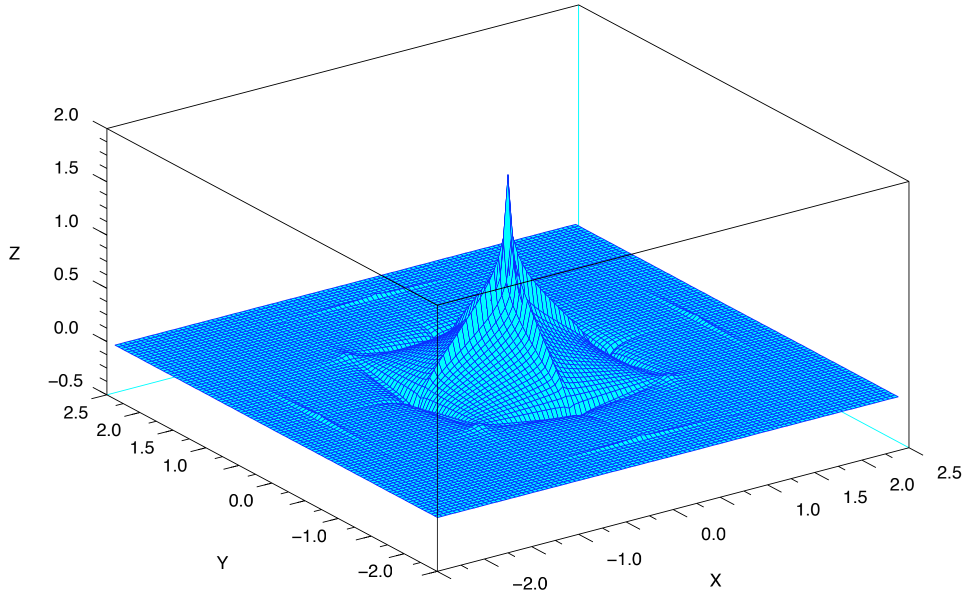
SL and LG, cubic



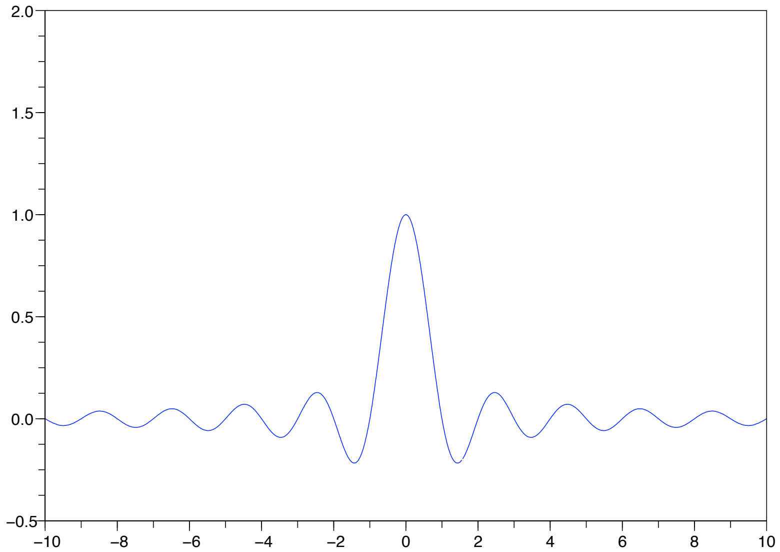
SL and LG, quintic



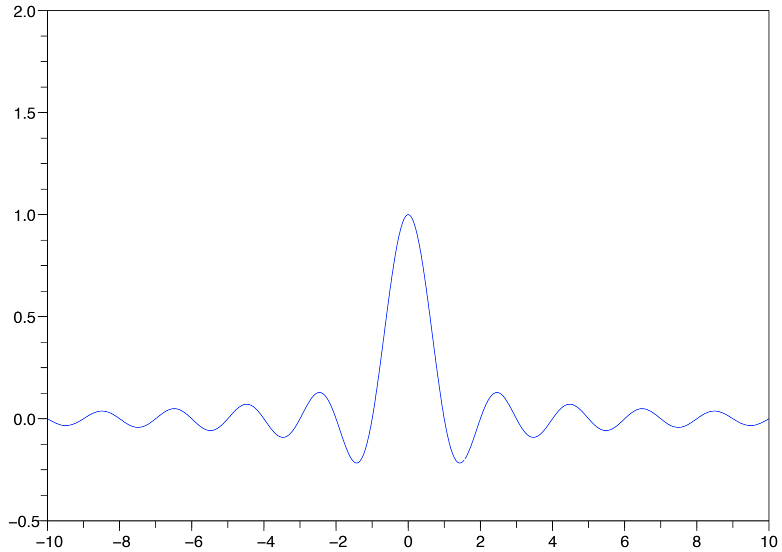
SL, cubic, $\mathbb{R}^2$



LG, cubic, $\mathbb{R}^2$



SL, Shannon wavelets



LG, Shannon wavelets

Situations covered by this result:

• High–order Lagrange interpolations which can be shown to have a positive Fourier transform (tested for $n \leq 13$):

$$\widehat{\psi}^{(n)}(\omega) = p(\omega^2)\,\frac{\sin\left(\frac{\omega}{2}\right)^{n+1}}{\left(\frac{\omega}{2}\right)^{n+1}}$$

with $p(\omega^2)$ a polynomial of degree $[n/2]$ with positive coefficients.

• Interpolatory wavelets, usually defined to be positive definite functions (e.g., in the case of the Shannon wavelet, $\widehat{\psi}(\omega) = 1_{(-\pi,\pi)}(\omega)$).

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Recasting SL as LG schemes – general linear case

• In this case, the previous technique can only be applied if basis functions are advected without deformations

• Such a LG type scheme ("area weighting LG scheme") has actually been proposed and analysed by Morton, Priestley and Süli (M^2AN, '88) with Lipschitz continuous, compactly supported bases

• Morton, Priestley and Süli's technique is not directly applicable to our basis functions, which may only be characterized by means of their Fourier transform

• Roughly speaking, the extension of this analysis requires bounded variation base functions with suitable decay conditions (details still to be checked), but this is the case for the LG bases corresponding to high–order Lagrange interpolation.