Grundy Colorings of Graphs and Reverse Mathematics

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- All graphs are "countable".
- A coloring is a function $f : V(G) \longrightarrow \omega$, where $x \sim y \implies f(x) \neq f(y)$.
- The chromatic number of G is $\chi(G)$, the least n such that there is a coloring $f : V \longrightarrow n$.

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where $\chi(\mathbf{K}) = \sup\{\chi(G) : G \in \mathbf{K}\}$.

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Equivalently, **K** is natural iff $\mathbf{K} = \mathbf{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a set of finite, connected graphs. Here, $\mathcal{F}$ is the set of forbidden graphs, which are those finite embeddable graphs in any $G \in \mathbf{K}$.

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In practice, (but it's not a theorem)

$$\chi(\mathbf{K}) = n < \omega \implies \mathsf{WKL}_0 \vdash \chi(\mathbf{K}) = n\ .$$

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It's easy to prove that $\chi(\mathbf{I}_n) = n$. Same proof shows that $\mathsf{WKL}_0 \vdash \chi(\mathbf{I}_n) = n$. In fact, $\mathsf{WKL}_0 \vdash \forall x\big(\chi(\mathbf{I}_x) = x\big)$.

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The On-Line chromatic number of G, $\chi_{\mathrm{OL}}(G)$, is the least n for which Painter has a winning strategy. Let $\chi_{\mathrm{OL}}(\mathbf{K}) = \sup\{\chi_{\mathrm{OL}}(G) : G \in \mathbf{K}\}$.

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$$\chi_{\mathrm{OL}}(\mathbf{K}) \leq n < \omega \implies \mathrm{RCA}_0 \vdash \chi(\mathbf{K}) \leq n.$$

For example, Kierstead & Trotter showed that $\chi(\mathbf{I}_n) = 3n - 2$, and it easily follows from their proof that

$$\mathrm{RCA}_0 \vdash \forall x \geq 1[\chi(\mathbf{I}_x) \leq 3x - 2].$$

However, there is a reversal [see my paper in Simpson's 2001].

THEOREM: $n < \chi_{\mathsf{OL}}(\mathbf{K}) \implies \mathsf{RCA}_0 \vdash \left[\chi(\mathbf{K}) \leq n \rightarrow \mathsf{WKL}_0\right]$.

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For example, if $2 \leq n < \omega$, then the following are equivalent over RCA_0:

- WKL_0;
- $\chi(\mathbf{I}_n) = n$;
- $\chi(\mathbf{I}_n) \neq 3n - 2$.

The Grundy number of G is $\Gamma(G) =$ the *largest* n such that there is an onto Grundy coloring $f : V(G) \longrightarrow n$.

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Easily, $\chi(G) = \gamma(G) \leq \Gamma(G)$, and for all $n < \omega$,

$$\mathrm{RCA}_0 \vdash \forall G[\chi(G) \leq n \rightarrow \gamma(G) = \chi(G)].$$

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For example, if $2 \leq n < \omega$, then there is c_n, $4n - 9 \leq c_n \leq 8n$, such that the following are equivalent over RCA_0:

- ACA_0;
- $\gamma(\mathbf{I}_n) = n$;
- $\gamma(\mathbf{I}_n) \neq c_n$.