Tao's correspondence principle, a finitary mean ergodic theorem and conservation results for Ramsey's theorem for pairs

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

Proof interpretations as tool for generalizing proofs

$$\begin{array}{ccccc} P & \xrightarrow{\mathcal{I}} & P^{\mathcal{I}} & & \\ G \downarrow & & \downarrow \mathcal{I}^G & & \\ P^G & \xrightarrow{G^{\mathcal{I}}} & (P^{\mathcal{I}})^G & = & (P^G)^{\mathcal{I}} \end{array}$$

- Generalization $(P^{\mathcal{I}})^G$ of $P^{\mathcal{I}}$: **easy!**
- Generalization P^G of P: **difficult!**

T. Tao: $P =$ **'soft analysis'**, $P^{\mathcal{I}} =$ **'hard analysis'**.

Monotone convergence principle (PCM)

Consider

$$A :\equiv \forall x\, \exists y\, \forall z\, A_{qf}(x,y,z), \quad A_{qf} \text{ **quantifier-free**.}$$

Example:

$$\forall k\, \exists n\, \forall m\, (|r_n - r_{n+m}| \leq 2^{-k}).$$

where (r_n) is a nonincreasing sequence in $[0,1] \cap \mathbb{Q}$.

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Problem: no computable f (E. Specker 1949).

2. Attempt: no-counterexample interpretation

Change

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to the equivalent **Herbrand normal form**

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We now ask for $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ s.t.

$$\forall x, g A_{qf}(x, \Phi(x,g), g(\Phi(x,g)))$$

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Solvable: Let $\tilde{g}(n) := n + g(n)$.

$$\Phi((r_n), k, g) := \min y \leq \max_{i \leq 2^k - 1} (\tilde{g}^{(i)}(0)) \left(|r_y - r_{\tilde{g}(y)}| \leq 2^{-k} \right).$$

Problems of the no-counterexample interpretation

N.c.i. weak enough to ensure an effective solution but except for $\forall\exists\forall$-sentences A **too weak** to provide the correct computational contribution of A in given proofs.

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IPP is strictly in between **$\exists$- and $\exists\forall$-induction**.

In particular: IPP equivalent (over RCA_0^*) to $B\Sigma_2^0$ (J. Hirst) and so can cause arbitrary **primitive recursive complexity**, but it has a trivial n.c.i.:

$$(\mathrm{IPP})^H \equiv$$
$$\forall n \geq 1 \forall f : \mathbb{N} \to n \forall F : n \to \mathbb{N} \exists i < n \exists m \geq F(i) \left(f(m) = i \right).$$

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Trivial n.c.i.-solution:

$$M(n, f, F) := \max\{F(i) : i < n\} \text{ and } I(n, f, F) := f(M(n, f, F))$$

are realizers for '$\exists m$' and '$\exists i$' in $(\text{IPP})^H$.

M, I **do not reflect** true complexity of IPP!

Gödel Functional Interpretation G (Gödel)

The axiom of **quantifier-free choice**

$$\text{QF-AC}: \ \forall x \exists y\, A_{qf}(x, y) \to \exists Y \forall x\, A_{qf}(x, Y(x)).$$

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$$(\text{IPP})^G \equiv \begin{cases} \forall f : \mathbb{N} \to n \forall K : n \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N} \exists i < n \exists g : \mathbb{N}^{\mathbb{N}} \\ \qquad\qquad (g(K(i,g)) \geq K(i,g) \wedge f(g(K(i,g))) = i). \end{cases}$$

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Solution nontrivial: uses a restricted form of so-called bar recursion which unrestricted interprets full analysis (P. Oliva 2006).

Monotone functional interpretation (MFI)of PCM

MFI (K.1996) extracts uniform monotone bounds.

PCM revisited: (a_n) nondecreasing sequence in $[0,1]$. MFI asks for Φ s.t. for all $k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N}$

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Corollary

(T. Tao's finite convergence principle, 2007)

$$\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 0 \leq a_0 \leq \ldots \leq a_M \leq 1 \exists N \in \mathbb{N}$$
$$(N + g(N) \leq M \wedge \forall n, m \in [N, N + g(N)](|a_n - a_m| \leq 2^{-k})).$$

In fact, we take $M := \tilde{g}^{(2^k)}(0)$.

Monotone interpretation of IPP

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This yields **a version** FIPP of Tao's 'finitary' IPP (equivalent to IPP over EA^2+WKL, i.e. roughly over WKL_0^*).

Formulations of FIPP in $\mathcal{L}(\mathrm{RCA}_0)$

Tao's first formulation FIPP1 for a version of FIPP (June 2007):

$$\mathrm{FIPP}_1 :\equiv \forall n \geq 1 \forall F \in AS \exists k \forall f : k \to n \, \exists c < n \, \exists A = f^{-1}(c) \left(|A| \geq F(A) \right),$$

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$F \in AS$ ('F asymptotically stable') iff $F : \mathcal{P}_{fin}(\mathbb{N}) \to \mathbb{N}$ and for all chains $A_1 \subseteq A_2 \subseteq$ of finite sets

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Here finite sets are given by their codes so that F is represented as a function $\mathbb{N} \to \mathbb{N}$.

The reasoning based on monotone functional interpretation yields instead the following weaker formulation:

$$\mathrm{FIPP}_2 :\equiv \forall n \geq 1 \forall F \in AS \exists k \forall f : k \to n \exists c < n \exists A \subseteq f^{-1}(c) \left(|A| \geq F(A)\right).$$

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Proposition (J. Gaspar, Aug. 2008)

FIPP_1 is false!

In reaction to Gaspar's counterexample to FIPP_1 Tao suggested the following corrected version

$$\mathrm{FIPP}_3 :\equiv \forall n \geq 1 \forall F \in ASNIS \exists k \forall f : k \to n \exists c < n \exists A = f^{-1}(c) \left(|A| \geq F(A)\right),$$

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where $F \in ASNIS$ ('F asymptotically stable near infinite sets') means that F is eventually constant on any family (A_n) of finite sets that converges to an infinite set A in the sense

$$\forall B \left(B \text{ finite } \to \exists k \forall m \geq k \left(A_m \cap B = A \cap B\right)\right).$$

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Proposition (J. Gaspar/K. 2008)

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Since $\mathsf{WKL}_0 \nvdash \mathsf{IPP}$ (J. Hirst) the first equivalence is nontrivial while for FIPP_3 we only know the equivalence

$$\mathsf{ACA}_0 \vdash \mathsf{IPP} \leftrightarrow \mathsf{FIPP}_3,$$

which is trivial as ACA_0 proves both principles.

Analysis in terms of ‘correspondence principles’

From Tao’s discussion of correspondence principles we got the following ‘continuous uniform bounded principle CUB’

$$\text{CUB}: \ cont(A) \wedge \forall f : \mathbb{N} \to n \, \exists x \, A(f,x) \to \exists z \forall f : \mathbb{N} \to n \, \exists x \leq z \, A(f,x),$$

where $cont(A)$ is the formula

$$\forall f : \mathbb{N} \to n \, \forall z \, \exists y \, \forall g : \mathbb{N} \to n$$
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Φ-CUB is CUB restricted to formulas $A \in \Phi$.

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Proposition (J. Gaspar/K.)

1. Over RCA_0 : Σ^0_1-CUB $\leftrightarrow$ WKL.
2. Over RCA_0 : Π^0_1-CUB $\leftrightarrow$ ACA.
3. RCA+CUB $=$ full second order arithmetic.

Logical metatheorems for uniformity in the absence of compactness

- **concrete Polish (P) and compact (K) metric spaces** are represented via $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$. Macros '$\forall x \in P, y \in K$'.

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  Crucial: no separability assumptions.

A formal system for analysis

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

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$\mathcal{A}^{\omega}[X, \langle\cdot,\cdot\rangle]$ results by adding constants and axioms expressing that $(X, \langle\cdot,\cdot\rangle)$ is a (real) Hilbert space.

$f : X \to X$ nonexpansive: $\|f(x) - f(y)\| \leq \|x - y\|$.

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then one can extract a **computable functional** $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t. for all $x \in P, y \in K, b \in \mathbb{N}$

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Similar results for metric, hyperbolic, CAT(0), normed and uniformly convex spaces.

Proof Mining in Ergodic Theory

Let X be a **Hilbert space**, $f : X \to X$ **linear and nonexpansive**.

$$A_n(x) := \frac{1}{n+1} S_n(x), \text{ where } S_n(x) := \sum_{i=0}^{n} f^i(x) \quad (n \geq 0).$$

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$$A_n(x) := \frac{1}{n+1} S_n(x), \text{ where } S_n(x) := \sum_{i=0}^{n} f^i(x) \quad (n \geq 0).$$

Theorem (von Neuman Mean Ergodic Theorem)

For every $x \in X$, the sequence $(A_n(x))_n$ converges.

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Based on logical metatheorem discussed above:

Theorem (K./Leuştean, to appear in Ergodic Theor. Dynam. Syst.)

Assume that X is a uniformly convex Banach space, η is a modulus of uniform convexity and $f : X \to X$ is a nonexpansive linear operator. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta) \, \forall i, j \in [n, n + g(n)] \left(\|A_i(x) - A_j(x)\| < \varepsilon \right),$$

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where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$
$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16} \eta \left(\frac{\varepsilon}{8b} \right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$
$$h, \tilde{h} : \mathbb{N} \to \mathbb{N}, \ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \leq n} h(i).$$

Corollary (K./Leuştean 2008)

X Hilbert space and $f : X \to X$ nonexpansive linear operator. Let $b > 0$. Then for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

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where Φ is defined as above, but with $K := \left\lceil \frac{512 b^2}{\varepsilon^2} \right\rceil$.

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Discussion: The Hilbert space case has been treated (again based on our metatheorem) prior by Avigad-Gerhardy-Towsner (TAMS to appear). However, the bound obtained by Avigad et al. is less good and matches our bound only in the special case of isometric f.

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‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

Ramsey's theorem for pairs

Theorem (A. Kreuzer/K.)

For each **fixed** $n \geq 2$:

$$\mathrm{EA}^2 + \mathrm{WKL} \vdash \forall c : [\mathbb{N}]^2 \to n \left(\Pi^0_1\text{-AC}(\xi(c)) \to \mathrm{RT}^2_n(c)\right).$$

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Remark

One can also formulate things in $\mathcal{L}(RCA_0)$. Then EA^2+WKL can be replaced by WKL_0^*.

Theorem (K.1998)

Let $\mathcal{T}^\omega$:=E-G$_\infty$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL, ξ a closed term.

$$\begin{cases} \mathcal{T}^\omega \vdash \forall f : \mathbb{N}^{\mathbb{N}} \left(\forall k \in \mathbb{N}\, \Pi^0_1\text{-AC}(\xi(f,k)) \to \exists x \in \mathbb{N}\, A_{qf}(f,x)\right) \\ \Rightarrow \text{ one can extract a (Kleene-)primitive recursive functional } \Phi \text{ s.t.} \\ \text{PRA}^\omega \vdash \forall f : \mathbb{N}^{\mathbb{N}}\, A_{qf}(f,\Phi(f)). \end{cases}$$

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$$\Pi^0_1\text{-AC}(f): \ \forall m \exists n \forall p \, (f(m,n,p) = 0) \to \exists h \forall m,n \, (f(m,h(n),p) = 0)).$$

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In addition to Π^0_1-AC$(\xi(f,k))$ one may also have Δ^0_2-CA$(\xi'(f,k))$ and hence Σ^0_1-IA$(\xi'(f,k))$, BW$(\xi''(f,k)$ etc.

Both theorems together yield:

Theorem (A. Kreuzer/K.,Nov 2008)

Let $\mathcal{T}^\omega :=$E-G$_\infty$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL, ξ a closed term.

$$\begin{cases} \mathcal{T}^\omega \vdash \forall f : \mathbb{N}^{\mathbb{N}} \left(\forall k\, \Pi^0_1\text{-AC}(\xi_1(f,k)) \wedge \forall k\ \mathbf{RT}^2_n(\xi_2(f,k)) \right. \rightarrow \exists x\, A_{qf}(f,x)) \\ \Rightarrow \text{ one can extract a (Kleene-)primitive recursive functional } \Phi \text{ s.t.} \\ \text{PRA}^\omega \vdash \forall f : \mathbb{N}^{\mathbb{N}}\, A_{qf}(f,\Phi(f)). \end{cases}$$

For a schema S let S^- denotes its restriction to instances which only have number parameters.

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$\mathcal{T}^\omega$:=E-G$_\infty$A$^\omega$+QF-AC^{1,0}+QF-AC^{0,1}+WKL+Π^0_1-AC$^-$+RT$^{2-}_n$ is

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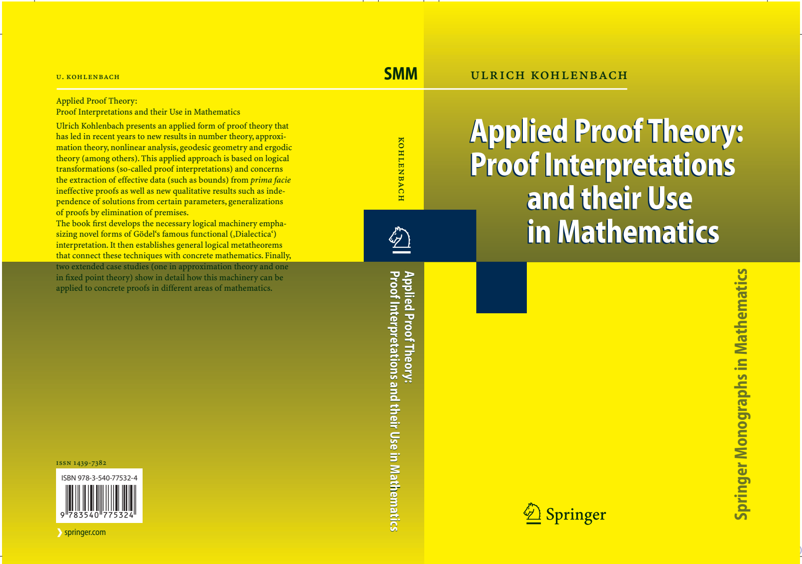
- Π^0_2-conservative over PRA,
- Π^0_3-conservative (but not Σ^0_3-cons.) over PRA+$I\Sigma^0_1$,
- Π^0_4-conservative over PRA+$B\Sigma^0_2$.

Similar results hold for the corresponding theories obtained by adding abstract metric, hyperbolic, normed and Hilbert spaces X.


U. KOHLENBACH

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Applied Proof Theory: Proof Interpretations and their Use in Mathematics

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Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from *prima facie* ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.
The book first develops the necessary logical machinery emphasizing novel forms of Gödel's famous functional (,Dialectica') interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.

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