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Computability and Orders on Structures

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Orders on magmas

- Magma $(M, \cdot)$ is (partially) *left-orderable* if
  there is a linear (partial) ordering $<$ on M that is left invariant:
  $(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y]$

- M is *bi-orderable* (*orderable*) if
  $(\forall x, y, z)[x < y \Rightarrow z \cdot x < z \cdot y \wedge x \cdot z < y \cdot z]$

- $LO(M)$ the set of left orders on M
  $RO(M)$ the set of right orders on M
  $BiO(M)$ the set of bi-orders on M

- Given a left order $<_l$ on a group G, we have a right order $<_r$:
  $x <_r y \Leftrightarrow y^{-1} <_l x^{-1}$

- G is left-orderable group $\Rightarrow$ G is *torsion-free*
  torsion-free: $(\forall x \in G - \{e\})[order(x) = \infty]$
  $e < x \Rightarrow x < x^2 < \cdots < x^n$

- Every torsion-free nilpotent group is orderable.

- Torsion-free, but not left-orderable group:

$$G = \langle x, y \mid xy^2x^{-1}y^2 = e,\ \ yx^2y^{-1}x^2 = e \rangle$$

- Let $<$ be a partial left order on a group G
  Positive partial cone: $P = \{a \in G \mid a \geq e\}$
  Negative partial cone: $P^{-1} = \{a \in G \mid a \leq e\}$

  1. $PP \subseteq P$ (P *sub-semigroup* of G)
  2. $P \cap P^{-1} = \{e\}$ (P *pure*)

- P with 1 & 2 defines a partial left order $\leq_P$ on G:

$$x \leq_P y \Leftrightarrow x^{-1}y \in P$$

  $x \leq_P y \Rightarrow x^{-1}y \in P \Rightarrow$
  $x^{-1}z^{-1}zy = (zx)^{-1}(zy) \in P \Rightarrow zx \leq_P zy$

- P with 1 & 2 defines a *left order* if
  3. $P \cup P^{-1} = G$ (P *total*)

- P with 1, 2 & 3 defines a *bi-order* if:
  4. $(\forall g \in G)[g^{-1}Pg \subseteq P]$ (P *normal*)
  For groups, orders often identified with their positive cones.

- Example: $G = \mathbb{Z} \oplus \mathbb{Z}$ bi-orderable with a positive cone

$$P = \{(a, b) \mid 0 < a \vee (a = 0 \wedge 0 \leq b)\}.$$

- Fundamental group of Klein bottle
  $G = \left\langle x, y \mid xyx^{-1}y = e \right\rangle$ left-orderable, but not bi-orderable.

  Positive cone $P = \{x^n y^m \mid n > 0 \vee (n = 0 \wedge m \geq 0)\}$
  defines a left order on G.

  If $<$ a bi-order, then $y > e$ or $y < e$.
  $y > e \Rightarrow y^{-1} = xyx^{-1} > e$

- A magma $(Q, *)$ is a *quandle* if:
  1. $(\forall a)[a * a = a]$ (idempotence);
  2. for every $b \in Q$, the mapping $*_b : Q \to Q$ defined by $*_b(a) = a * b$ is bijective;
  3. $(\forall a, b, c)[(a * b) * c = (a * c) * (b * c)]$ (right self-distributivity).

- A quandle Q is called *trivial* if the operation $*$ is defined by $(\forall a, b)[a * b = a]$.
  Every linear ordering of elements of Q is right invariant.

- For a group G, the *conjugate* quandle $\mathsf{Conj}(G)$ is one with domain G and the operation $*$ given by $a * b = b^{-1}ab$.
  Then every bi-order on G induces a right order on $\mathsf{Conj}(G)$.
  Possible to have $BiO(G) = \emptyset$ and $RO(\mathsf{Conj}(G)) \neq \emptyset$.

Topology on $LO(M)$

- Topology defined on $LO(M)$ by subbasis $\{S_{(a,b)}\}_{(a,b)\in(M\times M)-\Delta}$ where $\Delta = \{(a,a) \mid a \in M\}$:

$$S_{(a,b)} = \{R \in LO(M) \mid (a,b) \in R\}.$$

- (Dabkowska, Dabkowski, Harizanov, Przytycki, Veve, 2007)

  Let M be a magma with cardinality $|\mathcal{M}| = m \geq \aleph_0$.
  Then $LO(M)$ is a closed subspace of the Cantor cube $\{0,1\}^{\mathfrak{m}}$.
  In particular, $LO(M)$ is a compact space.

- If M is a countable magma, then $LO(M)$ is metrizable.

- If $M = G$ is a group, we showed how we could also use Conrad's theorem to establish that $LO(G)$ is compact.

- (Conrad, 1959) A partial left order P can be extended to a total left order on G iff for every $\{x_1, ..., x_n\} \subset G \backslash \{e\}$ there are $\epsilon_1, ..., \epsilon_n$, $\epsilon_i \in \{1, -1\}$, such that

$$e \notin sgr((P \backslash \{e\}) \cup \{x_1^{\epsilon_1}, ..., x_n^{\epsilon_n}\}),$$

  where $sgr(A)$ is the sub-semigroup of G generated by A.

- (Dabkowska, 2006)
  The space $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set.

- (Sikora, 2004)
  The space $LO(\mathbb{Z}^n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Solomon, 1998)
  For every orderable computable group G, there is
  a computable binary tree $\mathcal{T}$ and a Turing degree preserving bijection
  from $BiO(G)$ to the set of all infinite paths of $\mathcal{T}$.

- Hence, by the Low Basis Theorem of Jockusch and Soare,
  $\mathcal{T}$ has a *low* infinite path.

- Hence $BiO(G)$ contains an order of *low* Turing degree.

- (Downey, Kurtz, 1986)
  There is a computable torsion-free abelian group with
  no computable order.

- *Turing degree spectrum* of left-orders on computable G :

$$DgSp_G(LO) = \{\mathsf{deg}(P) \mid P \in LO(G)\}$$

$\mathsf{deg}(P) = \mathsf{deg}(\leq_P)$

$\mathcal{D}$ = the set of all Turing degrees

- (Solomon, 2002)
$DgSp_G(LO) = \mathcal{D}$
for a torsion free abelian group G of finite rank $n > 1$.

- (Solomon, 2002)
$DgSp_G(LO) \supseteq \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{0}'\}$
for a torsion free abelian group G of infinite rank.

- A group G for which every partial (left) order can be extended to a total (left) order is called *fully orderable* (*fully left-orderable*).

  Torsion-free abelian groups are fully orderable.

- (Dabkowska, Dabkowski, Harizanov, Togha, ta)
  Let G be a computable, *fully* left-orderable group and
  $\mathbf{d}$ a Turing degree such that:
  (a) No left order on G is determined uniquely by its finite subset $A \subset G \backslash \{e\}$;
  (b) For a finite $A \subset G \backslash \{e\}$, the problem '$e \in sgr(A)$' is $\mathbf{d}$-decidable;
  (c) $DgSp_G(LO)$ closed upward.
  Then

$$DgSp_G(LO) \supseteq \{\mathbf{a} \in \mathcal{D} \mid \mathbf{a} \geq \mathbf{d}\}$$

  and $LO(G)$ is homeomorphic to the *Cantor set*.

Orders on free groups F_n

- (Dabkowska, Dabkowski, Harizanov, ta)
  For a free group F_n of rank $n > 1$, we have $DgSp_{F_n}(BiO) = \mathcal{D}$.

  Proof idea:

  Lower central series of F_n: $\gamma_1(F_n) \geq \cdots \geq \gamma_i(F_n) \geq \cdots$
  Recall $\gamma_1(F_n) = F_n$, $\gamma_{i+1}(F_n) = [\gamma_i(F_n), F_n]$

  Construct bi-orders on F_n using orders on $\gamma_i(F_n)/\gamma_{i+1}(F_n)$.
  $\bigcap_{i=1}^{\infty} \gamma_i(F_n) = \{e\}$
  $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$ for some k_i

- Free groups are not fully left-orderable.

- (Dabkowska, Dabkowski, Harizanov, Togha, ta)
  Let G be a computable group, $\mathbf{d}$ a Turing degree,
  $\mathbb{P} = \{p_i\}_{i \in \omega}$ a $\mathbf{d}$-computable strong array of finite subsets of $G \backslash \{e\}$
  such that for every $p \in \mathbb{P}$, we have $e \notin sgr(p)$ and

  (i) there are $a \in G \backslash \{e\}$ and $q, r \in \mathbb{P}$ such that
  $q \supseteq p \wedge r \supseteq p$ and $a \in q \wedge a^{-1} \in r$;

  (ii) for each $a \in G \backslash \{e\}$ there is $q \in \mathbb{P}$ such that
  $q \supseteq p$ and $a \in q \vee a^{-1} \in q$.

  Then $(\forall \mathbf{x} \geq \mathbf{d})(\exists \mathbf{z} \in DgSp_G(LO))[\mathbf{x} = \mathbf{z} \vee \mathbf{d}]$.

- If $DgSp_G(LO)$ is closed upward, then
  $\{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\} \subseteq DgSp_G(LO)$.

- If $\mathbf{d} = \mathbf{0}$, then $DgSp_G(LO) = \mathcal{D}$.

- (Navas-Flores, 2008)
  The space $LO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

- Conjecture (Sikora, 2004)
  The space $BiO(F_n)$ for $n > 1$ is homeomorphic to the Cantor set.

- (Linnell, 2006)
  The space of left orders of a countable orderable group is either finite or contains a homeomorphic copy of the Cantor set.

- There are countable groups with infinitely countably many bi-orders.