# Risk Transfer and Adverse Selection in Principal-Agent Models

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# Weather Derivatives, 1

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- The fastest growing sector in the Chicago Mercantile Exchange is that of the CME Weather products.
- In general, weather derivatives cover low-risk, high-probability events. Weather insurance, on the other hand, typically covers high-risk, low-probability events.
- Lately, hedge funds have been adding weather derivatives to their portfolio, which has helped boost the overall market.

# Weather Derivatives, 2

Due to the highly localized nature of the weather phenomena affecting different economic agents, there is a real need for over-the-counter products, for example:

- A Colorado peach farmer concerned about some early frosts ruining her crop before the harvest at the end of September. Currently the CME only offers futures and options on Frost Days for Amsterdam-Schiphol, Netherlands (WMO 06240).

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- A Colorado peach farmer concerned about some early frosts ruining her crop before the harvest at the end of September. Currently the CME only offers futures and options on Frost Days for Amsterdam-Schiphol, Netherlands (WMO 06240).
- An insurance company based in Oregon has historical data that suggests the incidence of claims recorded during heavy Winter storms is abnormally high, so the company would like to limit its risk exposure during these events. Currently the CME only offers Snowfall options for New York and Boston.

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We will present a scheme that could be followed to price over-the-counter weather derivatives.

# The risk measure,1

**Definition 1** *A risk measure* $\rho : \mathbb{L}^{\infty}(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ *is said to be* ***law invariant*** *if*

$$\rho(X) = \rho(Y)$$

*for any two random variables* $X$ *and* $Y$ *which have the same law .*

# The risk measure,2

The benchmark law invariant risk measure is $\rho(X) = AV@R_\lambda(X)$ ($\lambda \in (0,1]$) which can be expressed as

$$AV@R_\lambda(X) = \sup_{Q \in \mathcal{Q}_\lambda} E_Q[-X],$$

where

$$\mathcal{Q}_\lambda = \left\{ Q << P \mid \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}$$

# Formulation of the Problem, 1

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- A set $\Theta := [a, 1]$ of agent types, where $\theta \in \Theta$ is the risk tolerance coefficient of an agent, which is private information. We require $a > 0$.
- The distribution of the agents' types has a density $d\mu$, which is known by the Principal.

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We will assume the principal is selling claims of the form

$$X = f(W) \quad \text{for a price} \quad \pi(X)$$

and $(X, \pi(X))$ denotes a contract. These are offered in a take-it-or-leave-it basis. Agents have mean-variance preferences, namely, an agent of type $\theta \in \Theta$ evaluates position $X$ via

$$U(\theta, X) = E[X] - \theta Var[X].$$

# The Agents' Problem, 1

When deciding over contracts of the form $(X, \pi(X))$, the agents face the problem of finding

$$v(\theta) := \sup_{X \in \mathcal{X}} \{U(\theta, X) - \pi(X)\} = U(\theta, X(\theta)) - \pi(X(\theta)),$$

where

$$\mathcal{X} := \{X \in \mathbb{L}^{\infty}(\Omega \times \Theta, P \times \mu) \mid X \text{ is } \sigma(W) \times \mathcal{B}(\Theta) \text{ measurable}\}.$$

Notice that $v$ is a convex function, since it is the supremum of affine functions (on $\theta$).

# The Agents' Problem, 2

We say that a set of contracts (or catalogue) is incentive compatible (**IC**) if, for any $\theta$, $\theta' \in \Theta$,

$$U(\theta, X(\theta)) - \pi(X(\theta)) \geq U(\theta, X(\theta')) - \pi(X(\theta'))$$

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and we say the contracts are individually rational (**IR**) if

$$U(\theta, X(\theta)) - \pi(X(\theta)) \geq 0 \quad \text{for all} \quad \theta \in \Theta$$

where the reservation utility for all agents has been normalized to $0$.

# The Principal's Problem, 1

For each contract $(X, \pi(X))$ issued by the principal, she receives the amount $\pi(X)$, and she is subject to the liability $X$. The aggregate risky position for the Principal stemming from the exchanges with the agents is

$$\int_{\Theta} (\pi(X(\theta)) - X(\theta)) d\mu(\theta).$$

The Principal's objective is to devise a pricing schedule $\pi$ as to minimize

$$\rho\left(G(W) + \int_{\Theta} (\pi(X(\theta)) - X(\theta)) d\mu(\theta)\right)$$

subject to $(X, \pi)$ $\in$**IC**$\cap$**IR.**

# The Principal's Problem, 2

**Lemma 1** *The catalogue* $(X, \pi)$ *is Incentive Compatible iff* $v$ *is proper, convex, non-increasing and* $-Var[X(\theta)] \in \partial v(\theta)$.

Note that

$$\pi(X(\theta)) = E[X(\theta)] - \theta Var[X(\theta)] - v(\theta)$$

so...

# The Principal's Problem, 3

Minimizing

$$\rho\left(G(W) + \int_\Theta (\pi(X(\theta)) - X(\theta))d\mu(\theta)\right)$$

over the set of Incentive Compatible and Individually Rational contracts $(X, \pi)$ is equivalent to minimizing

$$\rho\left(G(W) + \int_\Theta (E[X(\theta)] - \theta Var[X(\theta)] - v(\theta) - X(\theta))d\mu(\theta)\right)$$

over $\{(v, X) \mid v \geq 0, v'' \geq 0, v'(\theta) = -Var[X(\theta)]\, a.e.\}$.

# The Principal's Problem, 4

Note that $X(\theta) := E[X(\theta)] - X(\theta)$ is a zero-mean process that satisfies $-Var[X(\theta)] = v'(\theta)$, which together with the fact that $\rho$ is monetary yields

$$\inf_{(v,X)\in\mathbb{A}} \rho\left(W + \int_\Theta X(\theta)d\mu(\theta)\right) + \int_\Theta \big(v(\theta) - \theta v'(\theta)\big)\, d\mu(\theta),$$

where $\mathbb{A}$ is the set

$$\{(v, X) \mid v \geq 0,\ v'' > 0,\ E[X(\theta)] = 0,\ -Var[X(\theta)] = v'(\theta)\, a.e.\}\,.$$

Notation:

$$I(v) := \int_\Theta \big(v(\theta) - \theta v'(\theta)\big)\, d\mu(\theta).$$

# Existence, 1

**Theorem 1** *Existence holds for the Principal's problem.*

## Outline of the proof

We assume for simplicity that $d\mu = d\theta$. Let

$$\langle \cdot, \cdot \rangle := \int\int_{\Theta \times \Omega} XY \, d\theta dP.$$

For a given
$v \in C^* := \{u : [a, 1] \to \mathbb{R} \mid u \text{ is convex}, \; u \geq 0, u' \leq 0\}$, consider

$$\mathcal{X}_v := \{X \in \mathcal{X} \mid E[X(\theta)] = 0, \, Var[X(\theta)] \leq -v'(\theta) \, a.e.\}$$

Notice that if $X \in \mathcal{X}_v$, then $\|X\|_2^2 \leq v(a)$.

# Existence, 2

**Proposition 1** *Let* $R(X) := \rho\left(G(W) - \int_\Theta X(\theta)d\theta\right)$, *then the following bounds hold for any pair* $(X, v)$ *that is acceptable from the point of view of the principal*

$$-E_P[G(W)] + I(v) \leq R(X) + I(v) \leq \rho(G(W)).$$

*which in turn implies*

$$v(a) \leq \frac{1}{a}\left(\rho(G(W)) + E_P(G(W)\right) := K.$$

It follows from the former expression that $\mathcal{X}_v$ is a closed, bounded and convex set, and the convexity of $R(X)$ implies

$$\inf_{X \in \mathcal{X}_v} R(X) = R(X_v) \quad \text{for some} \quad X_v \in \mathcal{X}_v.$$

# Existence, 3

Consider $\tilde{Y} \in \mathcal{X}_v$, fix $\overline{\theta} \in \Theta$ and define

$$Y := \frac{\tilde{Y}(\overline{\theta})}{\sqrt{Var[\tilde{Y}(\overline{\theta})]}}$$

and

$$\tilde{\alpha}(\theta) := -Cov[X_v(\theta), Y] \pm \sqrt{Cov^2[X_v(\theta), Y] - v'(\theta) - Var[X_v(\theta)]},$$

then

$$Var[X_v(\theta) + \tilde{\alpha}(\theta)Y] = -v'(\theta)$$

# Existence, 4

**Proposition 2** *$\tilde{\alpha}$ is summable and therefore there is a $\theta^* \in (a, 1)$ such that*

$$\int_{\Theta_v \cap (a,\theta^*]} \tilde{\alpha}(\theta) d\theta - \int_{\Theta_v \cap (\theta^*,1]} \tilde{\alpha}(\theta) d\theta = 0.$$

Let

$$\alpha(\theta) := \begin{cases} \tilde{\alpha}(\theta), & \text{if} \quad \theta \leq \theta^*; \\ -\tilde{\alpha}(\theta), & \text{if} \quad \theta > \theta^*. \end{cases}$$

Then $Z_v := X_v + \alpha Y \in \partial \mathcal{X}_v = \mathbb{A}$ and $R(X_v) = R(Z_v)$. Therefore $Z_v$ satisfies

$$R(Z_v) = \inf_{X \in \mathbb{A}} R(X).$$

# Existence, 5

The Principal's problem can be restated as

$$\mathcal{P} = \inf_{v \in C^*} R(Z_v) + I(v)$$

**Lemma 2** *Let* $\{u_k\} \subset C^*$, *then there exists* $\{u_{j_k}\} \subset \{u_k\}$ *and* $\overline{u} \in C^*$ *such that*

$$u_{j_k} \longrightarrow \overline{u}$$

*uniformly on compact subsets of* $(a, 1]$.

# Existence, 6

Consider a minimizing sequence $\{v_k\}$ of the Principal's problem, i.e.

$$\mathcal{P} = \lim_{k\to\infty} R(Z_{v_k}) + I(v_k).$$

It follows from lemma 2 that there exists a subsequence $\{v_{ik}\}$ that converges uniformly on compact subsets of $(a, 1]$ to some function $\overline{v} \in C^*$. Rename this sequence $\{v_n\}$ to simplify notation. Since $v_n - \theta v_n' \geq 0$, Fatou's Lemma implies that

$$I(\overline{v}) \leq \liminf_{n\to\infty} I(v_n).$$

# Existence, 7

**Lemma 3** *The function* $v \to R(Z_v)$ *is lower semi-continuous with respect to uniform convergence on compact subsets.*

The lemma implies

$$R(Z_{\overline{v}}) + I(\overline{v}) \leq \liminf_{n \to \infty} R(Z_{v_n}) + I(v_n) = \mathcal{P},$$

hence the catalogue $(Z_{\overline{v}}, \overline{v})$ solves the Principal's problem.

# One T-claim, 1

Consider the situation where the principal writes one T-claim on her risk

$$f(W) \geq 0$$

and offers contracts of the form

$$(\alpha(\theta)f(W), \pi(\alpha(\theta)f(W)))$$

where $\alpha(\theta)$ solves

$$\sup_{\alpha} U(\theta, \alpha f(W)) - \pi(\alpha f(W)).$$

# One T-claim, 2

With this simple structure, we can characterize $\alpha(\theta)$ in terms of $v'(\theta)$, namely:

$$\alpha(\theta) = \sqrt{\frac{-v'(\theta)}{Var[f(W)]}}.$$

Therefore, the risky part of the principal's problem becomes:

$$\rho\left(G(W) - f(W)C(v)\right),$$

where

$$C(v) := \frac{1}{Var[f(W)]}\int_{\Theta}\sqrt{-v'(\theta)}d\mu(\theta).$$

# One T-claim, 3

Considering the fact that $\rho(\cdot)$ is a decreasing function, the principal should try to make $C(v)$ as small as possible while keeping $\widetilde{I}(v)$ (her aggregate payment) as large as possible. We solved the program

$$\sup_{v \in C^*} \int_\Theta -\sqrt{-v'(\theta)}d\mu(\theta)$$

subject to

$$\widetilde{I}(v) := \int_\Theta \left( \frac{E[f(W)]}{Var[f(W)]} \sqrt{-v'(\theta)} - v(\theta) + \theta v'(\theta) \right) d\mu(\theta) = A$$

# One T-claim, 4

We found $C(v)$ in terms of $A$, so the the principal's problem reduces to finding the minimum over $A \in \left[0, \frac{(EN)^2}{4MV}\right]$ of

$$\rho\left(G(W) - f(W)\frac{N^2E}{2MV} + f(W)\frac{N}{2M\sqrt{V}}\sqrt{\frac{(NE)^2}{V} - 4AM}\right) - A,$$

# One T-claim, 5

Once $A$ has been determined (this will depend on the choice of $\rho$) the principal will offer

$$X(\theta) = \frac{\Gamma}{2}\frac{f(W)}{2\theta - a}$$

for a price

$$\pi(X(\theta)) = \Gamma^2\left(\frac{1}{8}\frac{1}{2-a} - \frac{1}{\sqrt{2}}\frac{\theta}{(2\theta - a)^2}\right) + \frac{\Gamma}{2\theta - a}\left(\frac{E}{2} - \frac{\Gamma}{8}\right)$$

where

$$\Gamma = \frac{1}{2M}\left(\frac{NE}{\sqrt{V}} - \sqrt{\frac{(NE)^2}{V} - 4AM}\right)$$

# One T-claim, 6

Assume the Principal is exposed to temperature risk. She measures this exposure using $\rho(X) = AV@R_{0.05}(X)$, and she faces the following scenario:

- $W \in [0, 1]$, where $0$ represents the coldest possible Winter, and $1$ the warmest one. The temperature is assumed to be normally distributed with mean $1/2$ and variance $1/20$.
- The original exposure of the Principal is given by $0.1(W - 1.1)$, and $\rho(G(W)) = 0.0612$.
- $f(W) = (W - 0.5)^+$.

# One T-claim, 7

By proceeding as above the Principal's risk evaluation becomes $-0.6731$, and she offers

$$X(\theta) = \frac{0.5459}{2\theta - a} f(W)$$

for a price

$$\pi(\theta) = \frac{1.1921}{8(2-a)} - \frac{(1.1921)\theta - (0.22)(2\theta - a)}{\sqrt{2}(2\theta - a)^2}$$

NOTE: These figures were estimated using the “Mosek" optimization toolbox, considering 150 States of the world. The execution took $198.135$ seconds.

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- Non Mean-Variance agents (although the additive structure of the agents’ preferences is heavily used currently).
- The problem with multiple principals.

# Thank You!