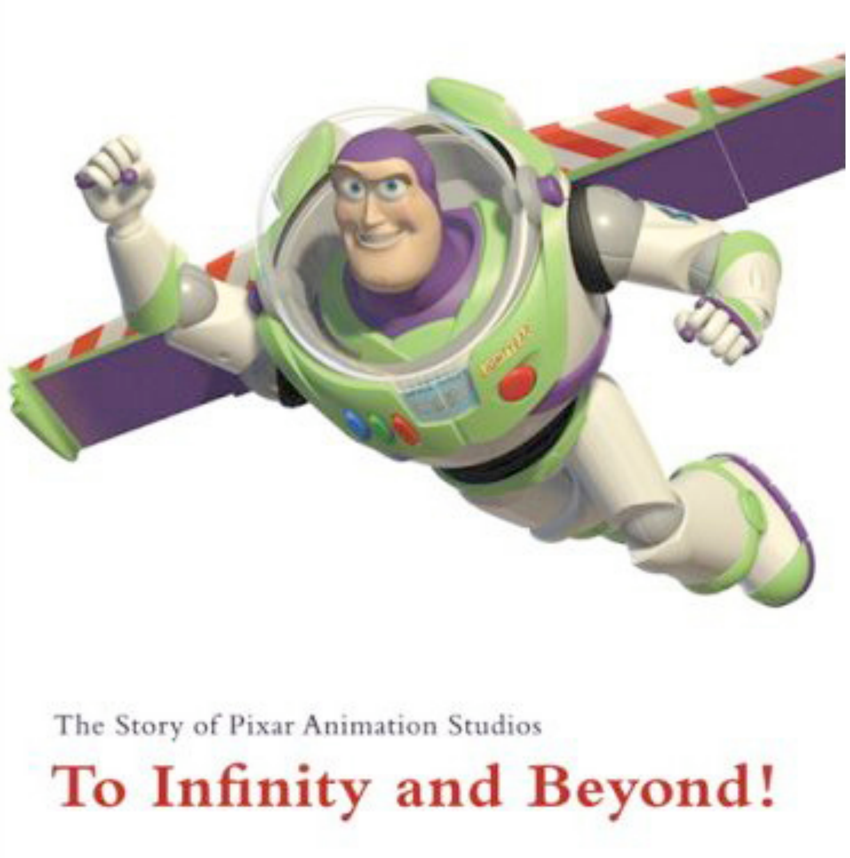


Tools to reduce infinite graph problems to its countable case

By François Laviolette

BIRS workshop on infinite graph
Banff
2007-10-14 to 2077-10-19

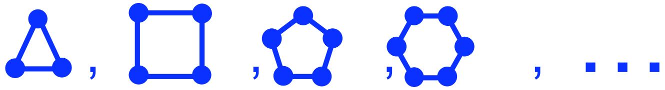
Many problems use infinite graphs in their representations.

To study such an infinite combinatorial structure, we :

- Analyze the topology of its one-way infinite paths (Theory of end, generalized depth-first search ideas,…
- Find a construction for the particular graphs related to the problem, and try to encode the most interesting of them by a finite structure.
- decompose and conquer.

Definitions and Notations

• The circuits:

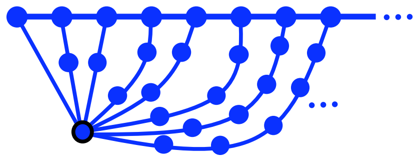


the cycles

et

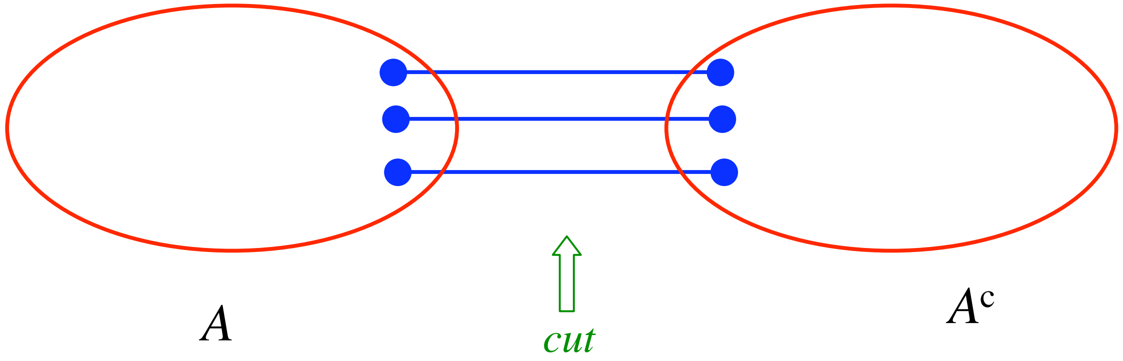
the double-ray

a ray: 

• A ray is *dominated* if: 

• The end: ‘the beginning of infinite graph theory’

- **Definition:** A *cut* in a graph G is a set of edges of G which separates a sub-graph A from its complement.



- **Definition 1:** A minimal cut (with respect to the inclusion) is called a *bond*.

- **Proposition:** a cut is a bond if and only if both the subgraph *A* and its complement are connected.

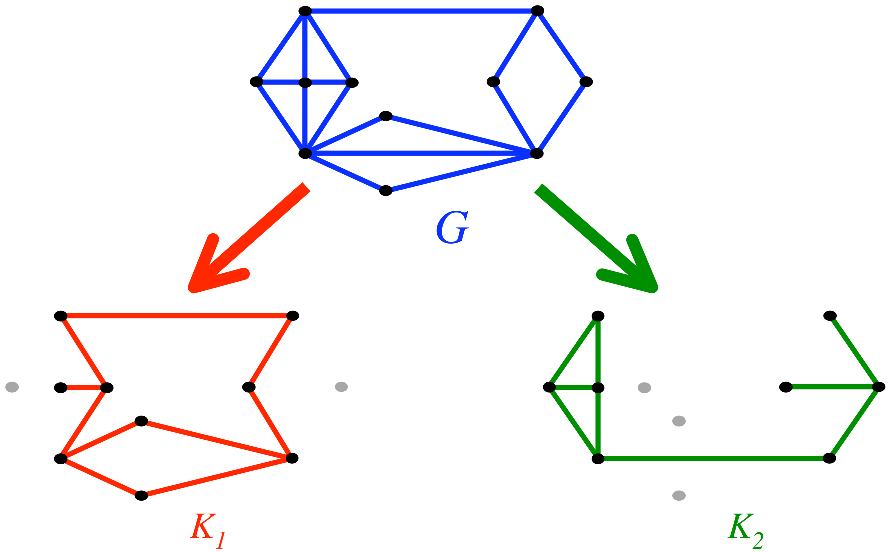
- **Proposition:** cuts are disjoint unions of bonds.

- **Theorem** (Menger): the edge-connectivity between two vertices *x* and *y* (i.e. the maximal "number" of edge-disjoint paths linking *x* and *y*) is equal to the minimal cardinality of a bond that separates *x* and *y*.

For any cardinal α, the relation "to be at least α-edge connected" induces an equivalence relation on the set of vertices of a graph.

- **Definition 2:** an equivalence class of this relation is called an *α-class*

Definition 3: A *decomposition* of a graph G is a family of connected subgraphs of G that are pairwise edge disjoint but whose union is G itself.



The subgraphs of the family are called the *fragments* of the decomposition.

Given any cardinal α,
an *α-decomposition* is a decomposition whose fragments are all of size ″ α..

Well known example : cycle decomposition.

Theorem (Euler, Hierholzer, Veblen):

Let G be a finite, connected graph. Then the following statements are equivalent:

1. *G admits an Euler tour;*
2. *no vertex of G has odd degree;*
3. *G has a cycle decomposition.*

Theorem ***Nash-Williams (1960)***

A graph has a cycle decomposition

iff

it has no finite cut of odd cardinality.

Idea of the proof of Nash-Williams's theorem *(The countable case)*

- We first note that:
  1. If a graph has no odd cut then each edge of it is contained in a cycle.

     "Those graphs have enough cycles"

  2. If we remove the edges of a (finite) cycle from a graph that has no odd-cut, the resulting graph will still have no odd-cut.

     "We have an invariant property"

- And we inductively construct a cycle-decomposition as follows:
  Let $e_1, e_2, e_3, \ldots$ be an enumeration of $E(G)$.
  Choose C_1, a cycle of G that contains e_1.
  Let i_2, be the smallest index such that $e_{i_2} \in E(G \setminus C_1)$.
  Choose C_2, a cycle of $G \setminus C_1$ that contains e_{i_2}.
  Let i_3, be the smallest index such that $e_{i_3} \in E(G \setminus (C_1 \cup C_2))$. Choose $C_3, \ldots$

- Clearly, $(C_i)_{i \in \omega}$ is a cycle-decomposition

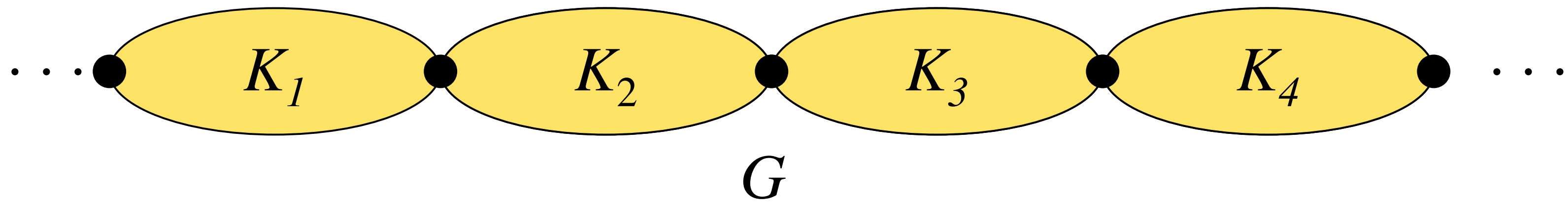
"And we are done (for the countable case)"

Definition 4: an α-decomposition Δ is *bond faithful* if

1. any bond of cardinality$\leq\alpha$ of G is totally contained in one fragment;
2. any bond of cardinality$<\alpha$ of a fragment is also a bond of G

In other words, (up to the cardinal α) the bond-structure of the graph can be recovered from the bond-structure of the fragments.

Here is an easy example:



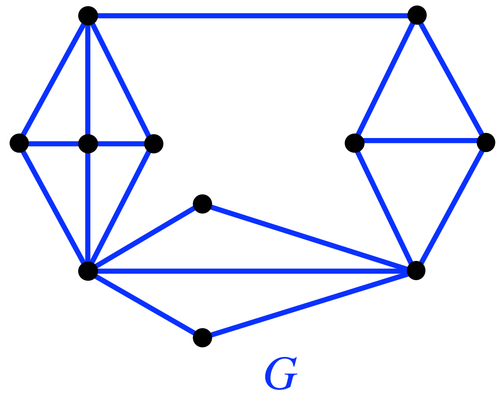
In a bond-faithful α-decomposition Δ, the following properties are always satisfied for any set *B* of edges of *G*:

- If $|B| < \alpha$ then
  B is a bond of G $\Leftrightarrow$ B is a bond of some fragment of Δ;

- If $|B| = \alpha$ then
  B is a bond of G $\Rightarrow$ B is a bond of some fragment of Δ;

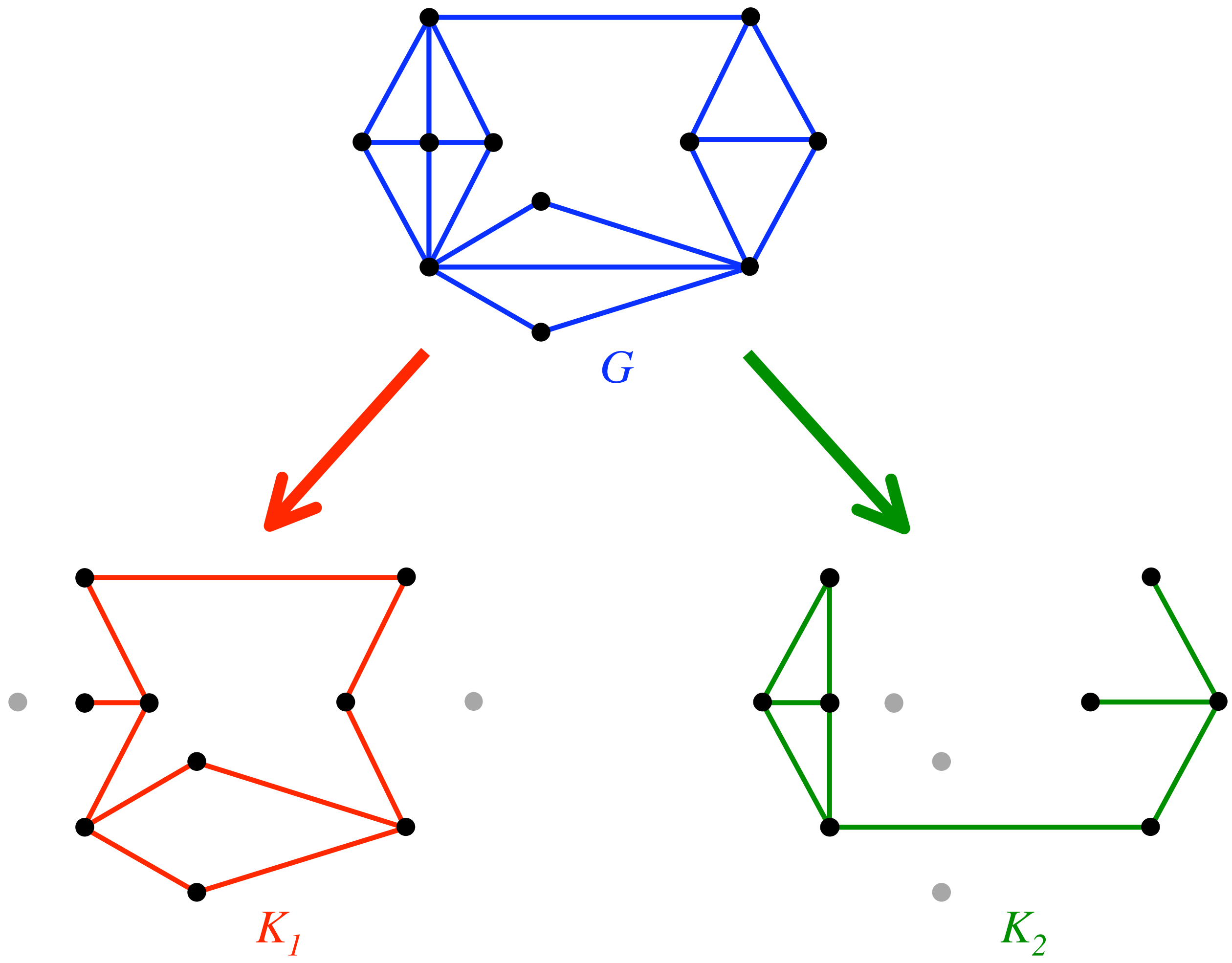
- If $|B| > \alpha$ then
  in any fragment H containing edges of B, B induces a cut of cardinality α in H.

Question: do such decompositions exist for any graph ?



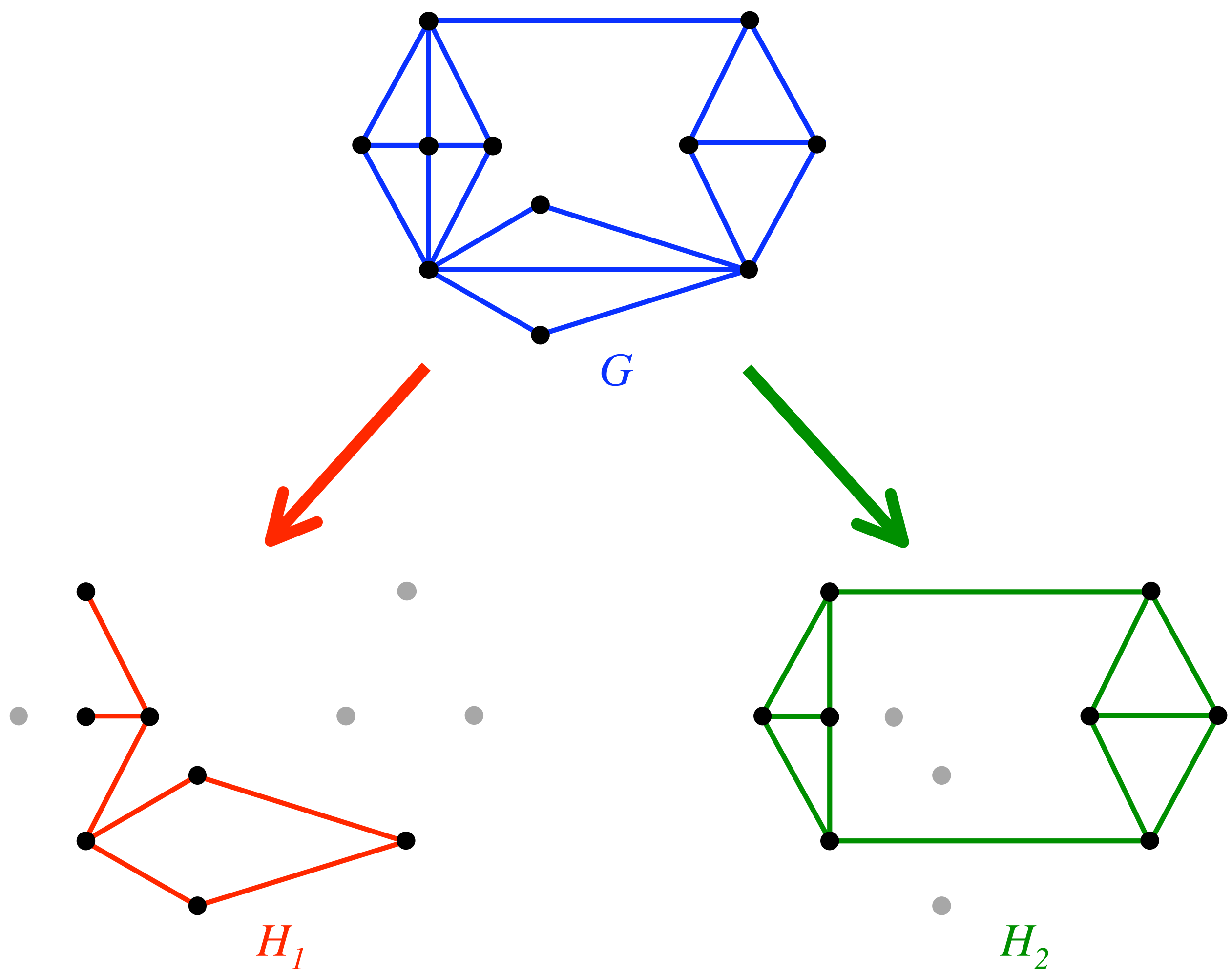
For this G, let's try for $\alpha=3$

Question: do such decompositions exist for any graph ?



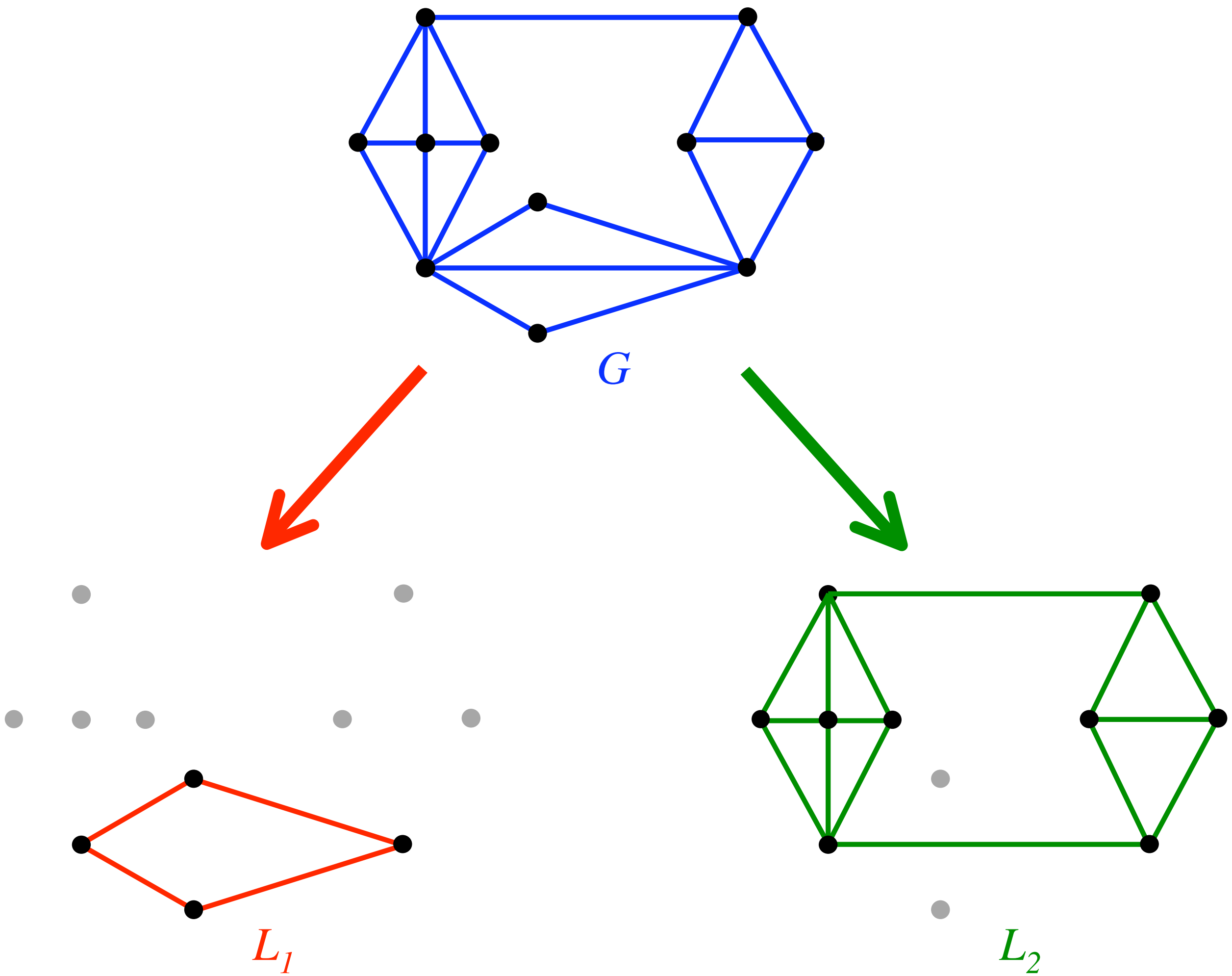
For this G, let's try for α=3 Attempt#1: No

Questions do such decompositions exist for any graph ?



For this G, let's try for $\alpha=3$ Attempt#2: No

Question: do such decompositions exist for any graph ?



For this G, let's try for α=3 Attempt#3: ??

Theorem:

Every graph admits a bond-faithful ω-decomposition.

In other words, it is always possible to decompose a graph G into countable fragments such that

1. every countable bond of G is a bond of some fragment;
2. The set of all the finite bonds of the fragments is exactly the set of all finite bonds of G.

Note: Under the Generalized Continuum Hypothesis assumption, this result can be generalized to:

Theorem: For all infinite cardinal α, every graph admits a decomposition into fragments of cardinality at most α that is bond-faithful up to α.

Proposition: *Every graph G is the edge disjoint union of two spanning graphs, say K and L, such that the edge-connectivity between any pair of infinitely edge-connected vertices is preserved, in G, K and L.*

Proposition: *Assuming GCH, every α-edge-connected graph contains α edge-disjoint spanning trees.*

Theorem : *Let W be the set of all the ω-classes of G. Then there exists a well ordering on W such that each equivalence class $w \in W$ can be separated from all the preceding ω-classes by a finite cut of the graph.*

The bond-faithful theorem :

Every graph admits a bond-faithful ω-decomposition.

Sketch of the proof :

STEP 1:

Every bridgeless graph admits an ω-decomposition whose fragments are all 2-edge-connected.

STEP 2:

Given any ω-decomposition Δ, then there exists an ω-decomposition Δ' that is coarser than Δ, and such that for any fragment H of Δ, the only bonds of H that are bonds of the corresponding fragment of Δ' are those that are bonds of G.

Proof (following) :

STEP 3:

Iterating step 2, we obtain an ω-decomposition that satisfies the property (ii). ***(Bonds in the fragments are bond in G).***

Also, if we manage to find some ω-decomposition that satisfies property (i). ***(Bonds in G are bond in the fragments).***

Applying STEP 3 to this particular decomposition will give what we want. ***(I.e..: a bond-faithful ω-decomposition)***.

STEP 4:

Theorem*: Let G be a graph, x a vertex of G and μ a regular uncountable cardinal.*

If x has degree≥ μ, then x is a cut vertex or is μ-vertex-connected to some other vertex y.

Proof (end)

STEP 5:

Let G/ω_1 represents the graph obtained from G by identifying vertices that belong to the same ω_1-classes.

Then, because of STEP 4, the blocks of G/ω_1 form a bond faithful ω-decomposition of G/ω_1.

STEP 6:

Construct a bond-faithful ω-decomposition of G from that bond-faithful ω-decomposition of G/ω_1. (The hard step)

Now, let us come back to Euler

Theorem (Euler, Hierholzer, Veblen):

Let G be a finite, connected graph. Then the following statements are equivalent:

1. *G admits an Euler tour;*
2. *no vertex of G has odd degree;*
3. *G has a cycle decomposition.*

In the infinite case those three statement are no longer equivalent !!

Generalizations of Theorem (*E*,*H*,*V*) to infinite graphs

(*Erdos, Grünwald et Vázsonyi*)
1938

G has an Euler tour

$\Updownarrow$

G is Eulerian, countable
and
∀ finite sub-graph *H*
G\E(*H*) has at most 2 infinite c.c.
and
∀finite Eulerian subgraph *K*
G\E(*K*) has at most 1 infinite c.c.

G has no vertex of odd degree

$\Updownarrow$

G admits a decomposition into fragments that has an Euler tour

(*Nash-Williams*)
1960

***G* has a decomposition into finite circuits.**

$\Updownarrow$

G has no bond of odd cardinality

***G* has a decomposition into circuits.**

$\Updownarrow$

?????????????????????????

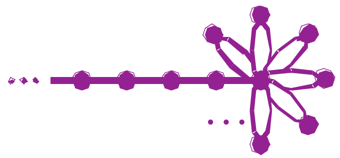
Note that: but: counter-examples:

to have a decomposition into finite **circuits**

$\Downarrow$ $\quad \not\Uparrow$

to Have a decomposition into **circuits**

$\Downarrow$ $\quad \not\Uparrow$



to have a decomposition into fragments that admit an Euler tour

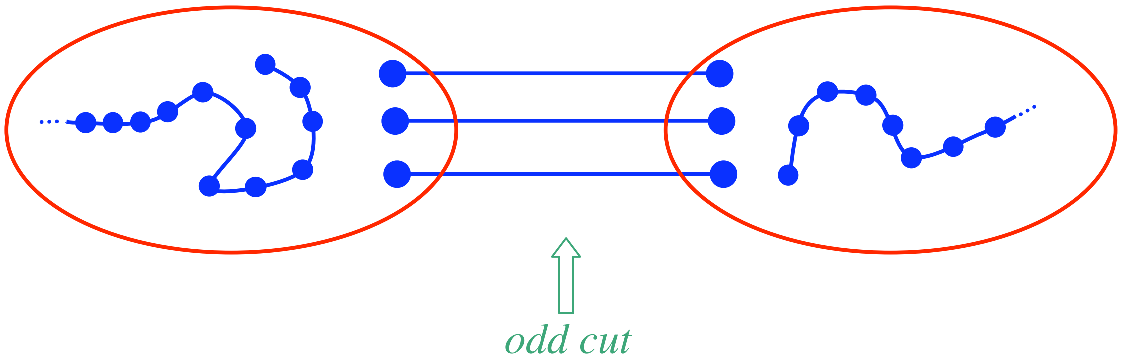
Decomposition into circuits --- (1)

Easy theorem :

G admits a decomposition into circuits

$\Downarrow$ but $\cancel{\Uparrow}$

for every odd cut, both the left side and the right side of the cut have rays



Decomposition into circuits --- (1)

Easy theorem :

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$\Downarrow$ but $\not\Uparrow$

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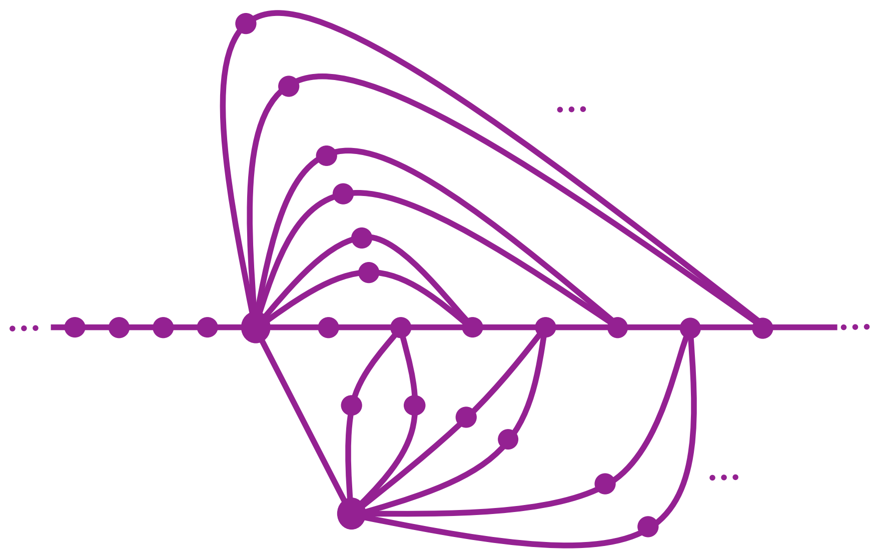
Counterexample:



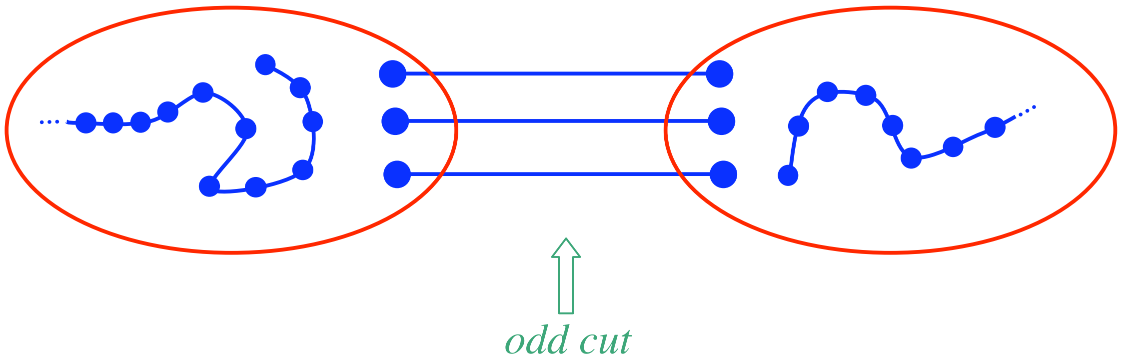
Decomposition into circuits --- (2)

Not so easy theorem :

G admits a decomposition into non dominated circuits

$\Downarrow\Uparrow$

for every odd cut, both the left side and the right side of the cut have non dominated rays



Decomposition into circuits --- (2)

Not so easy theorem :

G admits a decomposition into non dominated circuits

$\Downarrow\Uparrow$

for every odd cut, both the left side and the right side of the cut have non dominated rays

Idea of the proof of $\Uparrow$ *(The countable case)*

- in graph with this property, each edge of it is contained in a non dominated circuit.

"Those graphs have enough non dominated circuits"

- If we remove the edges of a non dominated circuits of such a graph, the resulting graph will still have the property.

"We have an invariant property"

Decomposition into circuits --- (2)

Not so easy theorem :

G admits a decomposition into non dominated circuits

$\Downarrow\Uparrow$

for every odd cut, both the left side and the right side of the cut have non dominated rays

Idea of the proof of $\Uparrow$ *(The uncountable case)*

- Apply the Bond Faithful theorem in the proper way :-)

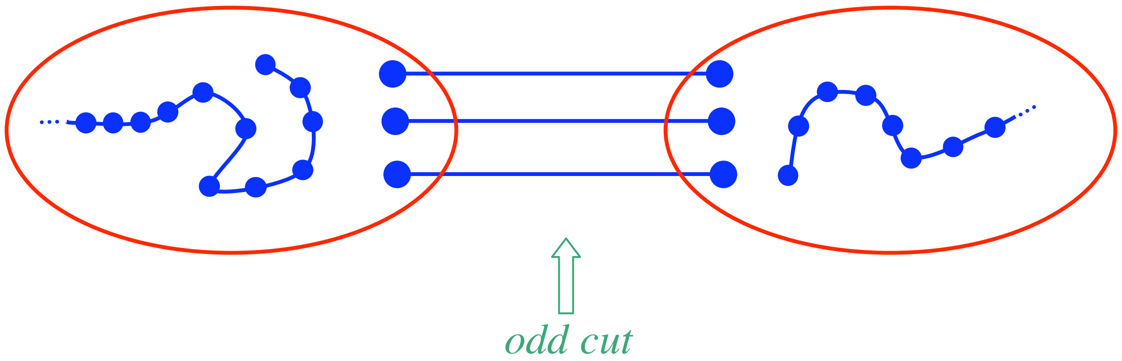
Decomposition into circuits --- (3)

The main theorem :

G admits a decomposition into circuits

$\Downarrow\Uparrow$

for every odd cut, both the left side and the right side of the cut have eligible rays



Decomposition into circuits --- (3)

The main theorem :

G admits a decomposition into circuits

$\Downarrow\Uparrow$

for every odd cut, both the left side and the right side of the cut have eligible rays

Question: What is an eligible ray?

Decomposition into circuits --- (3)

The main theorem :

G admits a decomposition into circuits

$$\Downarrow\Uparrow$$

for every odd cut, both the left side and the right side of the cut have eligible rays

Essentially, an *eligible* ray is a ray whose removal from the graph will not create odd bonds between vertices that originally were dominating the ray, and will not create new “odd-type” vertices.

Graphs that do not have eligible rays must have an odd cut for which one side is basically one of the four subgraphs: