Light Tailed Behaviour and Decay Rate for a General Type of Two-Dimensional Random Walk with Complex Boundaries

Yiqiang Q. Zhao
School of Mathematics and Statistics
Carleton University
(Canada-China Workshop on Industrial Mathematics, BIRS)
August 5—10, 2007

This talk is based on joint work with:

- Q.M. He, Dalhousie University
- Hui Li, Mount Saint Vincent University
- M. Miyzawa, Tokya University of Science

Abstract

Motivated by characterizing properties of rare events in stochastic models such as telecommunications systems, insurance policies, etc, in this talk, we present some key results for a general type of two-dimensional random walk with boundaries. This type of random walk can be modeled as a quasi-birth-and-death process with countably many background (phase) states. By using the matrix-analytic method, combined with probabilistic arguments, conditions for exactly geometric decay and for light-tailed but not exactly geometric decay are obtained.

Outline

- **Introduction**
  - QBD Process with Countably Many Phases
  - Issues of Interest
  - Selected Literature Review
- **Main Results**
- **Applications to Queueing Models**
  - Polling System
  - Gated Service Queues

Introduction (QBD process with countably many phase states)

Consider an irreducible, positive recurrent, and aperiodic QBD process, in discrete-time, with infinitely many phase (background) states. More sp

State spaces

$$S = \{(0, j): j \in S_0\} \cup \{(n, j): n = 1,2,\ldots, j = 0, 1,2,\ldots\}$$

S_0 is a countable set

Partition the transition matrix according to the level

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_0 & B & A & & \\ & C & B & A & \\ & & C & B & A \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

n is called the level variable and

j is called the background phase variable

A, B and C are matrices of infinite dimension

Introduction (Stationary Vector)

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n, \ldots)$$

Partitioned according to the level

$$\boldsymbol{\pi}_n = (\pi_{n,0}, \pi_{n,1}, \ldots, \pi_{n,j}, \ldots), \quad \boldsymbol{\pi}_0 = (\pi_{0,j})_{j \in S_0}$$

Matrix-geometric solution

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_1 R^{n-1}, \quad n \geq 1$$

$$R = A + RB + R^2 C$$

$$\boldsymbol{\pi}_0 = \boldsymbol{\pi}_0 B_0 + \boldsymbol{\pi}_1 C_0$$

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0 A_0 + \boldsymbol{\pi}_1 (B + RC)$$

$$\boldsymbol{\pi}_0 \mathrm{e} + \boldsymbol{\pi}_1 \left(\sum_{n=0}^{\infty} R^n \right) \mathrm{e} = 1$$

Introduction (Issues of Interests)

Characterization of tail asymptotics of both the joint distribution $\pi_{n,j}$ along direction n and the marginal distribution $\boldsymbol{\pi}_n \mathbf{e}$ as n $\rightarrow \infty$

- Exactly geometric decay rate
- Light tail behaviour without a geometric decay
- Upper and lower bounds (not in this talk)

Introduction (Selected Literature Review)

- **Complex analysis** (uniformization method, analytic continuation

The parallel queues feeded by arrivals with two types of demand and joint-the-shortest-queue

The tandem queue with coupled processors

Generalized joint-the-shortest-queue

Modified Jackson network

  - Takahaashi, Fujimoto and Makimoto (2001) (G
  - Haque (2003), Haque, Liu and Zhao (
  - Miyazawa (2004) (M/G/1)
  - Miyazawa and Zhao (2004) (GI/G/1)
  - Kroese, Scheinhardt and Taylor (2004), (QBD
  - Li, Miyazawa and Zhao (2007), Motyer and Ta

In literature, focus has been on

1. the joint distribution
2. exactly geometric decay along level direction
3. R is $1/\alpha$-positive for some $0< \alpha<1$.
4. R is irreducible

Main Results (Exact Geometric Decay)

If in addition, $\sum_{j=1}^{\infty} x_j < \infty$, the marginal also has an exactly geometric decay with same decay rate α.

$$\lim_{n\to\infty} \frac{\boldsymbol{\pi}_n \mathbf{e}}{\alpha^n} = \frac{c}{\alpha} \sum_{j=1}^{\infty} x_j$$

…s n→∞

R is 1/α-positive

(i) $\mathbf{x}R = \alpha\mathbf{x}$; $R\mathbf{y} = \alpha\mathbf{y}$ and $\mathbf{xy} < \infty$

(ii) $\frac{\pi_{1,i}}{x_i} < M < \infty$

$$\lim_{n\to\infty} \frac{\pi_{n,j}}{\alpha^n} = c x_j$$

Main Results (Exact Geometric Decay)

Application to the generalized joint shortest queue in which the difference of the two queues is taken as the level variable n and the minimum of two queues is background state j.

There exist an α, $0 < \alpha < 1$ … a positive row vector $\mathbf{x} = (x_0, x_1, \ldots)$ such that

(1) $\lim_{n\to\infty} \frac{R^n}{\alpha^n} = 0 \left(\lim_{n\to\infty} \frac{r_{i,j}^{(n)}}{\alpha^n} = 0 \right)$;

(2) $\mathbf{x}R = \alpha \mathbf{x}$;

(3) $\lim_{i\to\infty} \frac{\pi_{1,i}}{x_i} = c, 0 < c < \infty.$

$$\lim_{n\to\infty} \frac{\pi_{n,j}}{\alpha^n} = c x_j$$

$$\lim_{n\to\infty} \frac{\boldsymbol{\pi}_n \mathbf{e}}{\alpha^n} = \frac{c}{\alpha} \sum_{j=1}^{\infty} x_j$$

Main Results (Exact Geometric Decay)

If there exists $0 < \alpha < 1$, a positive column vector $\mathbf{y}$ such that

(1) $\lim_{n\to\infty} \frac{R^n}{\alpha^n} = 0$;

(2) $R\mathbf{y} = \alpha\mathbf{y}$;

(3) $\lim_{i\to\infty} \frac{1}{y_i} = c$, and $\sum_{i=0}^{\infty} \pi_{1,i} y_i < \infty$

then

$$\lim_{n\to\infty} \frac{\boldsymbol{\pi}_n \mathbf{e}}{\alpha^n} = \frac{c}{\alpha} \sum_{i=1}^{\infty} \pi_{1,i} y_i$$

If *0<c<∞*, the marginal distribution $\boldsymbol{\pi}_n \mathbf{e}$ has exactly geometric decay as n $\to\infty$

Main Results (Light tail without a geometric decay)

Definition: We say $\pi_{n,j}$ has a light tail with decay rate α, $0 < \alpha < 1$,

if for each j, $\overline{\lim_{n\to\infty}} \frac{\log \pi_{n,j}}{n} = \log \alpha$

Theorem : $\pi_{n,j}$ does not have an exact geometric decay, but has a light tail with decay rate α if one of the following is true.

$$(i) \quad \lim_{n\to\infty} \frac{\pi_{n,j}}{\eta^n} = 0 \text{ for all } \eta \geq \alpha \text{ and } \overline{\lim_{n\to\infty}} \frac{\pi_{n,j}}{\eta^n} = \infty \text{ for all } \eta < \alpha$$

$$(ii) \quad \lim_{n\to\infty} \frac{\pi_{n,j}}{\eta^n} = 0 \text{ for all } \eta > \alpha \text{ and } \overline{\lim_{n\to\infty}} \frac{\pi_{n,j}}{\eta^n} = \infty \text{ for all } \eta \leq \alpha$$

Main Results (Light tail without a geometric decay)

If there exists $0 < \alpha < 1$, a positive row vector $\mathbf{x}$ such that

(1) $\lim_{n\to\infty} \frac{R^n}{\alpha^n} = 0$;

(2) $\mathbf{x}R = \alpha\mathbf{x}$;

(3) $\lim_{i\to\infty} \frac{\pi_{1,i}}{x_i} = 0$,

then $\lim_{n\to\infty} \frac{\pi_{n,j}}{\alpha^n} = 0$

If $\alpha = \gamma$, where γ is the convergence norm of R, then the joint distribution $\pi_{n,j}$ does not have exactly geometric decay as n $\to\infty$. That is,

$$\lim_{n\to\infty} \frac{\pi_{n,j}}{\gamma^n} = 0 \qquad \overline{\lim_{n\to\infty}} \frac{\log \pi_{n,j}}{n} = \log \gamma$$

$$\xi_{i,j} = \sup_{z\geq 0}\left\{ z : \sum_{n=0}^{\infty} r_{i,j}^{(n)} z^n < \infty \right\}$$

$$\xi = \inf_{i,j}\{\xi_{i,j}\} \qquad \gamma = \frac{1}{\xi}$$

Main Results (Light tail without a geometric decay)

Assume $\lim_{n\to\infty} \frac{R^n}{\gamma^n} = 0$. If either of the following two sets of conditions are satisfied, the marginal distribution $\boldsymbol{\pi}_n \mathbf{e}$ does not have exact geometric decay but has a light tail with decay rate γ.

There exists a positive row vector $\mathbf{x}$ such that

(i) $\mathbf{x}R \le \gamma\mathbf{x}$

(ii) $\lim_{i\to\infty} \frac{\pi_{1,i}}{x_i} = 0$

(iii) $\sum_{i=0}^{\infty} x_i < \infty$

There exists a positive column vector $\mathbf{y}$ such that

(i) $R\mathbf{y} \le \gamma\mathbf{y}$

(ii) $\lim_{i\to\infty} \frac{1}{y_i} = 0$

(iii) $\sum_{i=1}^{\infty} \pi_{1,i} y_i < \infty$

Main Results (Light tail without a geometric decay)

$$\lim_{n\to\infty} \frac{\pi_n e}{\gamma_n{}^n} = 0$$

$$\overline{\lim_{n\to\infty}} \frac{\log \pi_n e}{n} = \log \gamma_R$$

Application (Polling system)

- Consider an exhaustive polling system with one server switching between two waiting lines that contain type 1 and type 2 customers, respectively.
- There is no switching time
- At any time, if the server is serving a type *k* customer, *k* =1, 2, it will keep serving type *k* customers, and switch over to serving another type only as the line of the type *k* customers becomes empty.
- The server goes into idle state only there are no customers in the system; and it becomes activated immediately upon a new arrival.
- Assume that the arrival processes for both types of customers are Poisson and the service times are exponential with rates $\lambda 1$, $\lambda 2$, $\mu 1$ and $\mu 2$, respectively.

Application (Polling system)

- $q1(t)$ be the queue length of type 1 customers in the system at time t;
- $q2(t)$ be the queue length of type 2 customers in the system at time t;
- $S(t)$ be the status of the server at any time t, where

$$S(t) = \begin{cases} 0 & \text{when server is idle,} \\ 1 & \text{when server is serving type 1 customers,} \\ 2 & \text{when server is serving type 2 customers.} \end{cases}$$

$$\pi_{n,i,j} = \lim_{t\to\infty} P\{q_1(t) = n, S(t) = i, q_2(t) = j\} \qquad \boldsymbol{\pi}_n = [\boldsymbol{\pi}_{n,1}, \boldsymbol{\pi}_{n,2}]$$

$$\boldsymbol{\pi}_{n,1} = (\pi_{n,1,0}, \pi_{n,1,1}, \pi_{n,1,2}, \cdots \pi_{n,1,j}, \cdots)$$

$$\boldsymbol{\pi}_{n,2} = (\pi_{n,2,1}, \pi_{n,2,2}, \pi_{n,2,3}, \cdots \pi_{n,2,k}, \cdots)$$

Application (Polling system)

The joint distribution $\pi_{n,2,j}$ does not have an exactly geometric decay, but has a light tail with decay rate

$$\alpha = \frac{\lambda_1}{\lambda_1 + (\sqrt{\mu_2} - \sqrt{\lambda_2})^2}$$ as $n \to \infty$. More specifically,

$$\lim_{n\to\infty} \frac{\pi_{n,2,j}}{\alpha^n} = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\log \pi_{n,2,j}}{n} = \log \alpha$$

Application (Polling system)

The marginal distribution $\boldsymbol{\pi}_{n,2}\mathbf{e}$ does not have an exactly geometric decay, but has a light tail with decay rate

$$\alpha = \frac{\lambda_1}{\lambda_1 + (\sqrt{\mu_2} - \sqrt{\lambda_2})^2}$$ as $n \to \infty$. More specifically,

$$\lim_{n\to\infty} \frac{\boldsymbol{\pi}_{n,2}\mathbf{e}}{\alpha^n} = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\log \boldsymbol{\pi}_{n,2}\mathbf{e}}{n} = \log \alpha$$

Applications (Gated Random Order Service Queue)

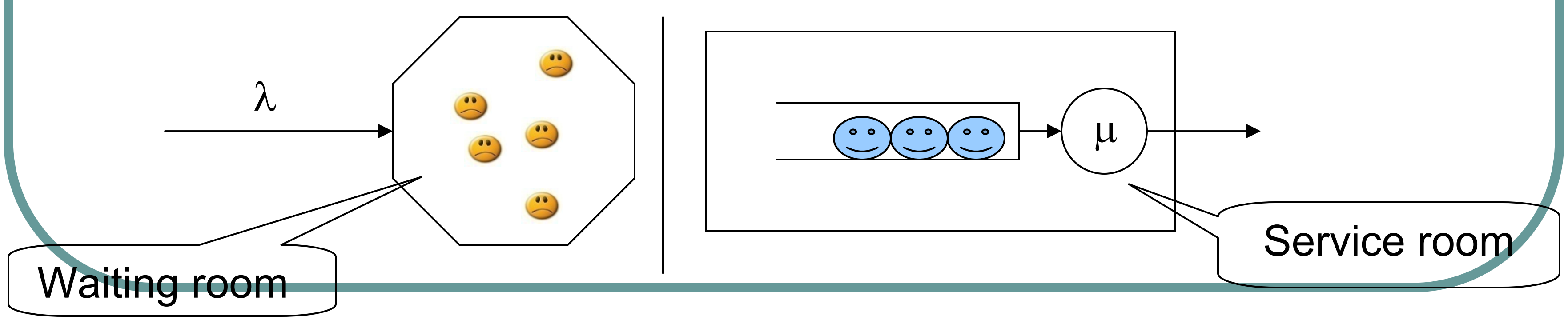
- Consider an M/M/1 queue with a service room and a waiting room
- Upon an arrival, if the service room is empty, the arriving customer goes directly into service room and receives its service immediately

$X_1(t)$ = the number of customers in the waiting room at time t;

(Level variable)

$X_2(t)$= the number of customers in the service room at time t,

(Background phase)



Applications (Gated Random Order Service Queue)

The joint distribution $\pi_{n,j}$ does not have an exactly geometric decay, but has a light tail with decay rate $\alpha = \frac{\lambda}{\mu}$ as $n \to \infty$. More specifically,

$$\lim_{n\to\infty} \frac{\pi_{n,j}}{\alpha^n} = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\log \pi_{n,j}}{n} = \log \alpha$$

Application (Gated Random Order Service Queue)

The marginal distribution $\boldsymbol{\pi}_{n,2}\mathbf{e}$ does not have an exactly geometric decay, but has a light tail with decay rate

$\alpha = \dfrac{\lambda}{\mu}$ as $n \to \infty$. More specifically,

$$\lim_{n\to\infty} \frac{\boldsymbol{\pi}_n \mathbf{e}}{\alpha^n} = 0 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\log \boldsymbol{\pi}_n \mathbf{e}}{n} = \log \alpha$$

Thank you!