An application of numerical bifurcation analysis

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Outline

- Introduction
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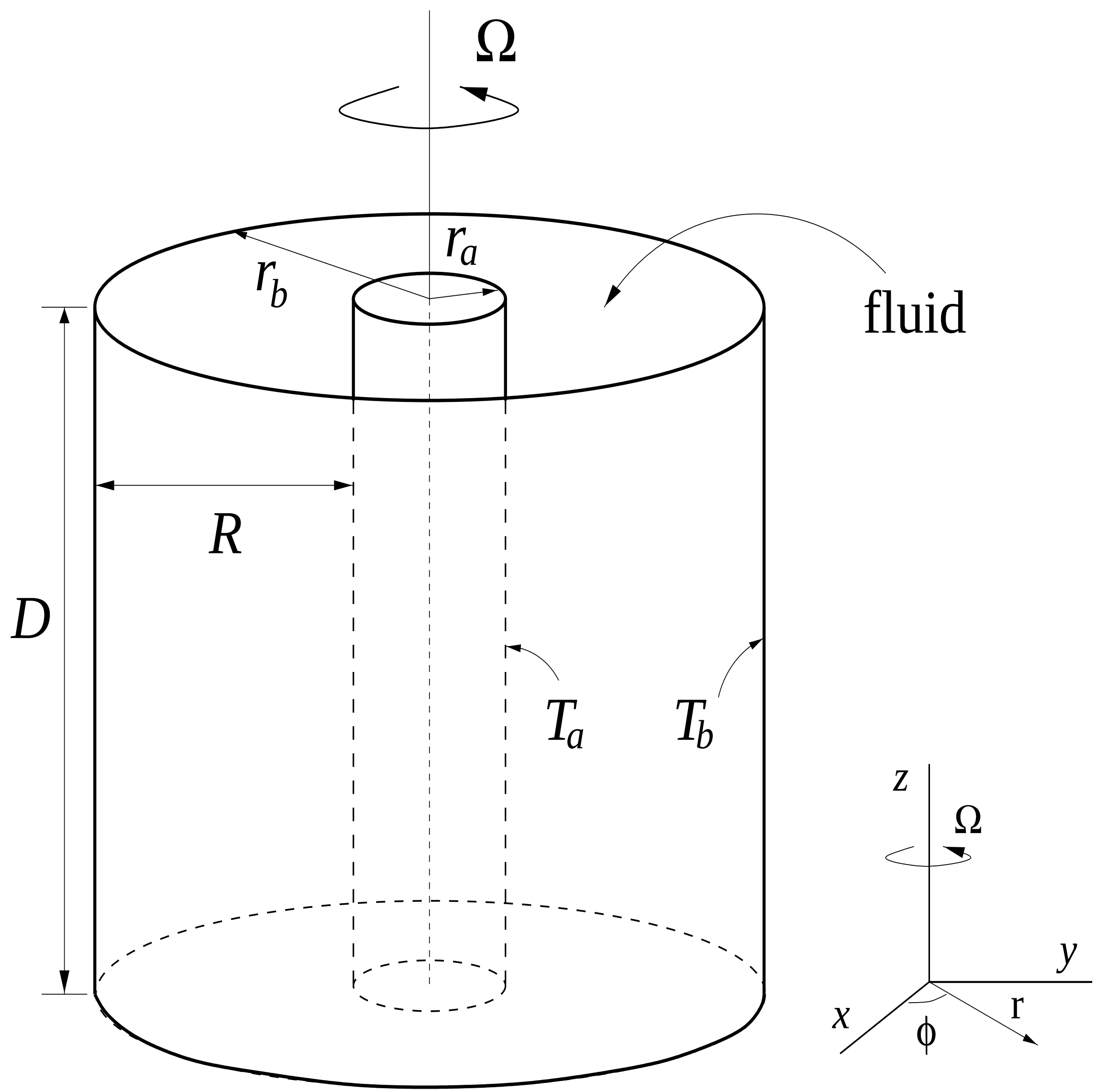
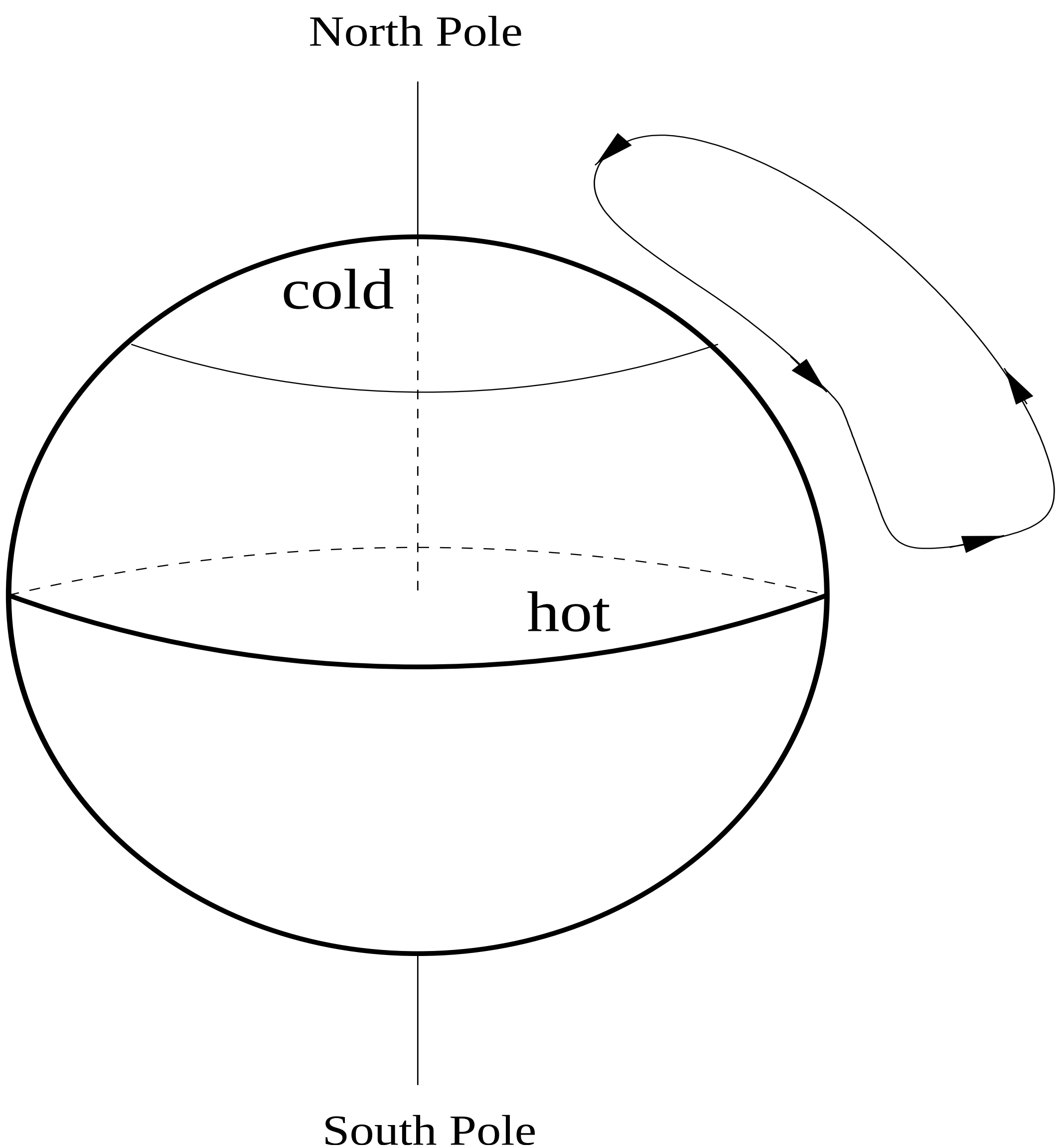
- Bifurcation analysis
  - Numerical continuation
  - Eigenvalue computation
- Examples
  - differentially heated rotating annulus
  - differentially heated rotating spherical shell
- Summary

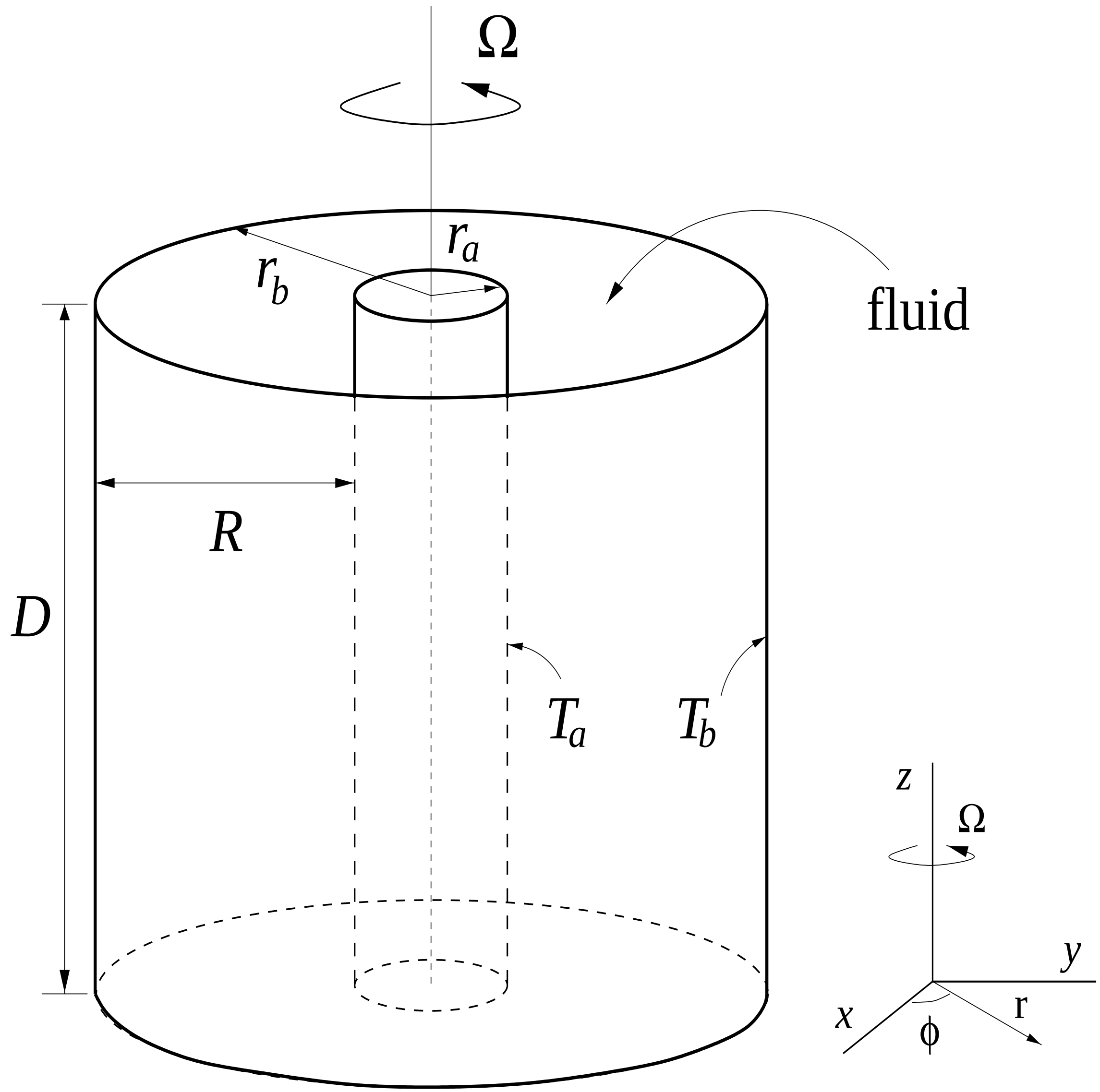
A differentially heated rotating annulus



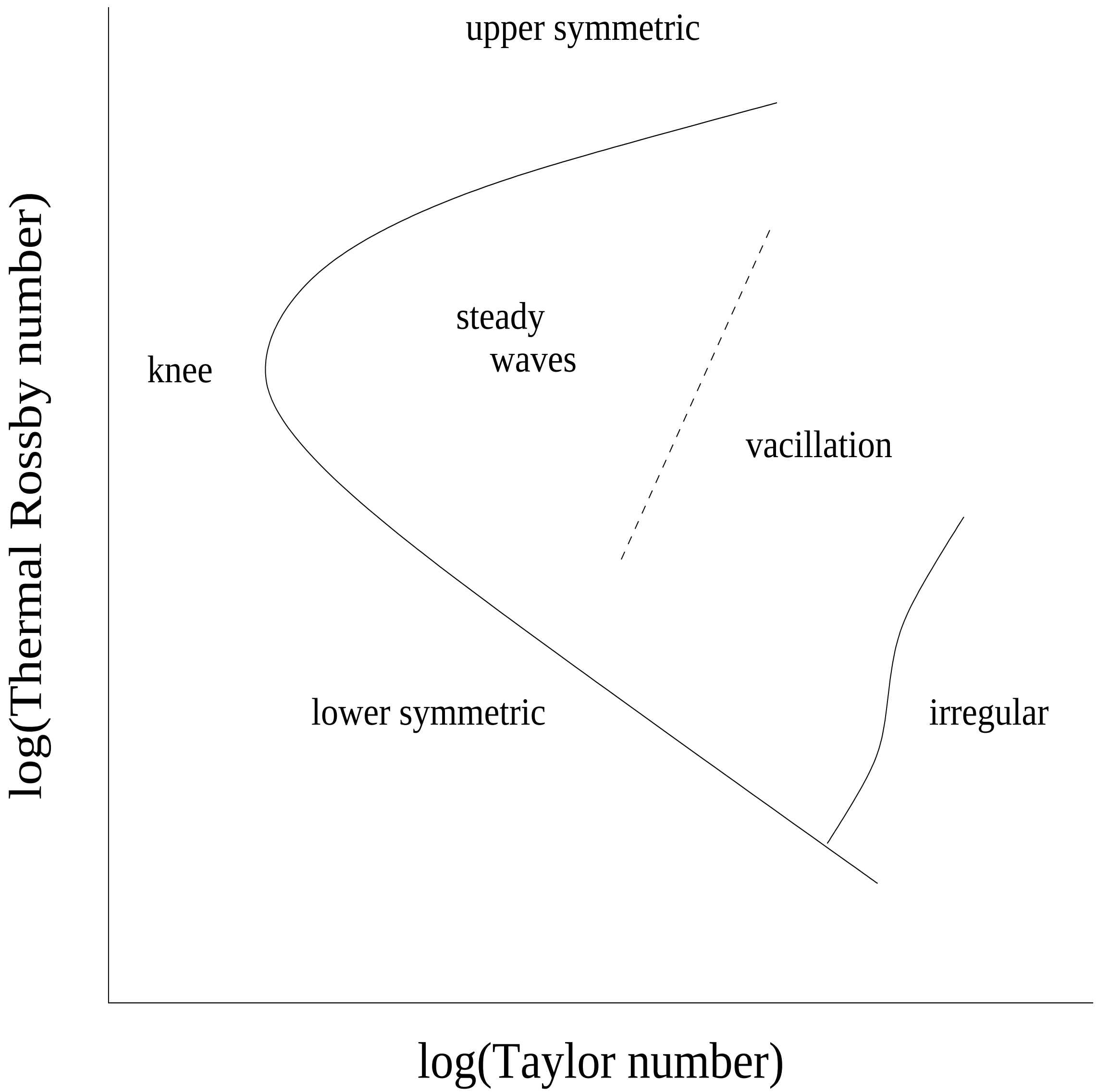
A differentially heated rotating planet



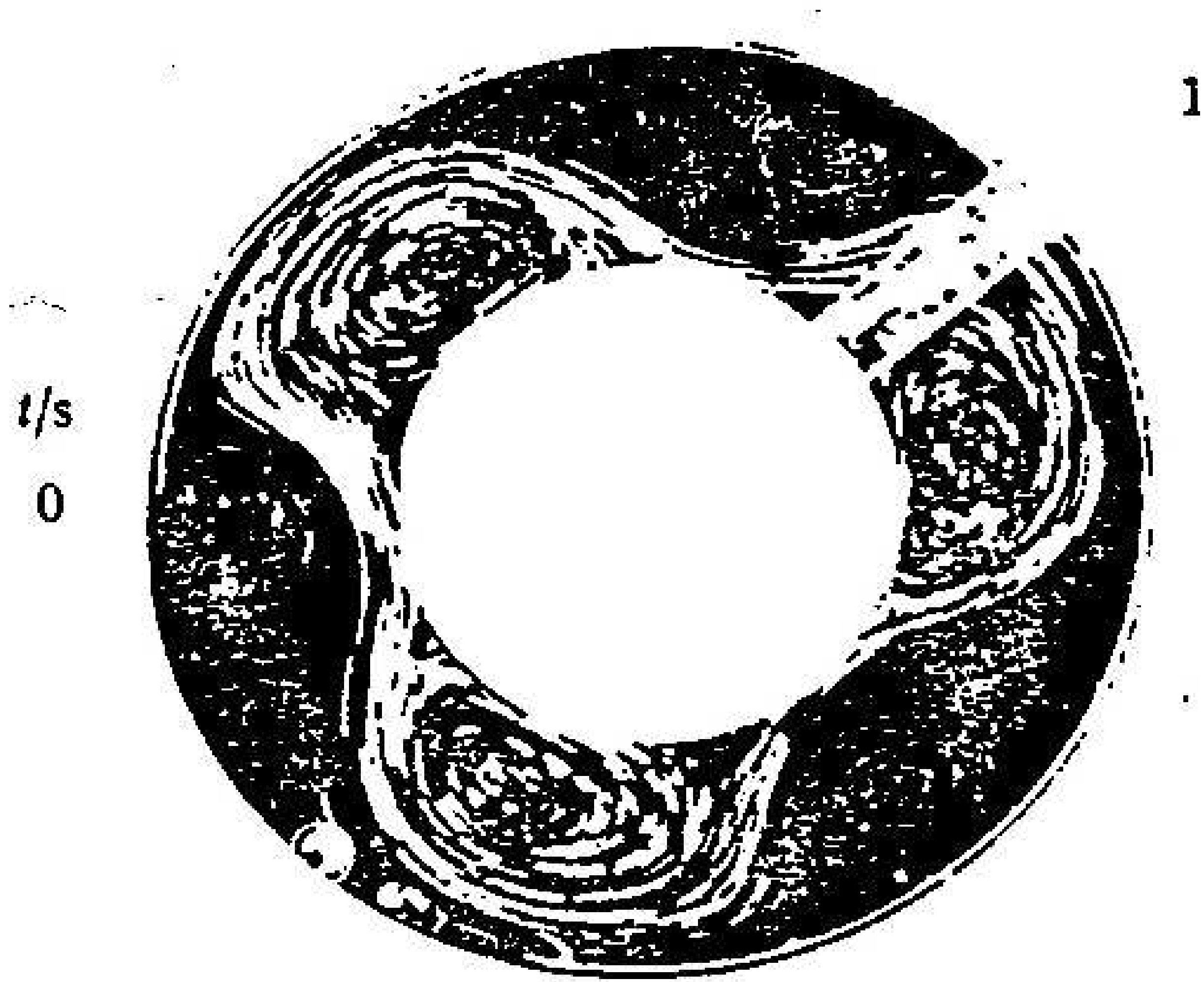
A differentially heated rotating annulus



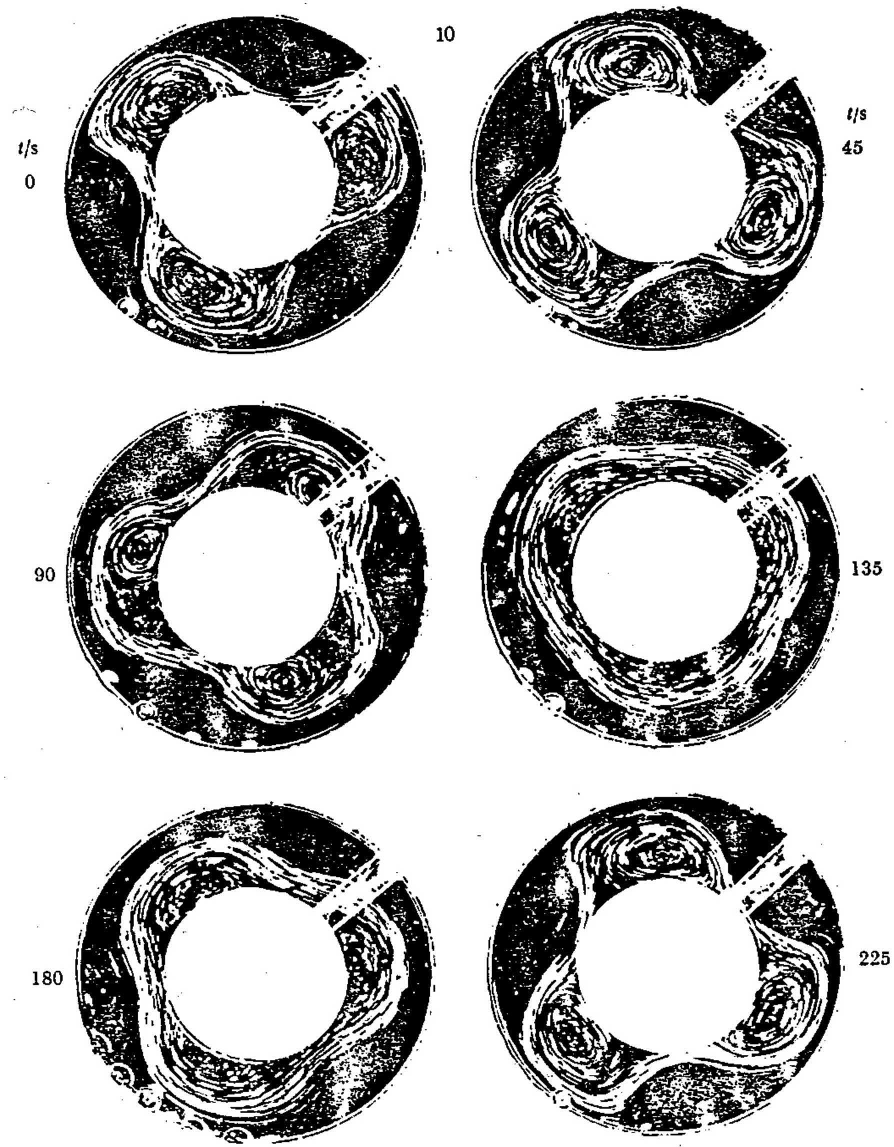
Regime diagram



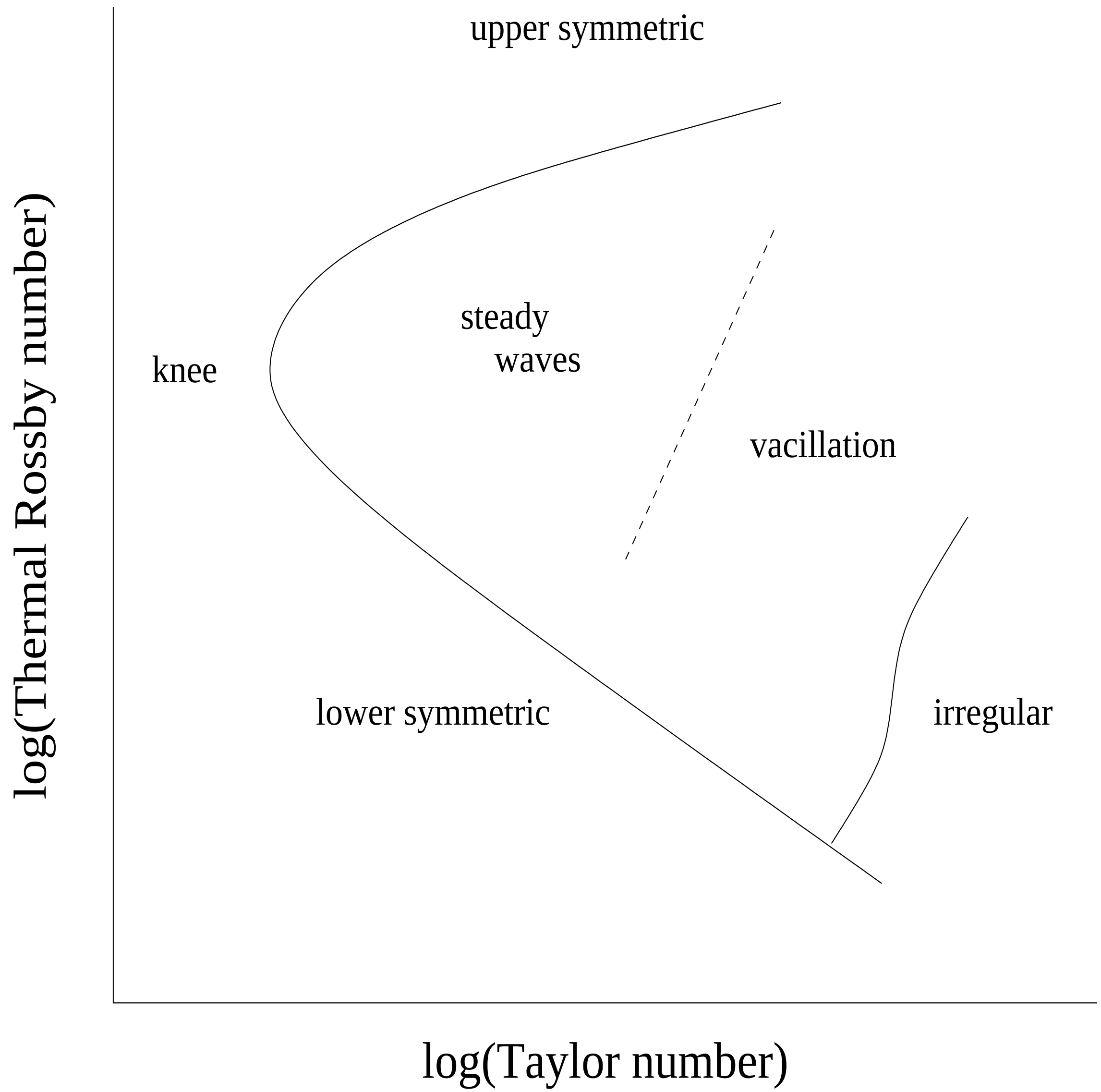
Wave flow in the annulus



Vacillating flow in the annulus



Regime diagram



Bifurcation analysis

- Nonlinear DE: $\frac{dx}{dt} = G(x, \alpha)$, $x \in \mathcal{R}^n$, $\alpha \in \mathcal{R}^1$.
- Steady solution $x_0 = x_0(\alpha)$ when: $G(x_0, \alpha) = 0$.
- Look for bifurcations from steady solution
  - linear stability of steady solution
  - from eigenvalues, λ, of the linearization of dynamical equation about the steady solution:

$$G_x(x = x_0, \alpha).$$

  - $Real\,(\lambda_j) < 0$ for all $j \rightarrow \quad x_0$ is linearly stable
  - $Real\,(\lambda_j) > 0$ for one $j \rightarrow \quad x_0$ is linearly unstable

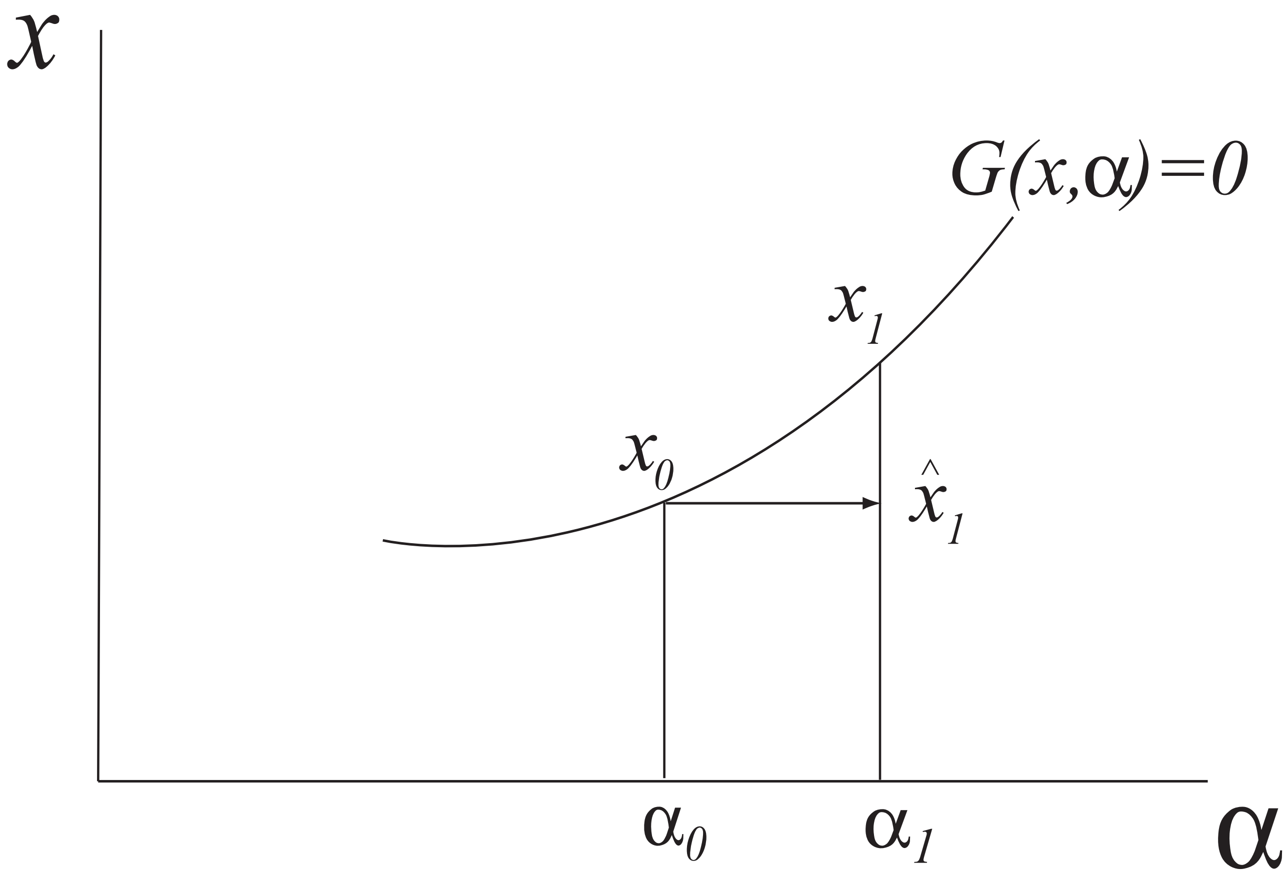
Numerical computations

- Steady solutions
  - use pseudo-arclength continuation
- Linear stability: eigenvalues
  - Implicitly restarted Arnoldi method
  - with Cayley transformations

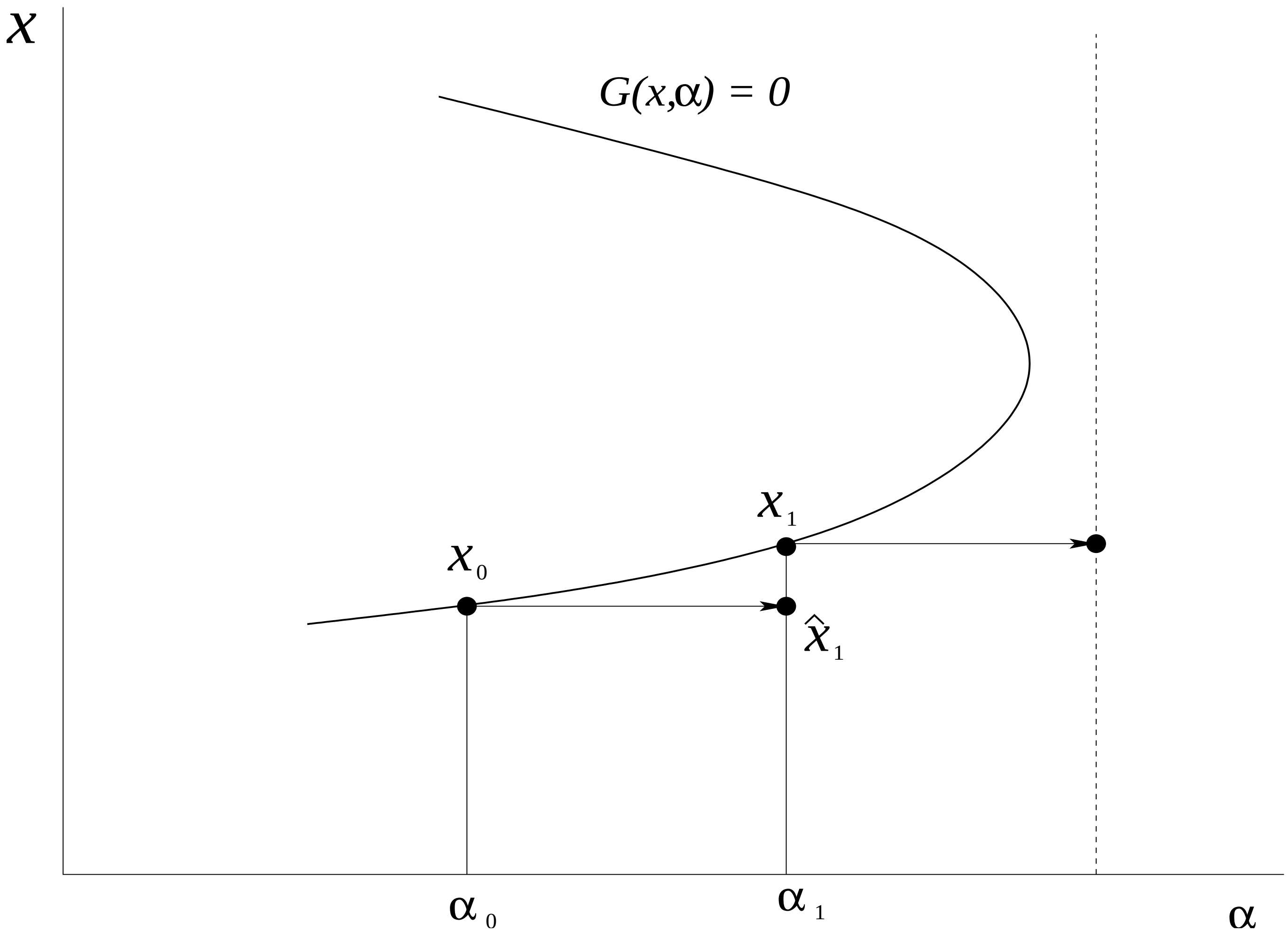
Steady solution: continuation

- Look for steady solutions
  - discretization reduces PDE to system of nonlinear algebraic equations
  - need to solve $G(x, \alpha) = 0$, $x \in \mathcal{R}^n$, $\alpha \in \mathcal{R}$
- Use Newton's method with continuation
  - need to have a good guess
  - assume we know x_0 at α_0 such that $G(x_0, \alpha_0) = 0$

Natural parameterization



Natural parameterization



Pseudo-arclength continuation

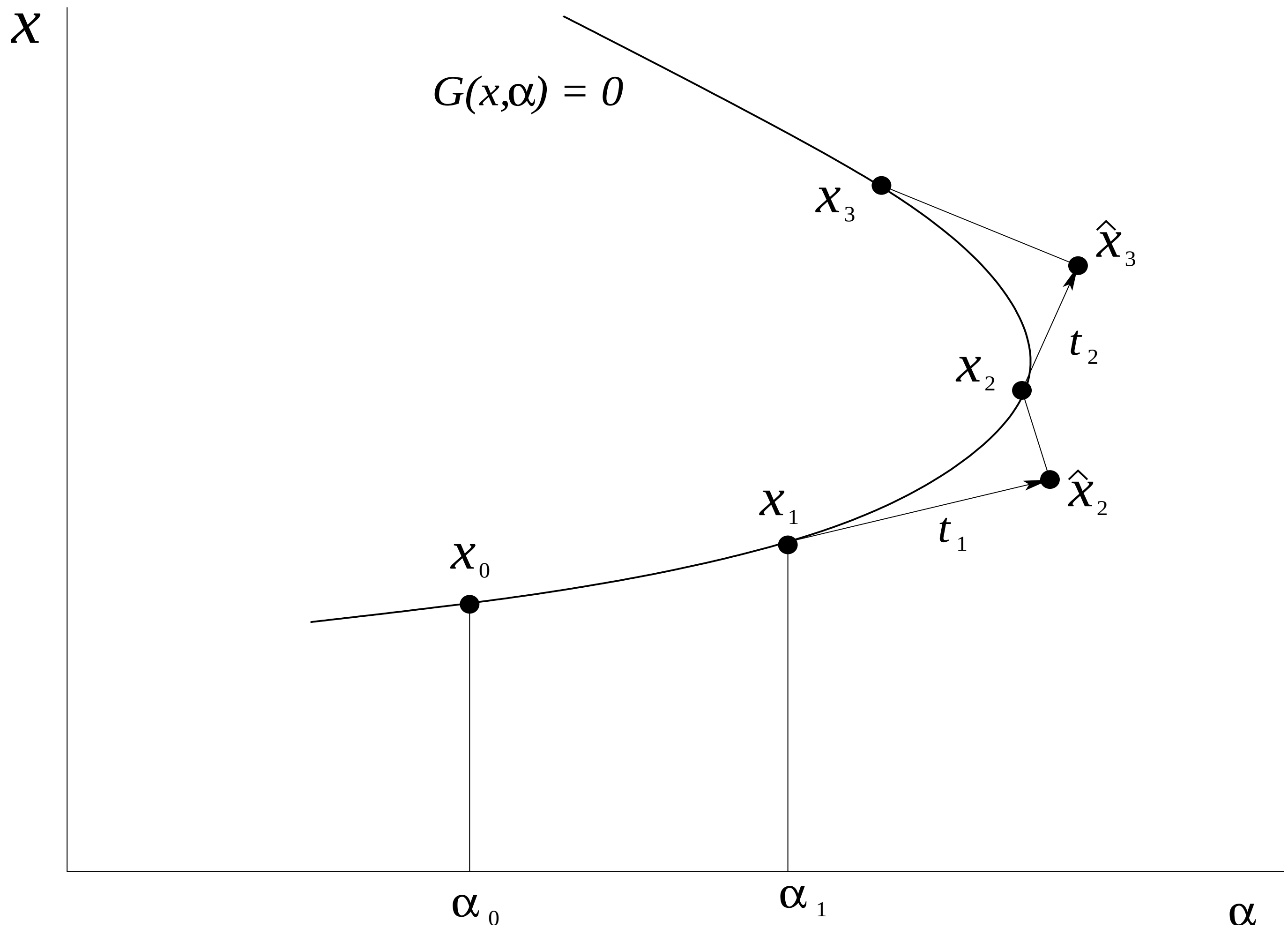
- Consider the parameter α as an unknown
- predictor: new guess $(\hat{x}_1, \hat{\alpha}_1)$ given by

$$\hat{x}_1 = x_0 + \frac{\Delta s}{\|t_0\|} t_0^{(x)}, \qquad \hat{\alpha}_1 = \alpha_0 + \frac{\Delta s}{\|t_0\|} t_0^{(\alpha)}$$

  - $t_0 = [t_0^{(x)} \;\; t_0^{(\alpha)}]$ is the tangent to the solution curve
  - the step size Δs measures arclength along tangent line
- for corrector, add an extra condition to get new system:

$$\begin{aligned} G(x, \alpha) &= 0 \\ f(x, \alpha) &= 0 \end{aligned}$$

Pseudo-arclength continuation



Eigenvalue approximation

- Eigenvalue problem
  - Linearize about steady solution
  - get generalized eigenvalue problems

$$\lambda \mathbf{B} \Phi = \mathbf{A} \Phi$$

  - discretization leads to matrix eigenvalue problems

Eigenvalue approximation

- For eigenvalues use ‘Implicitly restarted Arnoldi method’
  - iterative
  - memory efficient
  - finds extremal eigenvalues

Eigenvalue approximation

- Use generalized Cayley transform

$$\mathbf{C}(\mathbf{A},\mathbf{B}) = (\mathbf{A} - \sigma_1\mathbf{B})^{-1}(\mathbf{A} - \sigma_2\mathbf{B})$$

  - λ are eigenvalues from $\lambda\mathbf{B}x = \mathbf{A}x$
  - μ are eigenvalues from $\mu x' = \mathbf{C}x'$
  - $Real(\lambda) > \frac{\sigma_1 + \sigma_2}{2} \rightarrow |\mu| > 1$

Eigenvalue approximation

- Use generalized Cayley transform

$$\mathbf{C}(\mathbf{A},\mathbf{B}) = (\mathbf{A} - \sigma_1\mathbf{B})^{-1}(\mathbf{A} - \sigma_2\mathbf{B})$$

- Don't need to form the matrix $\mathbf{C}$ explicitly
  - only need the matrix-vector product $w = \mathbf{C}v$

$$w = \mathbf{C}v = (\mathbf{A} - \sigma_1\mathbf{B})^{-1}(\mathbf{A} - \sigma_2\mathbf{B})\,v$$

  - multiple by $(\mathbf{A} - \sigma_1\mathbf{B})$ get:

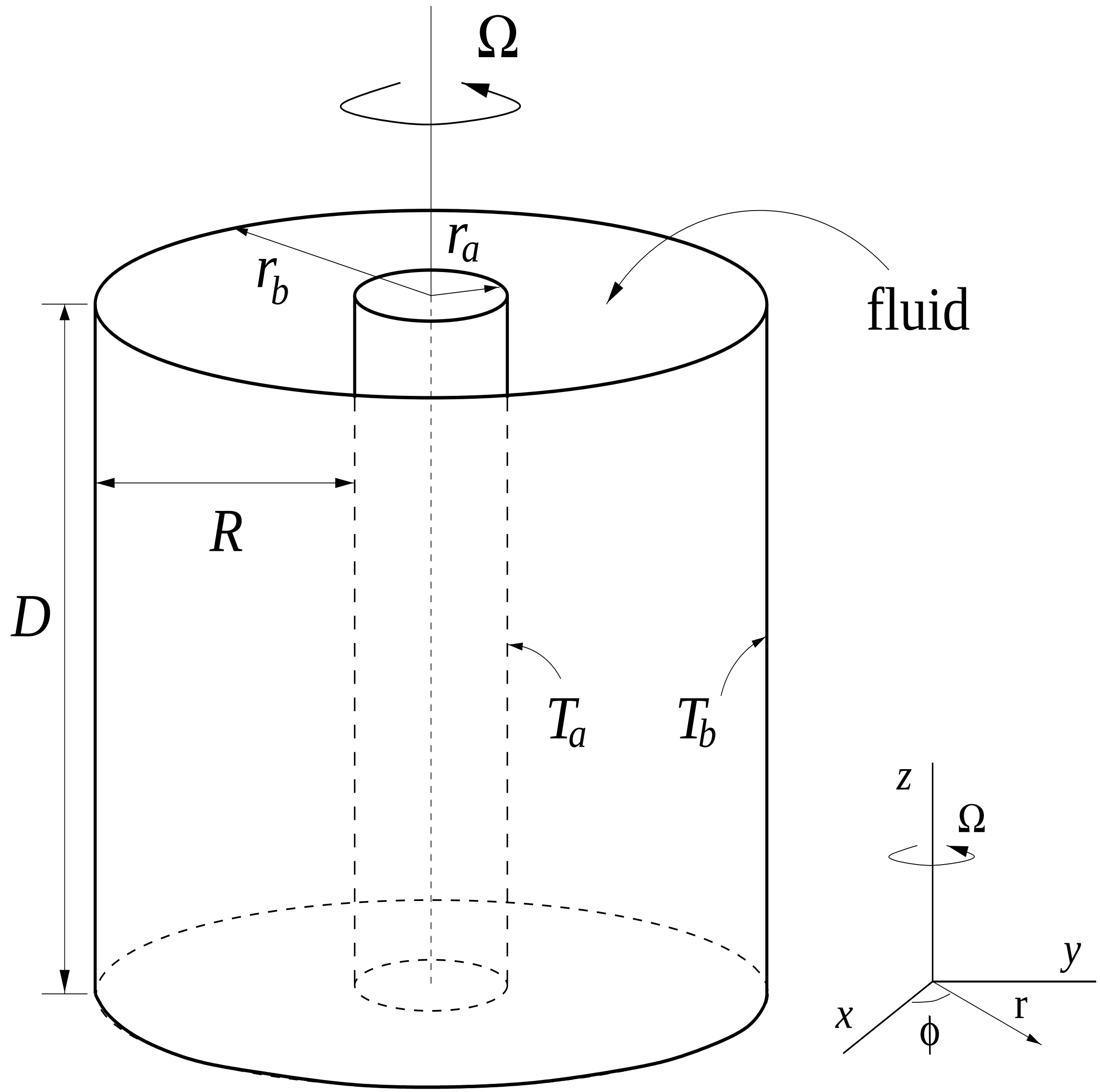
$$(\mathbf{A} - \sigma_1\mathbf{B})\,w = (\mathbf{A} - \sigma_2\mathbf{B})\,v$$

    i.e. a system of linear equations

Centre manifold reduction

- Apply centre manifold reduction at bifurcation points
  - gives a low-dimensional model of dynamics
  - get existence and stability of bifurcating solutions
  - gives results close to a bifurcation point (local dynamics)
- Write ODE (reduced equation) in normal form
  - compute the coefficients of the normal form equations
- Deduce dynamics of PDE from low-dimensional ODE

A differentially heated rotating annulus



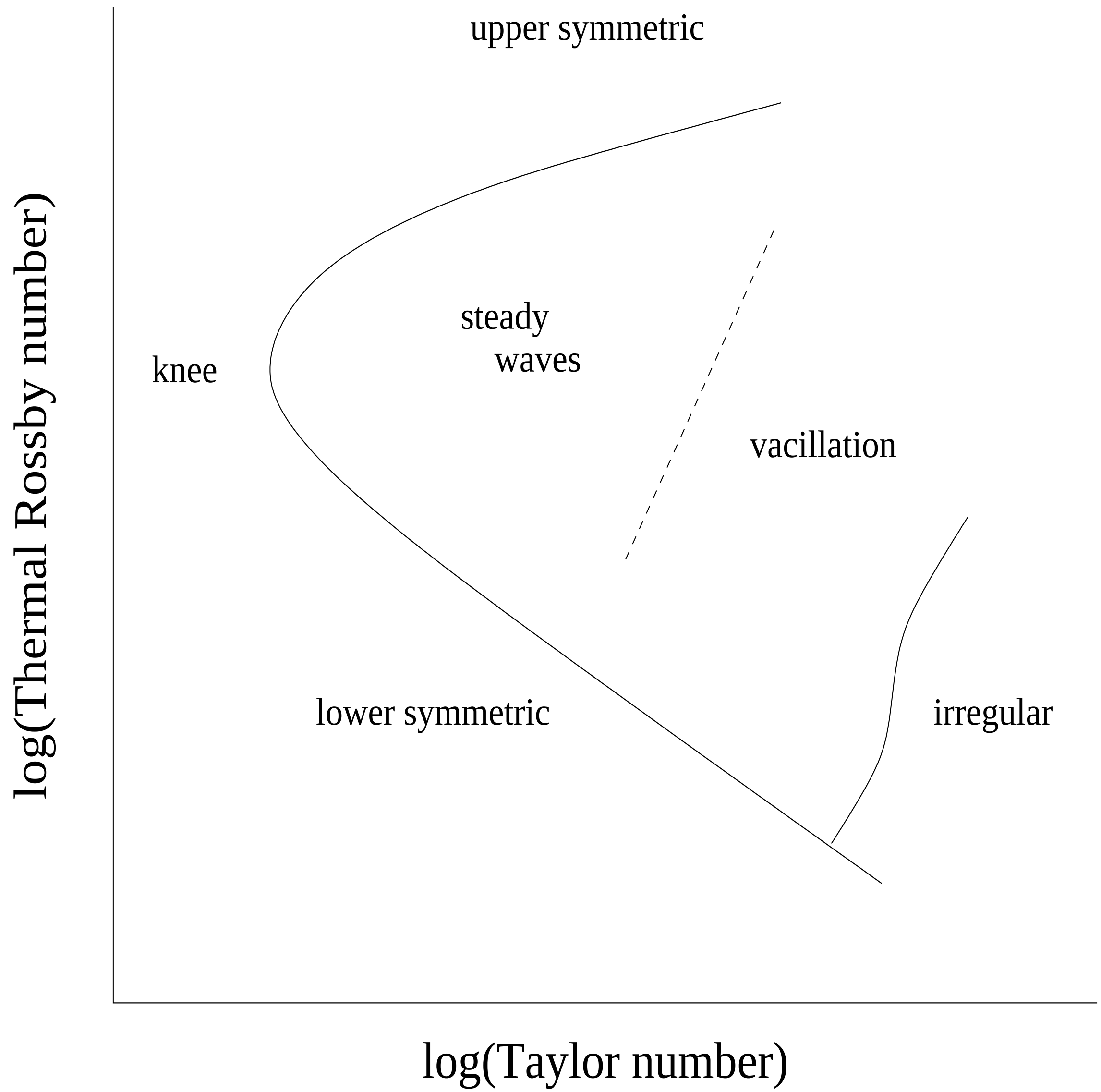
Model of fluid in the annulus

- Navier-Stokes equations in the Boussinesq approximation
- Cylindrical coordinates and rotating frame of reference
- No-slip boundary conditions
- Insulating top and bottom of annulus
- Differential heating: $\Delta T = T_b - T_a$
  inner cylinder cooled; outer cylinder heated
- Quantitatively accurate results

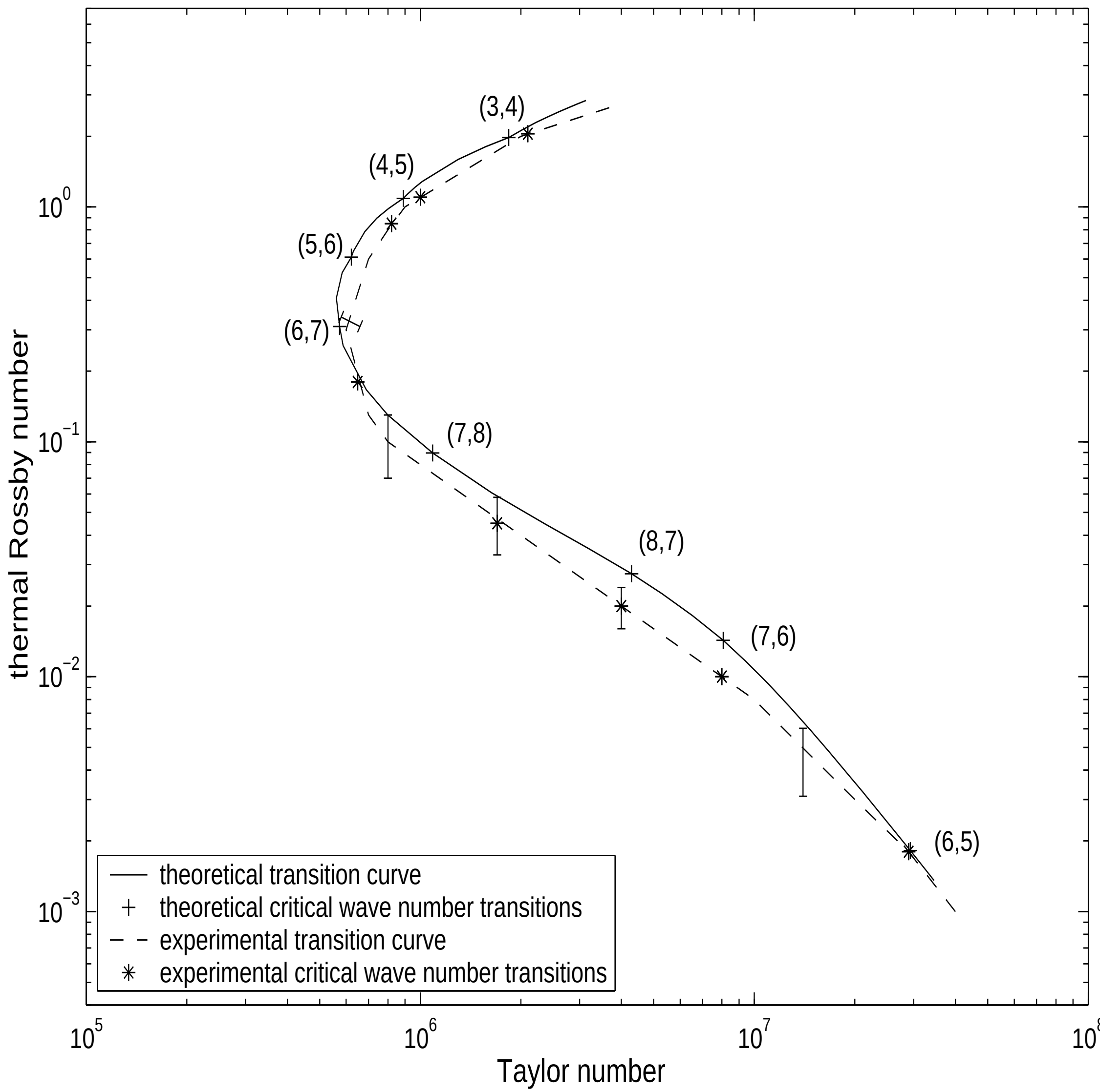
Analysis

- Look for steady flows invariant under rotation
  - primary transitions
  - reduces to problem in two-spatial dimensions
- Bifurcations from steady solutions

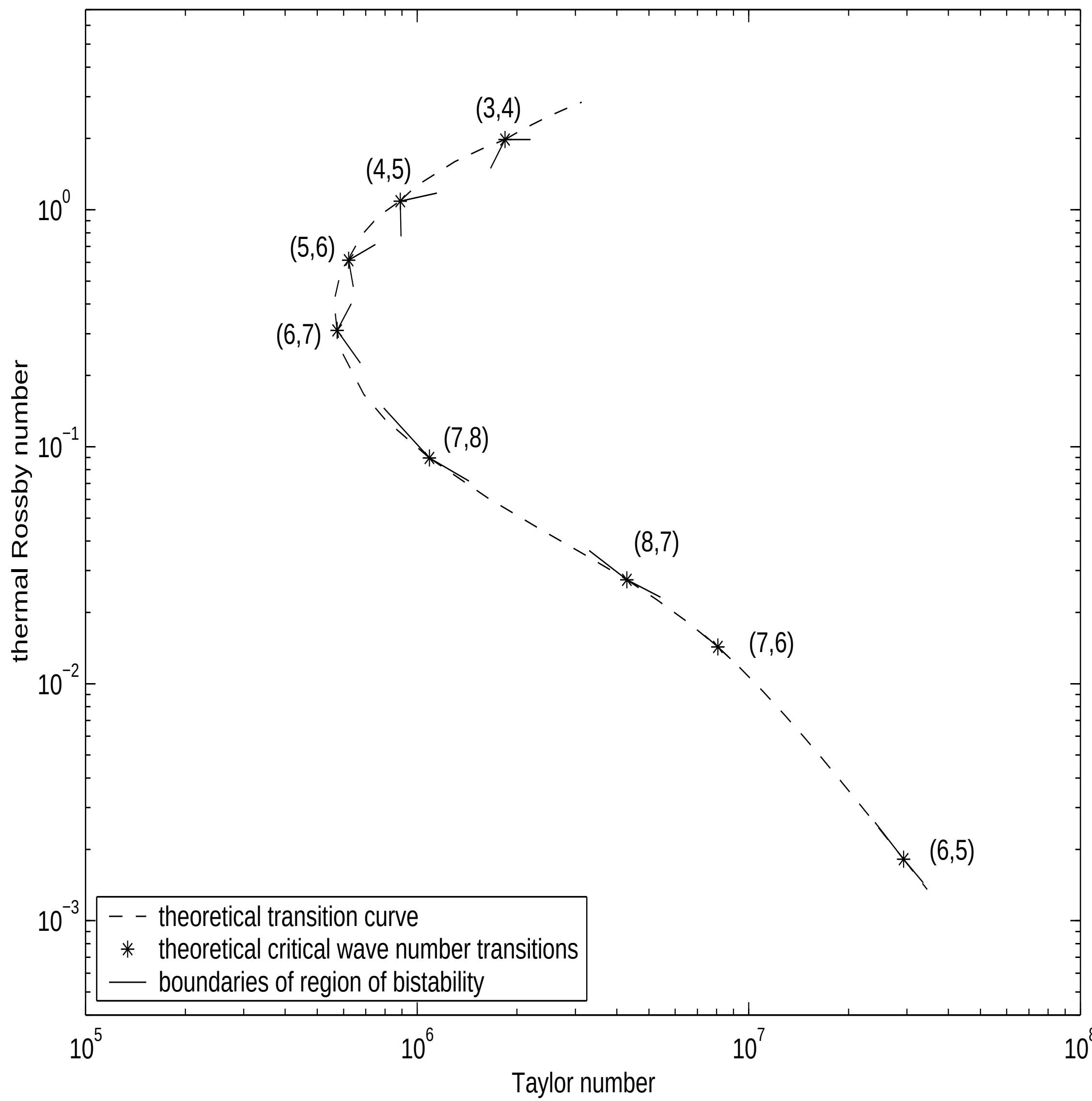
Regime diagram



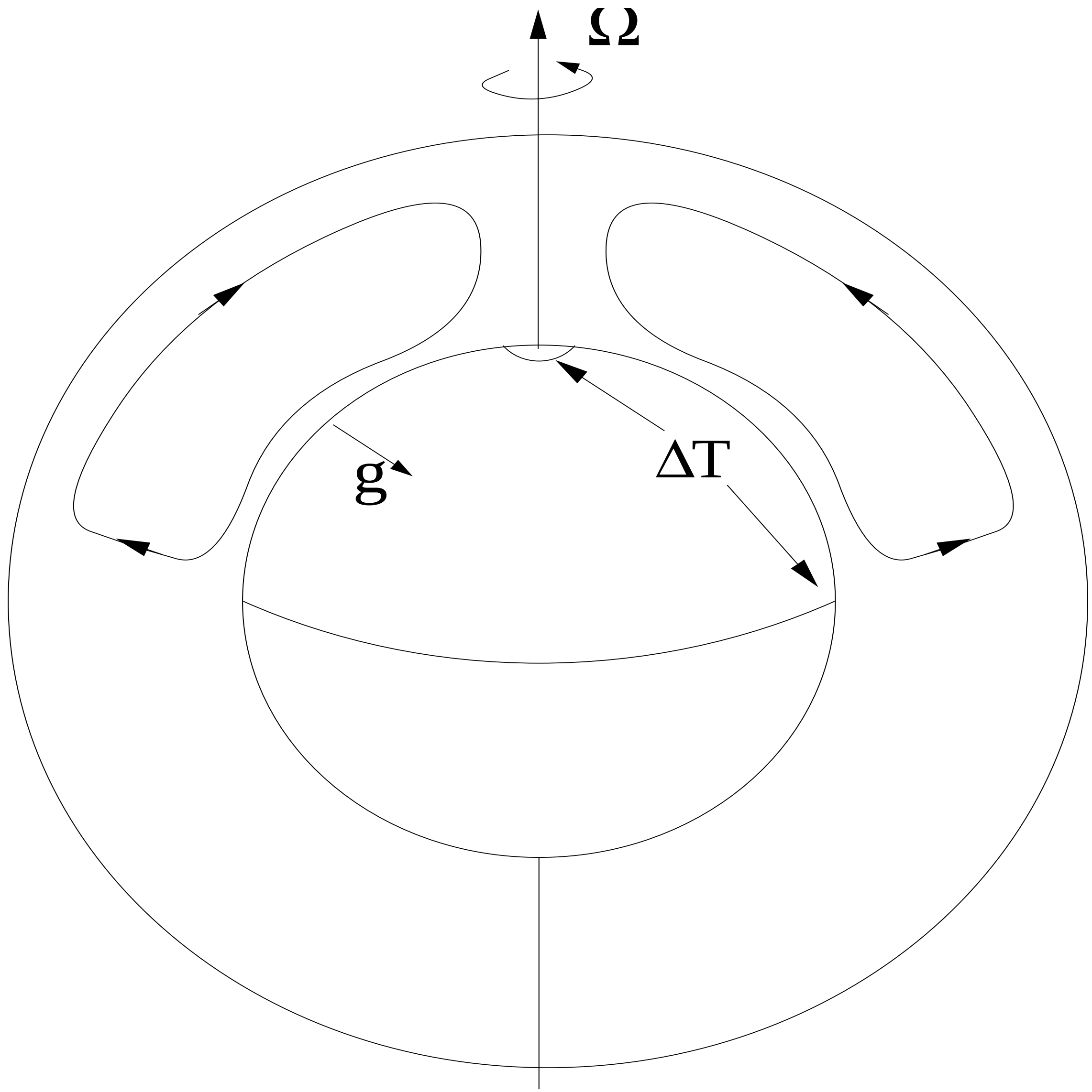
Transition curve



Regions of bi-stability



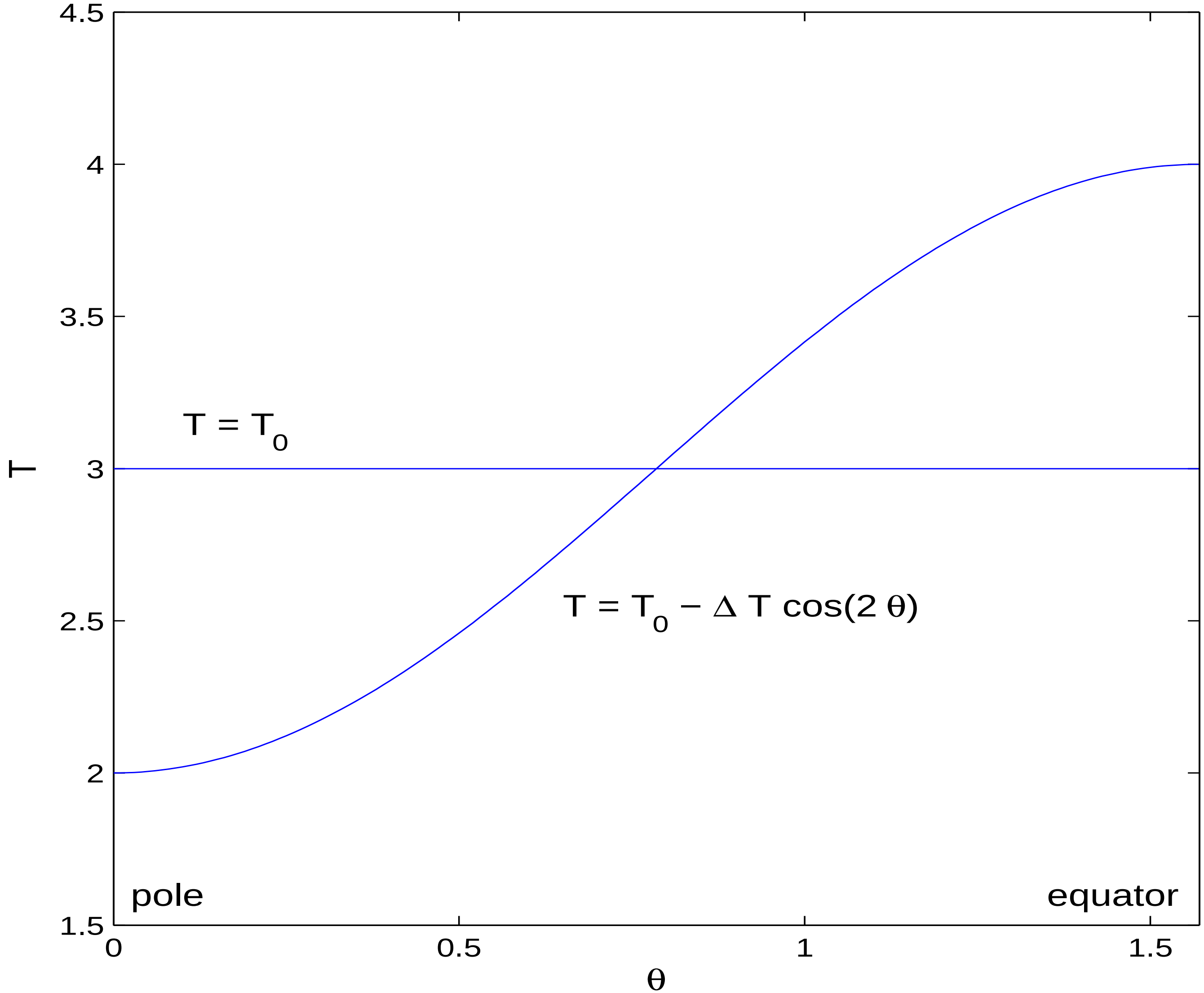
Spherical Shell



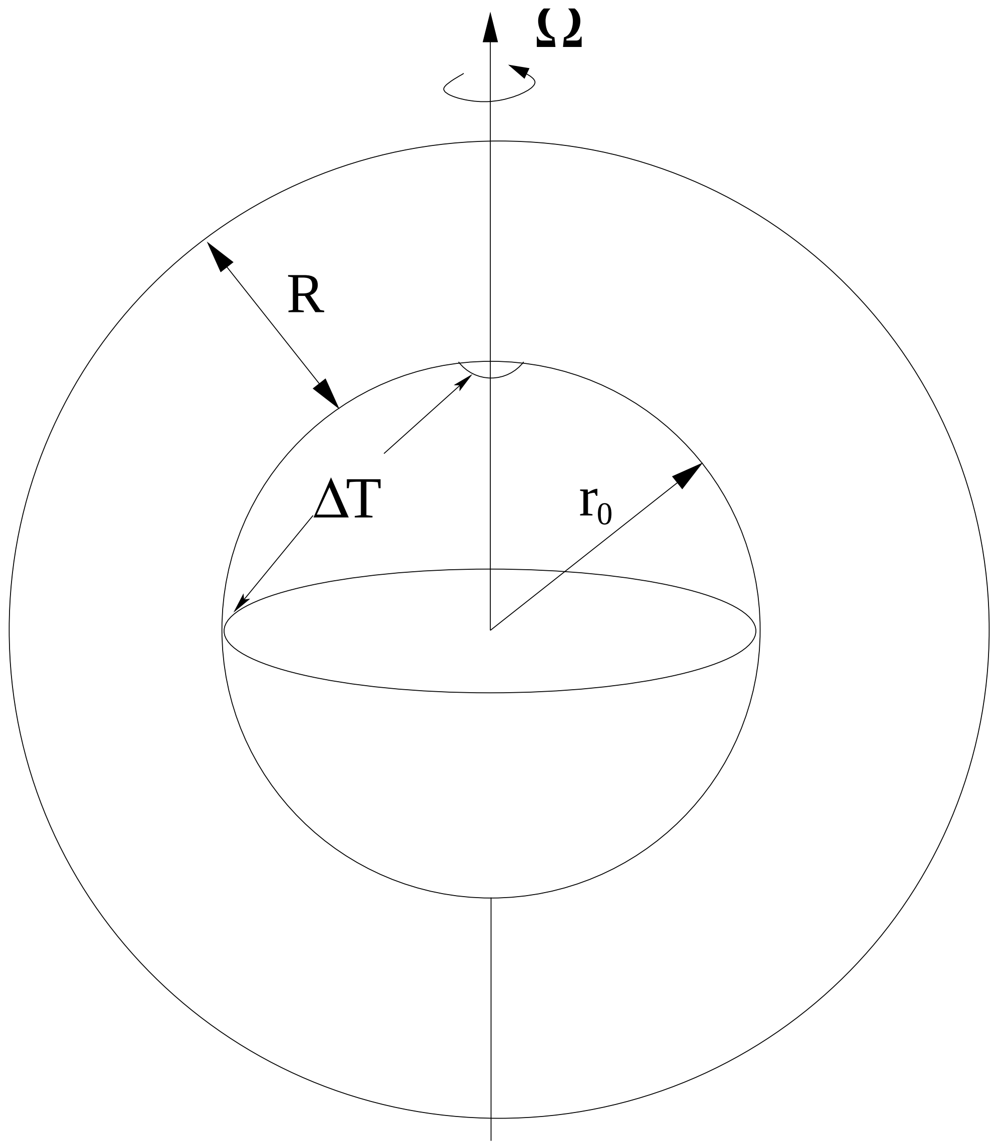
Model of fluid in a spherical shell

- Navier-Stokes equations in the Boussinesq approximation
- Spherical polar coordinates and rotating frame of reference
- No-slip boundary conditions at inner sphere
- Stress-free boundary condition at outer sphere
- Insulating outer sphere
- Differential heating imposed on inner sphere:
  at $r = r_0$, $T = T_0 - \Delta T \cos(2\theta)$.

Differential heating



Spherical shell

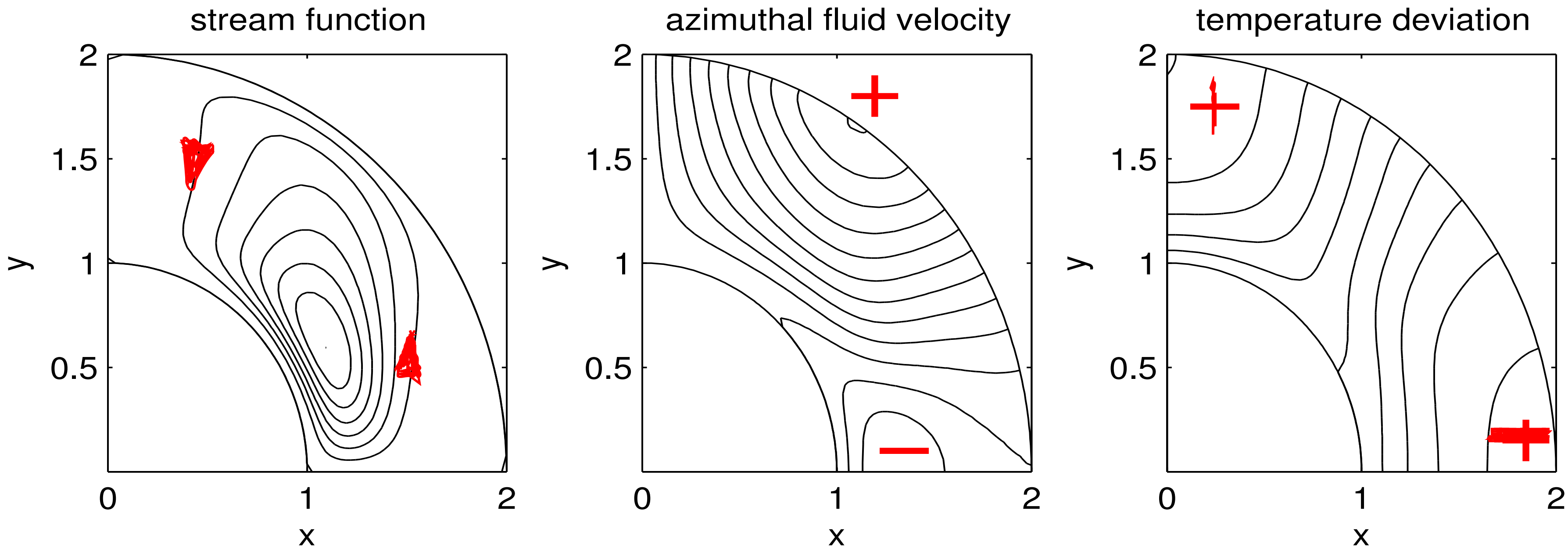


$\eta = \frac{R}{r_0}$

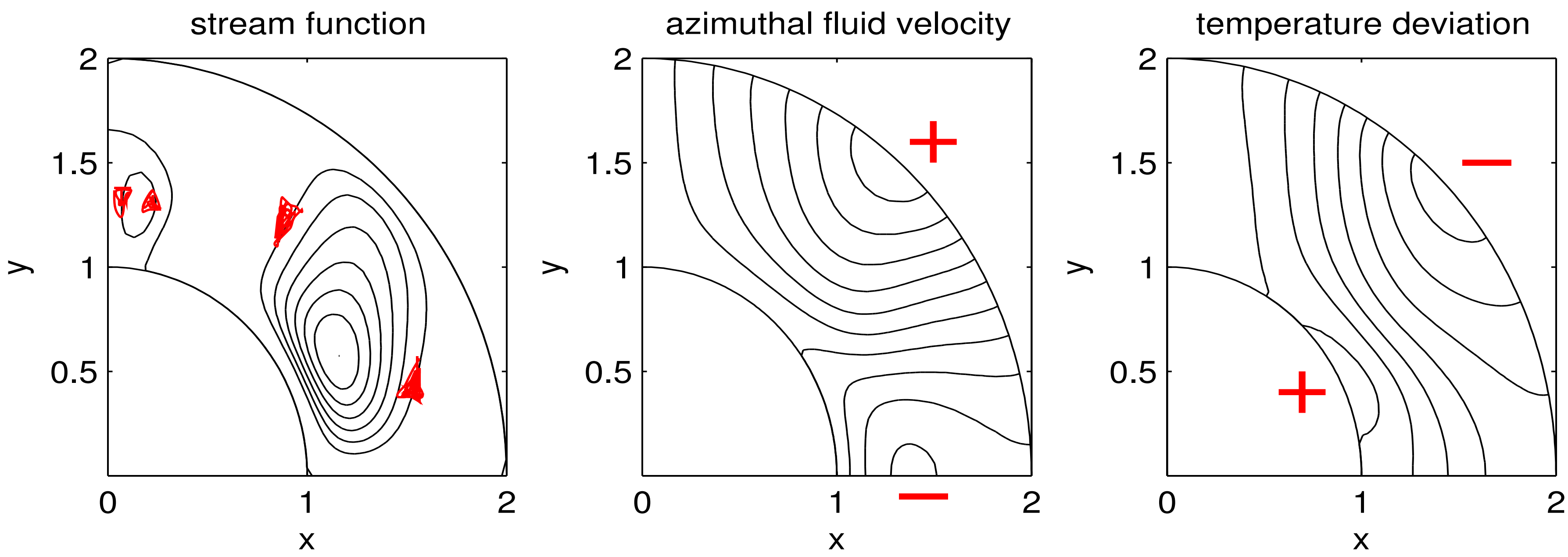
Analysis

- Look for steady flows invariant under rotation and reflection about equator
  - Reduces to problem in two-spatial dimensions
  - Introduces additional boundary conditions at pole and equator
- Bifurcations of steady solutions

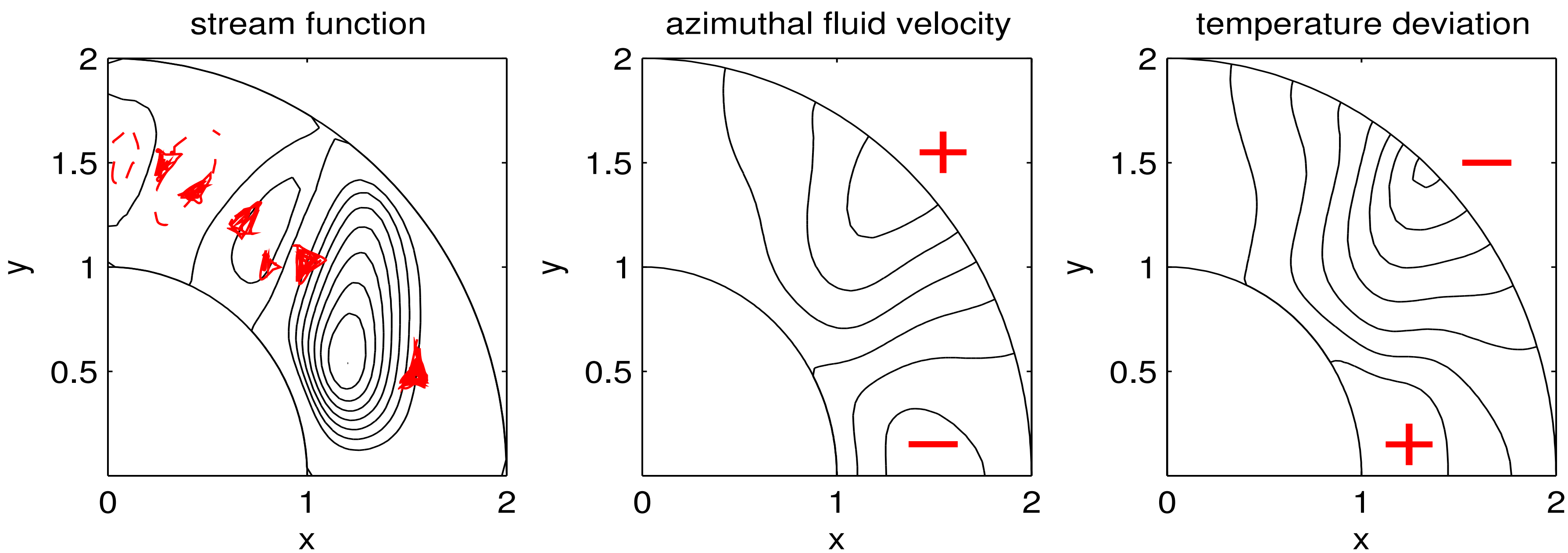
Steady Solution: $\eta = R/r_0 = 1/2, \Delta T = 0.004$



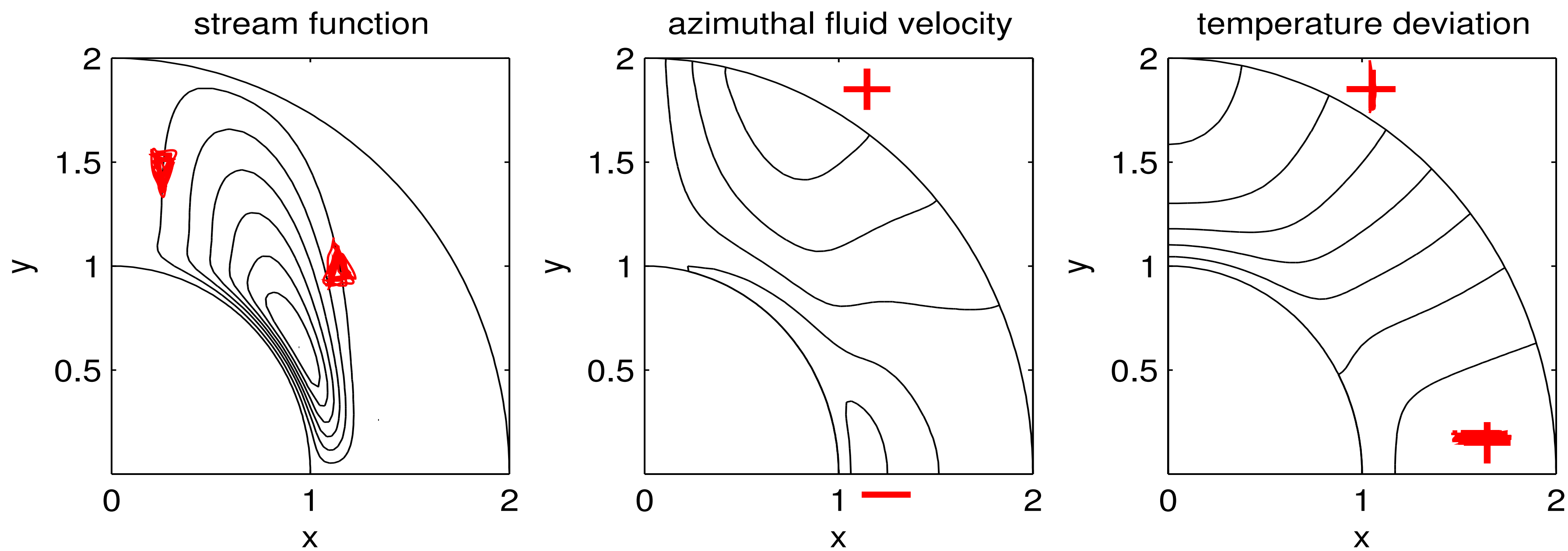
Steady Solution: $\eta = R/r_0 = 1/2, \Delta T = 0.026$



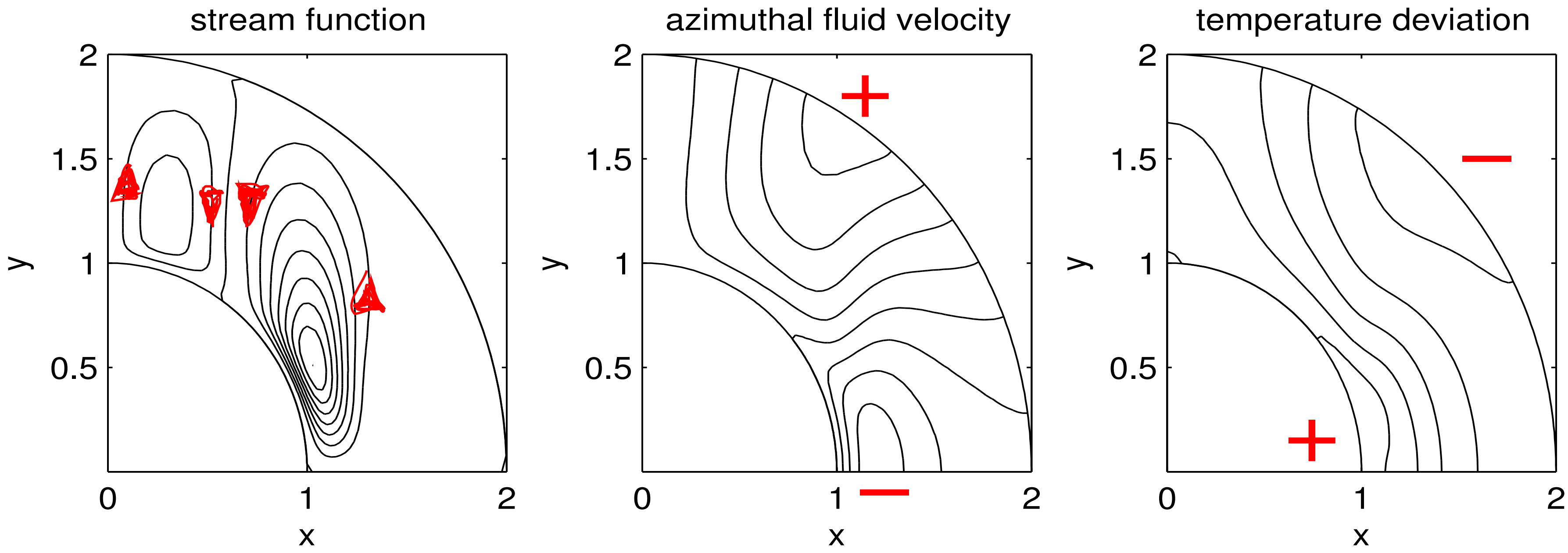
Steady Solution: $\eta = R/r_0 = 1/2$, $\Delta T = 0.0483$



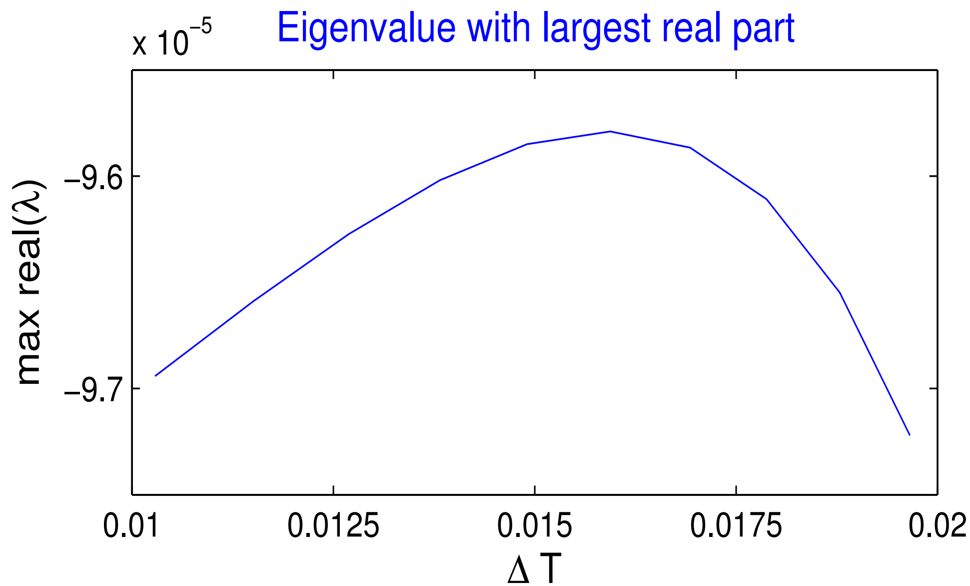
Steady Solution: $\eta = R/r_0 = 1, \Delta T = 0.002$

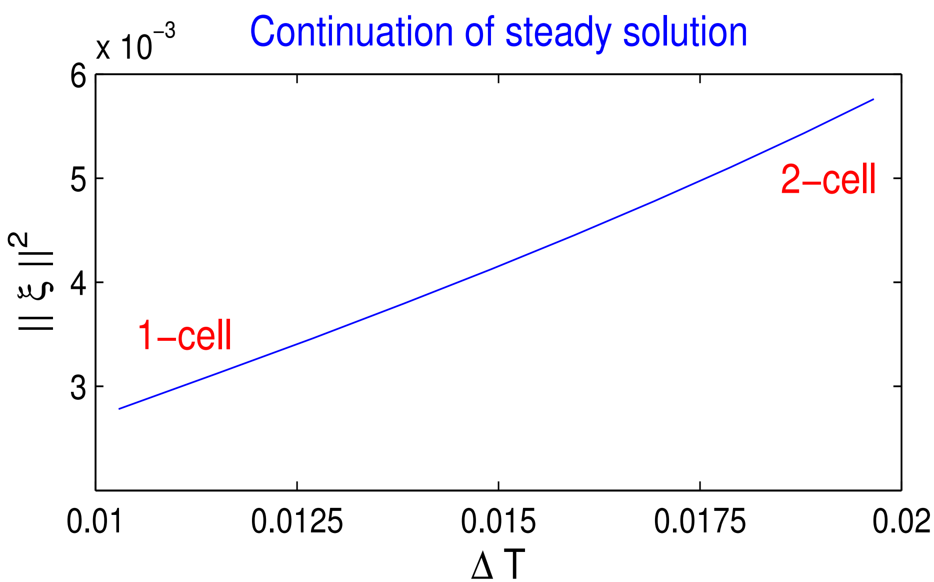


Steady Solution: $\eta = R/r_0 = 1, \Delta T = 0.029$

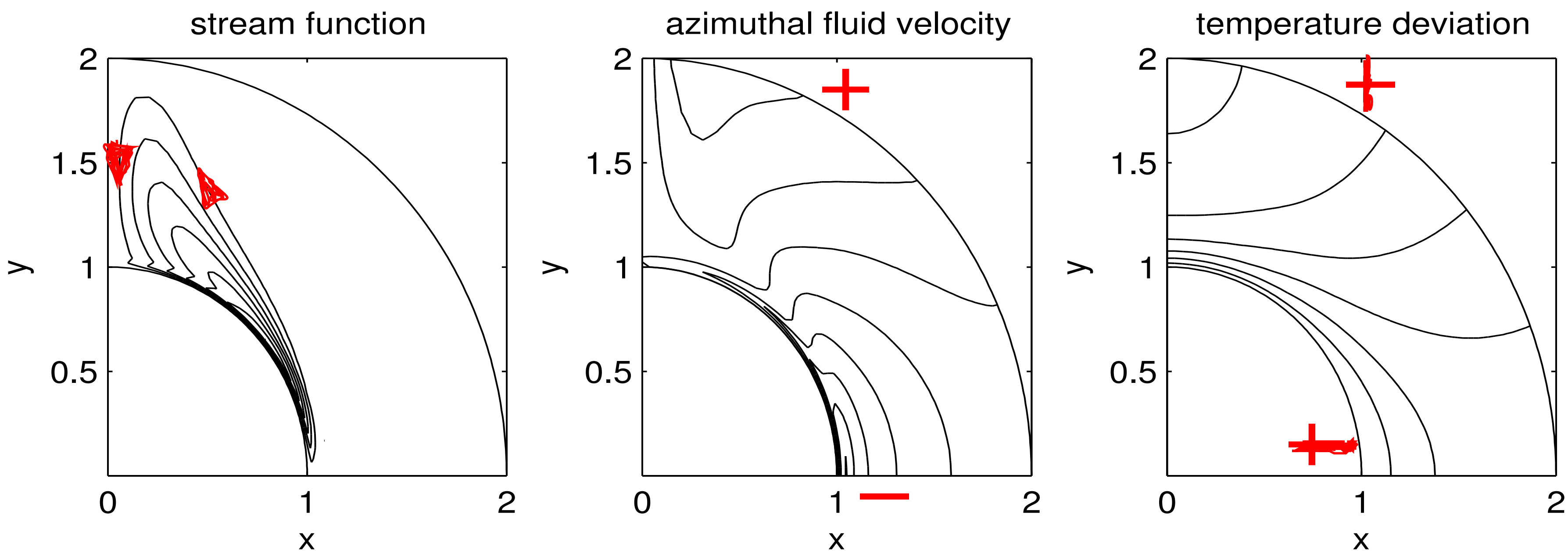


Bifurcation Diagram: $\eta = R/r_0 = 1$

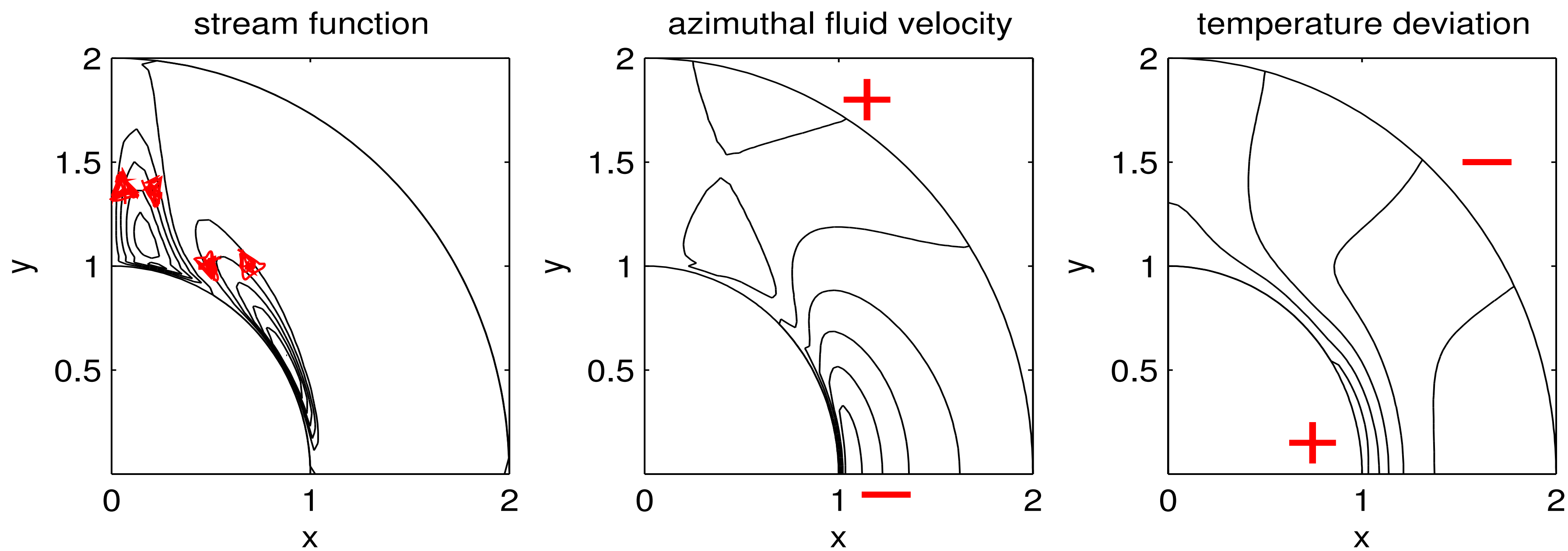




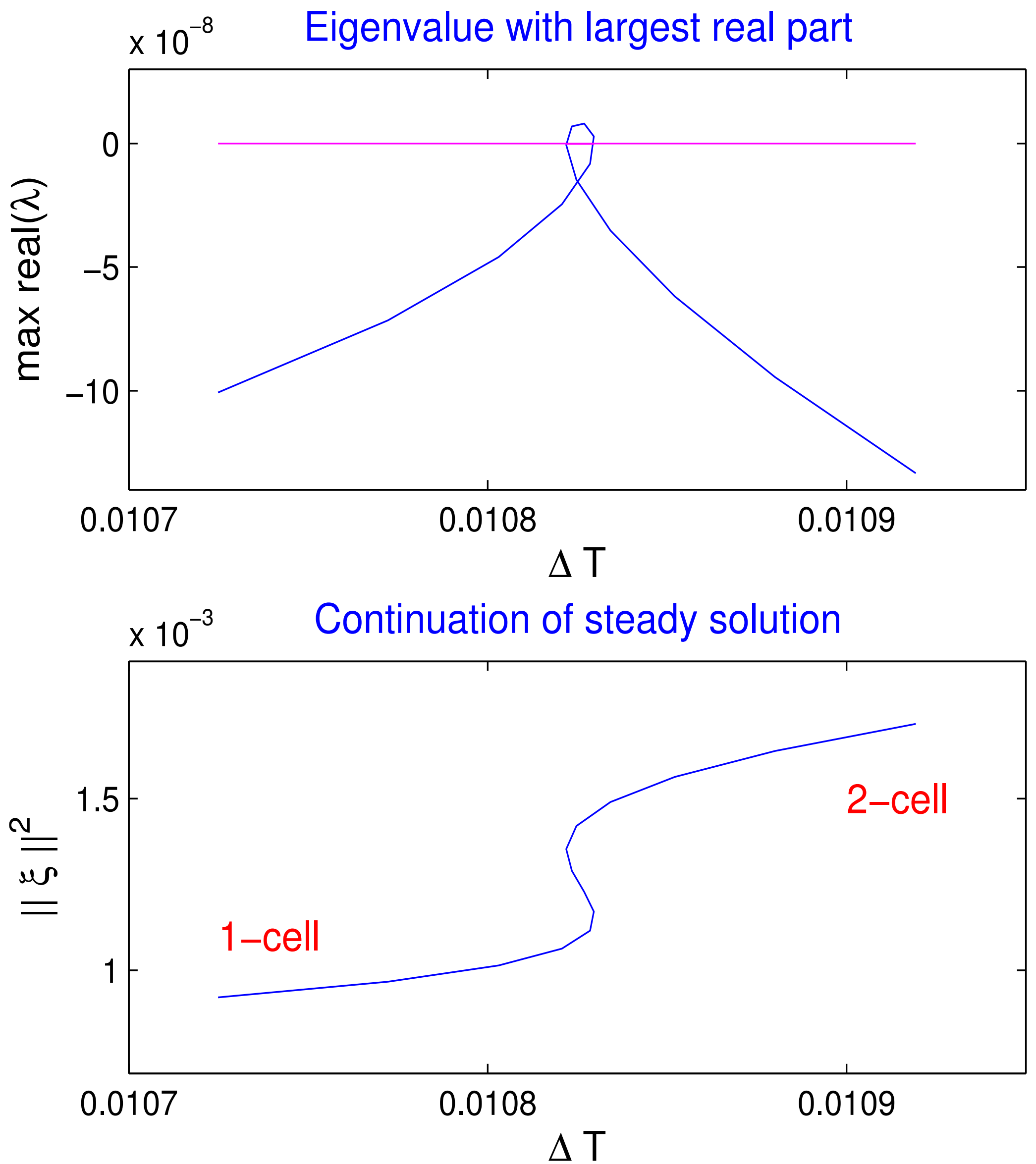
Steady Solution: $\eta = R/r_0 = 3.5$, $\Delta T = 0.001$



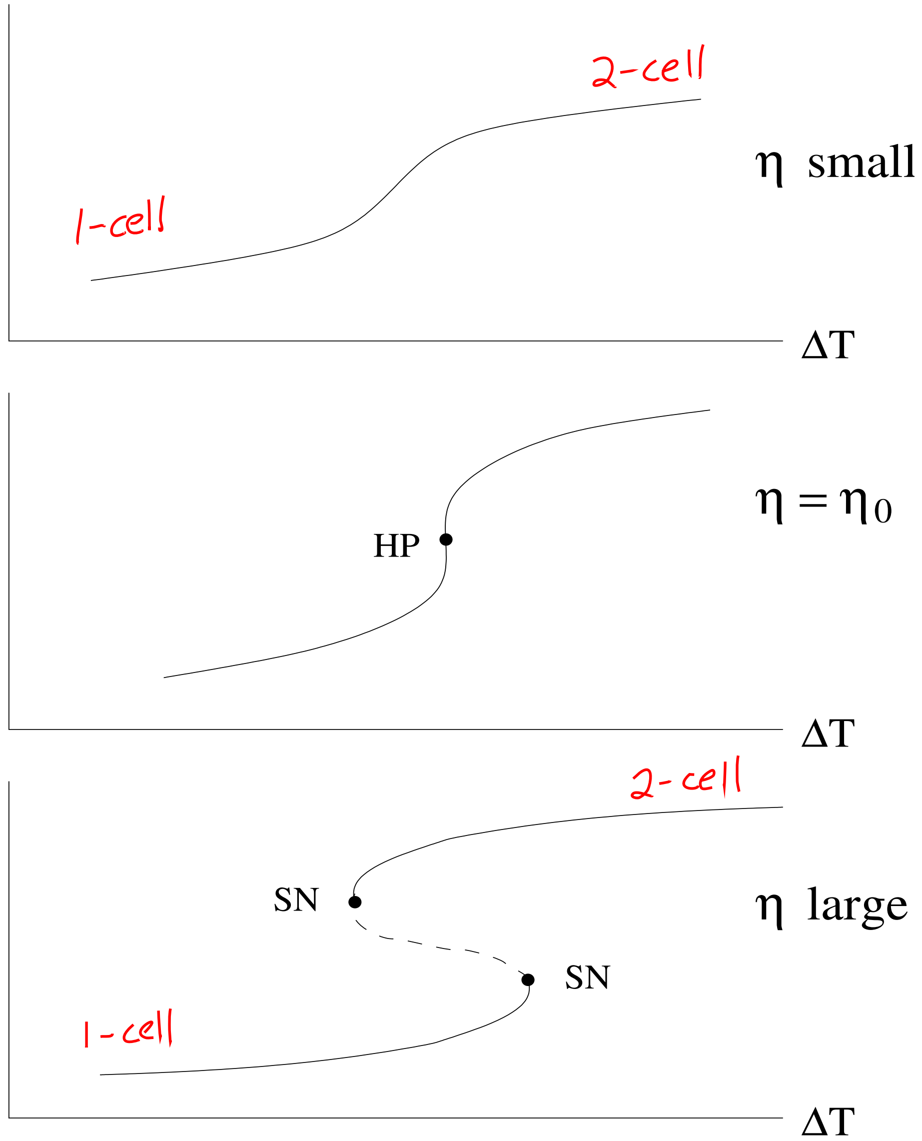
Steady Solution: $\eta = R/r_0 = 3.5$, $\Delta T = 0.019$



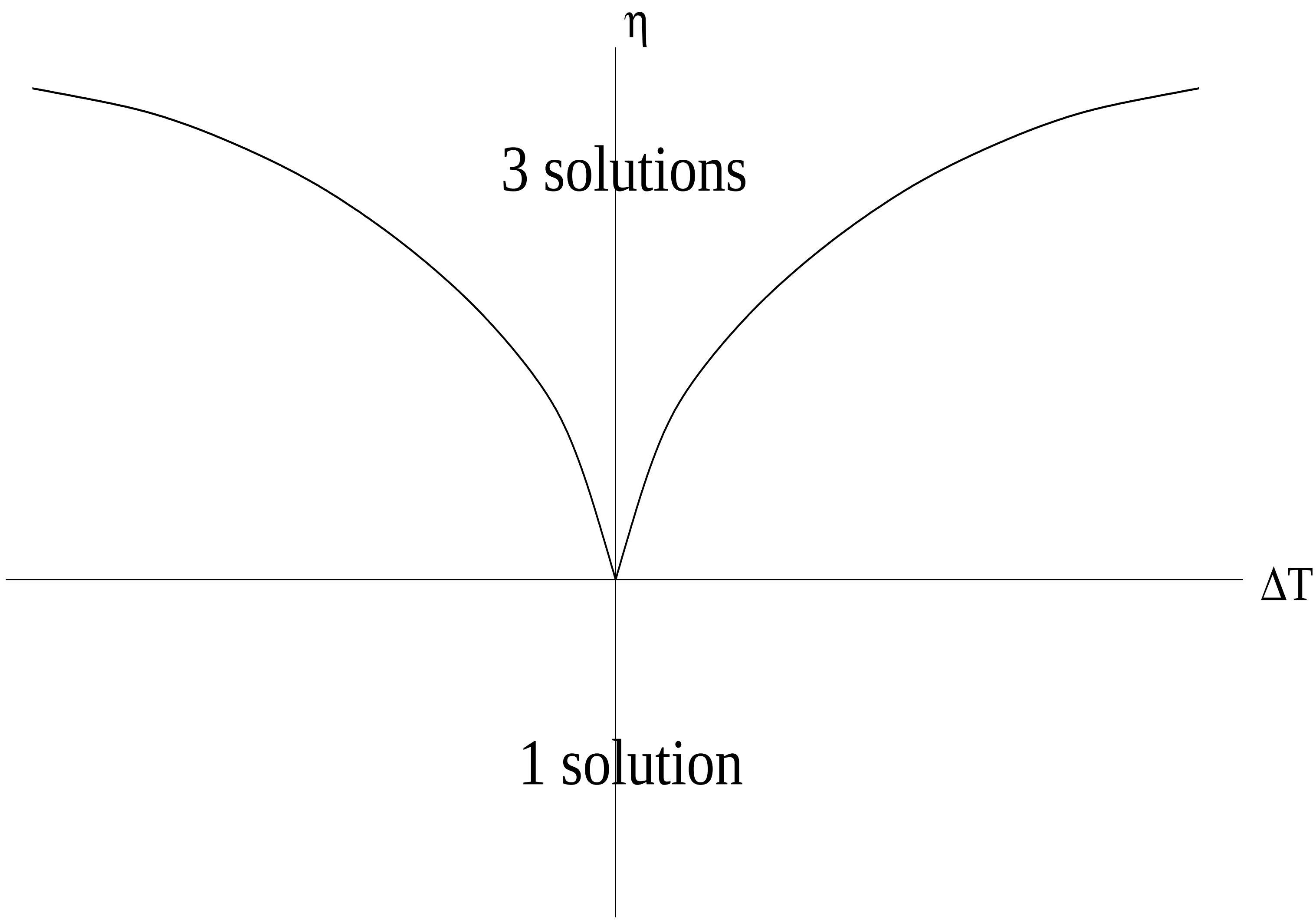
Bifurcation Diagram: $\eta = R/r_0 = 3.5$



Cusp bifurcation



Cusp bifurcation (schematic)



Computation of cusp point

- Codimension two bifurcation
  - Need two parameters: ΔT and η
- Write equations as:

$$\dot{U} = LU + N(U, U)$$

where U is dependent variable,
LU is linear part, $N(U, U)$ is nonlinear part,
and $\dot{U}$ is derivative with respect to time

Computation of cusp point

- Cusp point is characterized by:
  1. $LU_0 + N(U_0, U_0) = 0$
  2. zero eigenvalue of L_0 where $L_0 V = LV + N(V, U_0) + N(U_0, V)$
  3. vanishing of the coefficient of 2nd-order term of equation on centre manifold (or reduced equation)

Reduced equation

- Reduced equation

$$\dot{w} = \beta_1 + \beta_2 w + a w^2 + c w^3$$

where

$$a = 1/2 \left\langle \Phi^*, N(\Phi, \Phi) \right\rangle = 0$$

Φ is the eigenfunction corresponding to $\lambda = 0$,
Φ^* is the corresponding adjoint eigenfunction,
$\langle \cdot, \cdot \rangle$ is the inner product

Defining system

$$LU_0 + N(U_0, U_0) = 0, \quad g = 0, \quad g' = 0$$

where g and g' are scalars given by

$$L_0 V + gB = 0, \quad \langle C, V \rangle = 1$$

$$L_0 V' + g' B = -N(V, V), \quad \left\langle C, V' \right\rangle = 0$$

where B not in range of L_0,
and C not in range of the adjoint operator L_0^*.

- Solve to get $a = 0$ at $\eta = 3.46$, $\Delta T = 0.011$

Summary

- Application of numerical bifurcation analysis
  - compute flow regimes
  - compute details of flow transitions
- Could apply same ideas to industrial problems
- Applied to transitions from steady flows
- Could also apply similar ideas to transitions from periodic flows
  - HPC