Recent progress on nonlinear elliptic and parabolic problems and related abstract methods

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1 Overview

In recent years, significant progress has been made on the analysis of a number of important features of nonlinear partial differential equations of elliptic and parabolic type. These equations arise from mathematical biology, chemical reaction theory, material science, water waves and many other areas of science. For applications as well as for completeness of the mathematical theory, new methods have been established for the study of spatial and temporal patterns, sharp layers and spikes, blow-up, travelling waves, to mention but a few. These progress depend, to a large extent, on the interaction and further development of an extensive range of techniques, including topological and variational methods, bifurcation theory, singular perturbation, infinite dimensional dynamical systems, elliptic and parabolic estimates.

The workshop brought together some of the most distinguished mathematicians in the world and many of the top young researchers in the field. It provided in a very timely fashion a platform to facilitate the dissemination of the most recent research ideas and techniques in this active area of research. The talks during the workshop stimulated extensive discussions and collaborations, and we have every reason to expect more fruitful collaborations after the workshop.

Apart from the scientific programs, which are detailed below, Professor Jean Mawhin and Professor Henri Berestycki gave two very interesting informal presentations at an evening session to celebrate Professor Norman Dancer's 60th birthday.

2 Presentation Highlights

Walter Allegretto (Univ of Alberta), *On some parabolic equations motivated by biological problems*: We consider nonlinear parabolic equations that model some physical problems for which boundary effects are extremely important in determining the response of the system. Both theoretical properties and numerical simulations are presented. Examples from mathematical biology are used as illustrations.

Thomas Bartsch (Univ. Giessen) *On a parabolic semiflow with small diffusion*: We consider the singularly perturbed semilinear parabolic problem $u_t - d^2\Delta u + u = f(u)$ with homogeneous Neumann boundary conditions on a smoothly bounded domain $\Omega \subset R^N$. Here f is superlinear at 0 and ∞ and has subcritical growth. For small $d > 0$ we construct a compact invariant set X_d in the boundary of the domain of attraction of the asymptotically stable equilibrium 0. The main features of X_d are that it consists of nonnegative functions that are pairwise non-comparable, and that its topology is at least as rich as the topology of $\partial\Omega$ in a certain sense. This implies the existence of connecting orbits within X_d.

Peter Bates (Michigan State Univ) *Invariant Manifolds of Spikes*: Many singularly perturbed nonlinear elliptic equations have spike-like stationary solutions. These can be found through various methods, including Lyapunov-Schmidt schemes that, in the neighborhood of a proposed spike solution, decompose the operator equation into one that is restricted to a ”normal subspace” and one in a ”tangential subspace”. Here, these subspaces correspond to eigenstates of the operator, linearized at an approximate spike solution, and where ”tangential” means ”corresponding to eigenvalues near zero”, and ”normal” means ”complementary”. In this talk We will describe a more global decomposition in which the ”tangential subspace” is replaced by a finite-dimensional manifold of spike-like states and this manifold is invariant with respect to the corresponding nonlinear parabolic equation and also normally hyperbolic. The stationary spike-like states lie on this manifold.

Henri Berestycki (EHESS) *Generalized travelling fronts passing an obstacle*

Daniel Daners (Univ of Sydney) *The Faber-Krahn inequality for Robin Problems*: We present a Faber-Krahn inequality for the first eigenvalue of the Laplacian with Robin boundary conditions, asserting that amongst all Lipschitz domains of fixed volume, the ball has the smallest first eigenvalue. We discuss the idea of the proof as well as uniqueness of the minimizing domain. The method avoids the use of symmetrisation arguments. This is partly joint work with James Kennedy.

Marek Fila (Comenius Univ.) *Large time behaviour of solutions of a semilinear heat equation with a supercritical nonlinearity*: We consider global positive solutions of the Cauchy problem for a parabolic equation which has an ordered family of regular steady states and a singular steady state. All of them are stable in a suitable sense. We shall discuss the rate of convergence to these steady states which turns out to depend explicitly on the spatial decay rate of the initial function.

Nassif Ghoussoub (University of British Columbia) *Bessel potentials and optimal Hardy and Hardy-Rellich inequalities*: We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball Ω of radius R in R^n, $n \geq 2$, so that the following inequalities hold for all $u \in C_0^\infty(\Omega)$:

$$\int_\Omega V(x)|\nabla u|^2 dx \geq \int_\Omega W(x) u^2 dx$$

and

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \left(\frac{V(x)}{|x|^2} - \frac{v'(|x|)}{|x|}\right)|\nabla u|^2 dx.$$

We then identify a large number of such couples (V, W) – that we call Bessel pairs – and the best constants in the corresponding inequalities. This will allow us to complete, improve, extend, and unify most related results –old and new– about Hardy and Hardy-Rellich type inequalities which were obtained by Caffarelli-Kohn-Nirenberg, Brezis-Vázquez, Adimurthi-Chaudhuri-Ramaswamy, Filippas-Tertikas, Adimurthi-Grossi-Santra, as well as some very recent work by Tertikas-Zographopoulos, Liskevich-Lyachova-Moroz, and Blanchet-Bonforte-Dolbeault-Grillo-Vasquez, among others.

This is joint work with Amir Moradifam.

Massimo Grossi (Univ. of Rome I) *Existence results in the supercritical case*: We show some existence results for supercritical nonlinearities in the unit ball of R^n, for $n > 2$. The basic result concerns the existence of a positive solutions but also nodal solutions and related problems will be considered.

Changfeng Gui (Univ of Connecticut) *A Hamiltonian identity for PDEs and it's application*: In this talk We will present a new hamiltonian identity for PDEs and systems of PDEs. We will also show some interesting applications of the identity to problems related to entire solutions. In particular, we show the Young's law in triple junction configuration for a vector-valued Allen Cahn model in phase transition, and derive a necessary condition for the existence of saddle solutions for Allen-Cahn equation with asymmetric double well potential.

Francois Hamel (Universite Aix-Marseille III) *Uniqueness and further qualitative properties of monostable pulsating fronts*: The talk is concerned with qualitative properties of pulsating travelling fronts in a general periodic framework, for monostable reaction-diffusion equations. First, the uniqueness of pulsating fronts for a given speed is established for Kolmogorov-Petrovsky-Piskunov nonlinearities. These results provide in particular a complete classification of KPP pulsating fronts. To do so, the main tool is to prove the exponential behavior of the front when it approches its unstable limiting state. In the general monostable case, the logarithmic equivalent of the front is proved. For a noncritical speed, the decay rate is the same as in the KPP case. This talk is based on joint works with L. Roques.

Danielle Hilhorst (Univ. Paris-Sud) *Peak solutions of a chemotaxis-growth system*: We consider an elliptic reaction-diffusion system which both contains a nonlinear reaction term and a gradient one. In the case that the gradient term has a dispersal effect, we prove the existence of a boundary peak solution, and in both the cases that the gradient term has a dispersal or an aggregation effect we discuss the existence of an interior peak solution. This is joint work with N. Dancer and Shusen Yan.

Thomas Hillen (Univ of Alberta) *A classification of spikes and plateaus*: In this talk We will propose a simple classification of spike versus plateau local maxima (and minima). The classification uses non-local gradients and the fourth order derivative. It confirms the classical notion of plateaus for Cahn-Hilliard and spikes for Gierer-Meinhardt. Further, We will show that this classification can be used to study stability of spatial patterns.

Vera Mikyoung Hur (MIT) *Steady free-surface water waves with vorticity*: The mathematical problem for free-surface water waves embodies the equation of hydrodynamics, the concept of wave propagation, and the critically important role of boundary dynamics. We will give a precise account of its formulation and discuss its distinct features. Particular emphasis is given to the effects of vorticity. Existence theories of traveling waves will be presented with proofs, at least their ideas. Steady periodic waves on a deep water are traditionally referred to as Stokes waves, whose existence theory is established for a general class of vorticity distributions as an application of generalized degree theory of Healey-Simpson and Rabinowitz's global bifurcation theory. Solitary waves near a KdV soliton are constructed for an arbitrary vorticity profile via a Nash-Moser implicit function theorem. Other mathematical aspects of steady water waves, such as the Cauchy problem, symmetries, and stability/instability will be discussed, if time permits.

Meiyue Jiang (Beijing Univ.) *Semilinear Elliptic Equations with Indefinite Nonlinearities*: Let $\Omega \subset R^n$ be bounded domain with smooth boundary, we consider the Dirichlet problem:

$$\begin{aligned} -\triangle u &= \lambda u + a_+(x)|u|^{q-1}u - a_-(x)|u|^{p-1}u + h(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $a_\pm : \overline{\Omega} \to \mathbf{R}$ are continuous functions and $h : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a C^1 function which is sublinear in u at infinity, and λ is a real parameter, $1 < q < \frac{n+2}{n-2}$ and $p > 1$.

In this talk, based on computations of critical groups and Morse theory, we will present various existence and multiplicity results of solutions.

This is a joint work with K. C. Chang.

Congming Li (Univ of Colorado) *Classification of solutions to some integral systems*: We will present the recent work (joint with W. Chen, C. Jin, J. Lim, and B. Ou) on systems of integral equations related to the Hardy-Littlewood-Sobolev (HLS) inequality. The focus is on the Euler-Lagrange equations of the HLS. The main object is to classify all the nonnegative solutions through the study of symmetry, monotonicity, regularity and the asymptotic of the solutions. The integral form of the method of moving planes is the main tool.

Yuan Lou (Ohio State Univ) *Principal Eigenvalue and Eigenfunction of Elliptic Operator with Large Advection and its Application*: The asymptotic behavior, as the coefficient of the advection term approaches infinity, of the principal eigenvalue of an elliptic operator is determined. As an application a Lotka-Volterra reaction-diffusion-advection model for two competing species in a heterogeneous environment is investigated. The two species are assumed to be identical except their dispersal strategies: one disperses by random diffusion only, and the other by both random diffusion and advection along environmental gradient. When the advection is strong relative to random dispersal, both species can coexist. In some situations, it is further shown that the density of the species with large advection in the direction of resources is concentrated at the spatial location with maximum resources.

Hiroshi Matano (Univ of Tokyo) *Convergence and sharp thresholds for propagation in nonlinear diffusion problems*: We study the Cauchy problem for the equation

$$u_t = u_{xx} + f(u)$$

on R^1, where $f(u)$ is a locally Lipschitz continuous function satisfying $f(0) = 0$. We show that any nonnegative bounded solution with compactly supported initial data converges to a stationary solution as t tends to infinity. Moreover the limit is either a constant or a symmetrically decreasing stationary solution. Note that we assume no condition on the symmetry or monotonicity for the initial data.

We next consider the special case where $f(u)$ is either a bistable nonlinearity or a combustion nonlinearity, and prove the sharpness of transition between propagation and extinction, thus extending the earlier result of Zlatos for any compatcly supported nonnegative initial data.

Finally we discuss how one can extend the above results to the case where f has some sort of discontinuity.

This is a joint work with Yihong Du.

Jean Mawhin (Universite Catholique de Louvain) *Maximum and antimaximum principles around an eigenvalue with constant eigenfunction*: We consider the existence of a maximum and/or an antimaximum principle for the solutions of some abstract linear equations, around an eigenvalue with constant eigenfunction. Applications are given to various boundary value problems for ordinary and partial differential equations.

This is a joint work with J. Campos and R. Ortega.

Pavol Quittner (Comenius Univ.) *Very weak solutions of elliptic equations with nonlinear boundary conditions*: We study very weak solutions of linear (and nonlinear) elliptic equations in bounded domains with nonlinear Neumann boundary conditions. We find a sharp critical exponent with the following property: If the growth of the nonlinearity is subcritical then any very weak solution is bounded. We also find sufficient conditions for the existence of uniform a priori bounds of positive very weak solutions. In the supercritical case we prove the existence of multiple positive unbounded very weak solutions blowing up at a prescribed point of the boundary. This is a joint work with Wolfgang Reichel.

Paul Rabinowitz (Univ of Wisconsin-Madison) *On A class of infinite transition solutions for an Allen-Cahn model equation*: For an Allen-Cahn model equation, we discuss some recent results on the existence of a class of solutions which undergo an infinite number of spatial transitions.

Juan Luis Vazquez (Universidad Autonoma de Madrid) *Nonlinear elliptic and parabolic equations with "incompatible" measures as data*: We consider semilinear ellitpic equations like

$$-\Delta u + e^u = \mu$$

posed in R^2 with right-hand side a measure that contains Dirac deltas. If the intensity of one of these deltas is larger than 4π, then the problem cannot be solved. Parabolic equations shed light on this mystery by producing a natural generalized measure solution.

Wenxian Shen (Univ of Auburn) *Variational Principle for Spatial Spread and Propagation Speeds in Time almost and Space Periodic KPP Models*: The present talk is concerned with spatial spread and propagation speeds in time almost periodic and space periodic KPP models. A notion of spatial spread (propagation) speed interval in any given direction for a time almost periodic and space periodic KPP model is first introduced, which extends the concept of the spatial spread (propagation) speed for time independent or periodic KPP models and can be applied to more general time dependent KPP models. Then a variational principle for the spatial spread (propagation) speed is established. To prove the variational principle, we apply the recently developed principal spectrum theory for general time dependent linear parabolic equations. In addition, based on the variational principle, the influence of time and space variation on the spread speed is discussed. It is shown that the time and space variation speeds up the spatial spread.

Hal Smith (Arizona State Univ) *Applications of monotone systems theory to parabolic systems of partial differential equations*: I would summarize joint work with Moe Hirsch and with German Enciso.

Charles Stuart (EPFL) *A stable branch of solutions of a nonlinear Schrdinger equation*: In joint work with François Genoud, we consider the problem

$$\Delta u(x) + V(x)\,|u(x)|^{p-1}\,u(x) - \lambda u(x) = 0 \text{ with } u \in H^1(R^N)\backslash\{0\} \tag{1}$$

where $N \geq 3$ and $V \in C^1(R^N \backslash \{0\})$ is such that

$$\begin{aligned} (i) \quad & \text{for some } b \in (0,2), \quad |x|^b V(x) \text{ is bounded as } x \to 0 \\ & \text{and} \qquad |x|^b V(x) \to 1 \text{ as } |x| \to \infty, \\ (ii) \quad & 1 < p < 1 + \tfrac{2(2-b)}{N-2}. \end{aligned}$$

We prove that there exist $\lambda_0 > 0$ and $U \in C^1((0, \lambda_0), H^1(R^N))$ such that $(\lambda, U(\lambda))$ satisfies (1) for all $\lambda \in (0, \lambda_0)$. Furthermore, if $1 < p < 1 + \frac{2(2-b)}{N}$, the standing wave $e^{i\lambda t}U(\lambda)(x)$ is an orbitally stable periodic solution of the nonlinear Schrödinger equation

$$i\partial_t w + \Delta w + V(x) |w|^{p-1} w = 0.$$

We discuss also similar conclusions obtained recently by de Bouard-Fukuizumi and Jeanjean-Le Coz. A crucial ingredient is the non-degeneracy of the ground state of

$$\Delta u(x) + |x|^{-b} |u(x)|^{p-1} u(x) - u(x) = 0.$$

Susanna Terracini (Universit di Milano-Bicocca) *Spectral properties and uniqueness theorems related to optimal partitions*: We consider the spatial segregation in connection with the asymptotics of competition-diffusion systems as competition grows to infinity. We focus of the associated free boundary problem and we prove uniqueness of the partition with respect to the boundary data. Finally, we shall discuss optimal partition problems associated with eigenvalues and related spectral problems.

Juncheng Wei (Chinese Univ of Hong Kong) *Toda system, Allen-Cahn equation and nonlinear Schrodinger equations*: We show that there is a deep connection between the following Toda system

$$(1) \quad -f_j^{''} = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}}, j = 1, ..., K$$

and two well-known scalar autonomous PDES:

$$(2) \quad \Delta u + u - u^3 = 0 \ \ R^2,$$

$$(3) \quad \Delta u - u + u^p = 0, u > 0 \ \ R^2$$

More precisely, given any solution to the Toda system (I), we can find solutions to (2) or (3) of the form

$$u \sim \sum_{j=1}^{K} W(x - f_{\alpha,j}(y))$$

where $f_{\alpha,j} = f_j(\alpha y) + 2 \log \frac{1}{\alpha}$. Central to our construction is the use of Dancer's solutions. Other types of solutions to (3) as well as moduli space of solutions to (3) will also be discussed. (Joint work with M. del Pino, M. Kowalczyk and F. Pacard.)

Tobias Weth (Univ of Giessen) *A priori bounds and multiple existence of solutions to a non-cooperative elliptic system*: We report on recent joint work with J.C. Wei and E.N. Dancer on the Dirichlet problem for a non-cooperative two-component elliptic system in a smooth bounded domain. The system arises in the Hartree-Fock theory for a Bose-Einstein (double) condensate with repulsive interaction modeled by a parameter. We completely determine the parameter range for which the system admits a priori bounds for positive solutions. If the underlying domain is a ball, we also construct positive radial solutions with a prescribed number of intersections. If time permits, I will also address the asymptotic shape of solutions as the strength of the repulsive interaction tends to infinity.

Eiji Yanagida (Tohoku Univ) *Solutions with moving singularities for a semilinear parabolic equation*: For the Fujita-type equation, there exists a singular steady state that is radially symmetric with respect to the singular point. In this talk, we consider singular solutions that are obtained as perturbation of the singular steady state. In some parameter range, given any smooth curve and initial data in a wide class, we establish the local existence of a solution with a singularity moving along the curve. This is a joint work with Shota Sato.

Andrej Zlatos (Univ of Chicago) *Speed-up of Reaction-diffusion Fronts by Strong Flows*: We present recent results on speed-up of traveling fronts by strong flows in reaction-diffusion equations. We characterize periodic flows which can arbitrarily speed up fronts for general combustion-type reactions in two dimensions, as well as those which achieve the fastest speed-up rates for KPP reactions in any dimension. Relations to quenching of reactions and to homogenization in the associated passive scalar equations will also be discussed.