Numerical Methods for Degenerate Elliptic Equations and Applications 06w5095

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Saturday, December 9 – Thursday, December 14, 2006

1 Overview

Degenerate elliptic partial differential equations occur widely in many branches of applied math and engineering. The theory of viscosity solutions has been enormously successful in addressing the problems of existence, uniqueness and stability of solutions for this class of equations under minimal regularity requirements. Until recently, however, there has been much less success constructing solutions for many practical examples.

The development of techniques for effectively computing solutions to degenerate elliptic equations could have an huge impact in these application areas.

What stands in the way of developing effective computational techniques? The biggest obstacle is the high dimensionality of the domains on which practical problems must be solved. For example, the Hamilton-Jacobi-Bellman equations for optimal flight trajectories evolve on an underlying space of at least six dimensions. Stochastic control problems from mathematical finance may have twenty dimensions. New techniques need to be developed which overcome this obstacle.

1.1 Timeliness

This is an exciting time for numerical methods for degenerate elliptic PDEs. As computing power increases, application area researchers have been developing techniques specific to their problems. Recognition of the common elements in these algorithms has the potential to transfer knowledge between fields and develop more general methods.

Numerical analysts have in the past borrowed techniques from conservation laws to solve Hamilton-Jacobi equations. But new methods which address the class of degenerate elliptic equations directly have opened the way to solving more and more equations, including second order, fully nonlinear equations.

This conference aims to bring together specialists from these different areas with the goals of exchanging knowledge about and improving upon the many different algorithms. It will connect experts in numerics with application area specialists and viscosity solution theoreticians.

1.2 Importance

We mention a selection of applications, many of which will be touched upon by the conference. The theory of viscosity solutions appears in atmospheric and ocean studies, economics, mathematical ecology, statistical mechanics, optimal control, game theory, mathematical finance, and many other fields of science and engineering. Putting this theory into practice requires development of rigorous computationally effective methods comprehensive enough to include all these applications and simple enough to be adopted and adapted by non-specialists in the application fields.

Level Set Methods: Level set methods have enjoyed great success tracking the motion of interfaces in low dimensional spaces for fields such as combustion, fluid dynamics, phase transitions, etc. Flexible, high-order, dimension-by-dimension numerical schemes have been developed and have proven very effective.

Optimal control: Even when the linearity assumption is questionable, linear models are relied upon because problems with linear dynamics have exact solutions. This limitation is accepted because there are few practical solution methods for nonlinear control problems. But it has been known for decades that the Hamilton-Jacobi-Bellman equation solves the problem! Why aren't engineers using the Bellman equation? Because present methods are just too costly to utilize in the high dimensional spaces of interest. Nevertheless, there is much active research in accurate and/or scalable solutions to this equation.

Stochastic Control and Mathematical Finance: Many problems from mathematical finance lead to degenerate elliptic equations; for example, optimal portfolio selection and asset pricing. While the simplest problems may have exact solutions, more general problems require numerical approximation. Large collections of PDEs appear with similar properties which demand flexible, robust solution techniques. Often these problems have a high dimensional underlying space; for example, bond pricing may have 20 dimensions.

Robotics and Machine Learning: On the flip side, the robotics and machine learning literature is filled with investigations of policy search algorithms and reinforcement learning. Although not often formulated as such, the goal of such methods is approximation of a value function, a value function which is usually the solution of a Hamilton-Jacobi-Bellman equation. While the discretization of the underlying space is often coarse and lacks convergence arguments, these algorithms have been demonstrated in dimensions far beyond those that level set methods have achieved.

Image Processing: A digital image can be regarded as a function defined in two dimensions. Image transforms can be interpreted as operators on this space of functions. A large class of image transforms can be interpreted as short time solutions of degenerate elliptic PDEs. Applications appear in medical image processing (which includes volumetric images such as those from MRI), digital photography, and computer vision.

1.3 Conclusion

We propose to gather together applied mathematicians and numerical analysts with expertise in the field of viscosity solutions, and to add a collection of carefully chosen application area experts whose research is impacted by viscosity solutions. The success of level set methods has demonstrated to the mathematical community the unmet demand for accurate numerical approximation of these equations, but has only touched on a narrow class of problems. As witness to the broader need, engineers and scientists are developing problem specific solutions all the time. Theoreticians and numerical analysts are developing new solutions techniques which will benefit from contact with applied researches and their real world problems. Bringing these groups together can foster advances not only through modification of existing numerically proven algorithms to attack other problems, but also through analysis and formalization of existing problem specific algorithms. New and improved solution techniques for degenerate elliptic equations have a large potential to impact the wide variety of fields mentioned above, and many beyond.

1.4 Participants

Attached is a photo of the participants.

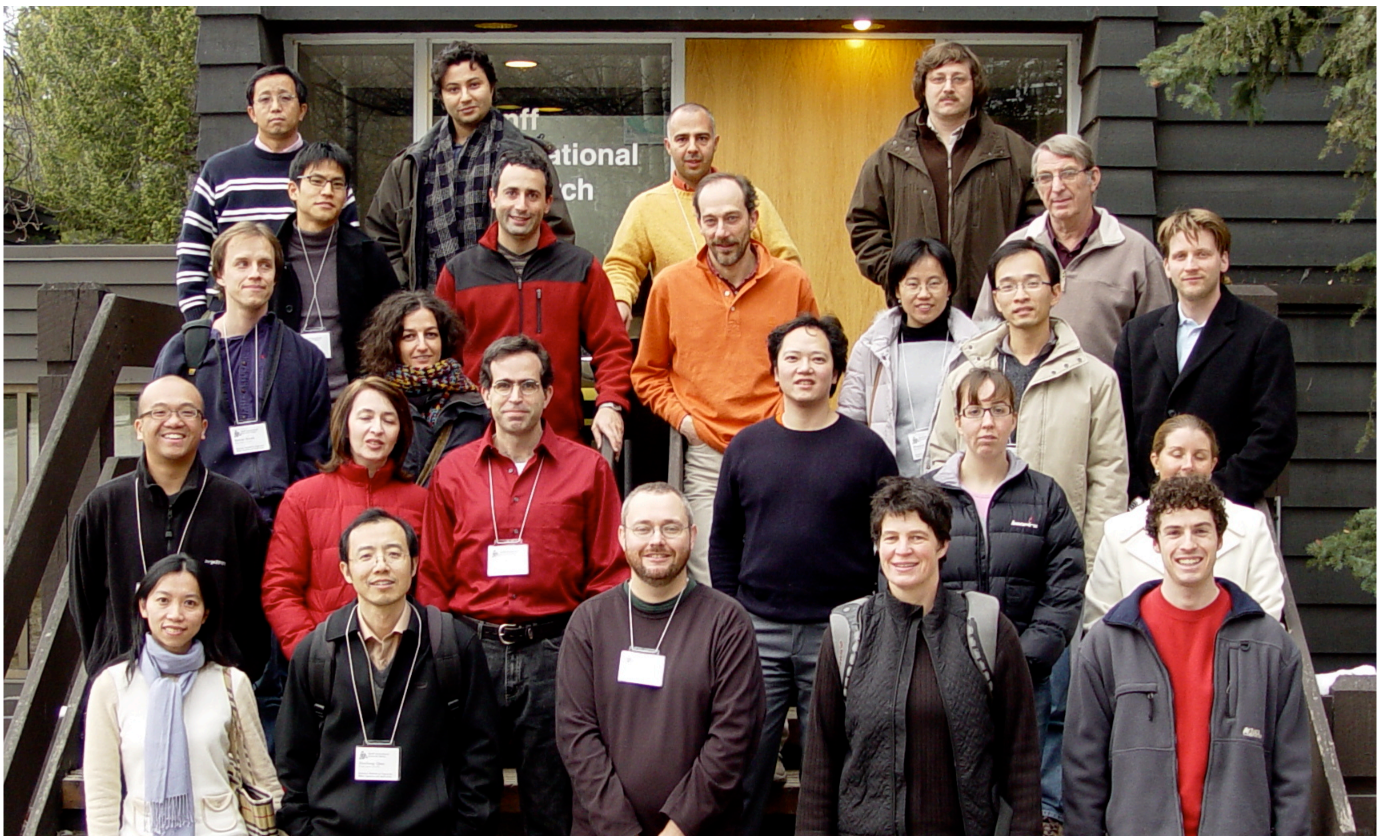

Figure 1: Photo of Participants: Numerical Methods for Degenerate Elliptic Equations and Applications 06w5095

2 Presentation Highlights

2.1 Advances in Advancing Interfaces: Interface Solvers in Geophysics, Semiconductors, LCD/LEM/Plasma Displays, and Logistical Transport

Speaker: J.A. Sethian

We discuss a variety of new algorithms and applications of interface schemes. In geophysics, we show how to derive equations for a propatating field in time-to-depth conversions required for accurate seismic imaging, and show how provide better initial guesses for depth migration. In two phase flow, we show how to couple interface solvers to projection fluid solvers to capture viscoelastic inkjets in the manufacturing process. In semiconductors, we show how to build algorithms for superconformal electrodeposition in damascene copper deposition, coupling PDE-based material transport solvers to level set methods (show in the figure below). And in logistical transport, we show how to build an algorithm for solving the continuous traveling salesmen problem, in which the goal is to find the shortest path which reaches all cities located in a domain-dependent cost function, in which the actual geodesics paths and link weights are unknown as the outset.

2.2 Fully nonlinear second order elliptic Partial Differential Equations: numerics and game interpretations

Speaker: Adam Oberman

The theory of viscosity solutions gives powerful existence, uniqueness and stability results for first and second order degenerate elliptic partial differential equations. The approximation theory developed by Barles and Sougandis in the early nineties gave conditions for the convergence of numerical schemes.

While there has been a lot of work on first order equations, there has been very little work on genuinely nonlinear or degenerate second order equations. This despite the fact that many of these equations are of the subject of current research and applications. For example: level set motion by mean curvature, the Infinity Laplacian, the PDE for the value function in Stochastic Control, the Monge-Ampere equation.

In this talk, we build convergent schemes for the aforementioned equations, and also for some less well-known or newer equations, including the Pucci Equations and a new PDE for the convex envelope.

Another subject of recent work has been finding non-generic stochastic control or game interpretations of the PDEs. While it has been known for some time that this is always possible, there have been recent interpretations for the motion by mean curvature (by Kohn and Serfaty) and for the Infinity Laplacian (by Peres-Schramm-Scheffield-Wilson). These interpretation are linked to simple numerical approximation schemes which we build.

We will also discuss a general framework for building these types of schemes. The majority of the work discussed can be found on the author's webpage.

2.3 A Simple Technique for Solving Partial Differential Equations on Surfaces

Presenter: Steve Ruuth Many applications require the solution of time dependent partial differential equations (PDEs) on surfaces or more general manifolds. Methods for treating such problems include surface parameterization, methods on triangulated surfaces and embedding techniques. In particular, embedding techniques using level set representations have received recent attention due to their simplicity. Level set based methods have several limitations, however. These include the inability to naturally treat open surfaces or objects of codimension two or higher. Level set methods also typically lead to a degradation in the order of accuracy when solved on a banded grid.

This talk describes an approach based on the closest point representation of the surface which eliminates these and other limitations. A noteworthy feature of the method is that it is remarkably simple, requiring only minimal changes to the corresponding three-dimensional codes to treat the evolution of partial differential equations on surfaces.

2.4 On Wavelet-Based Numerical Homogenization

Presenter: Alina Chertock

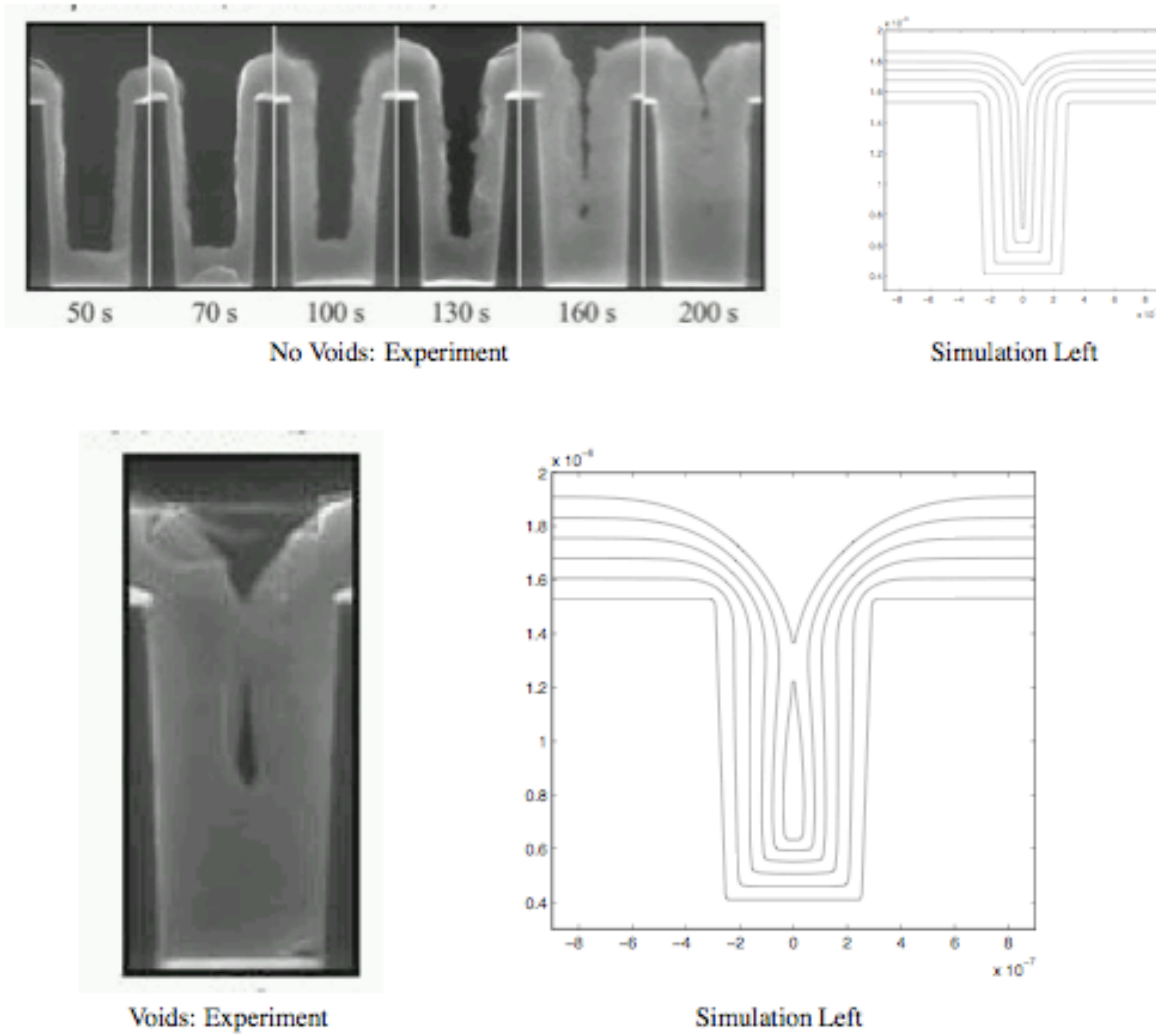

Figure 2: Superconformal electrodeposition in damascene copper deposition, coupling PDE-based material transport solvers to level set methods.

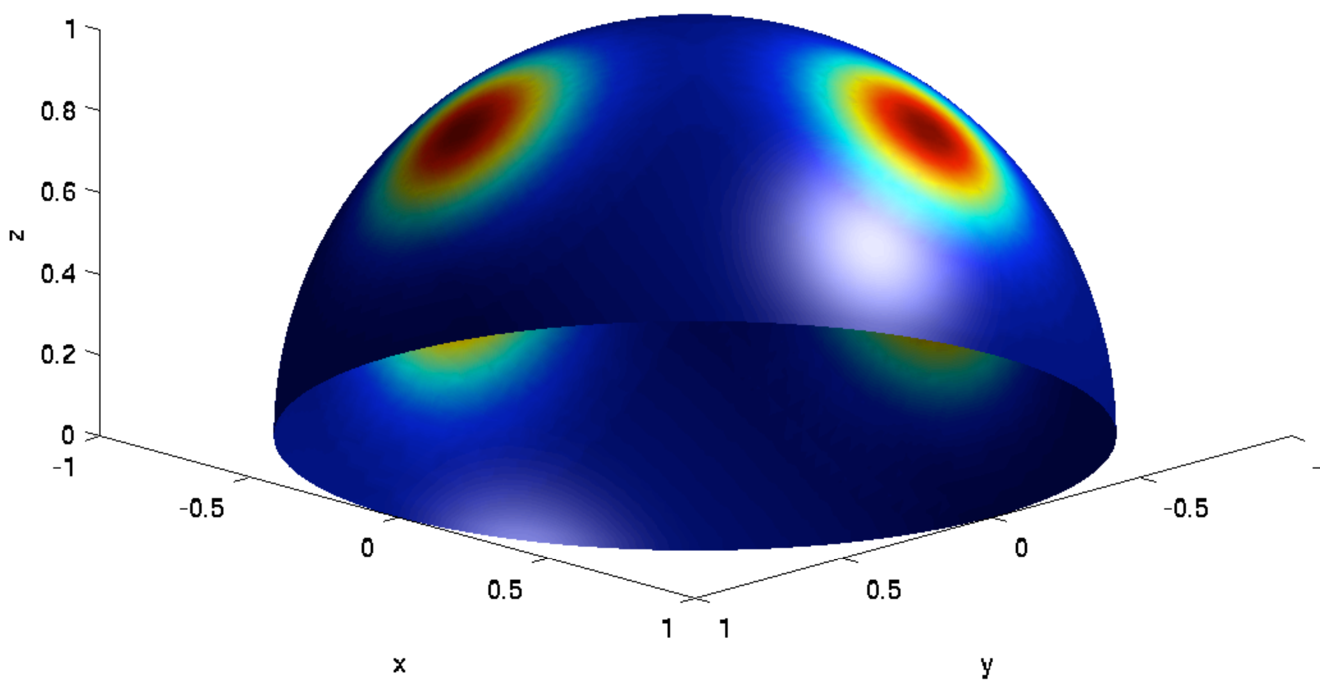

Figure 3: Steady state pattern from a Turing model on the surface of a hemisphere.

Recently, a wavelet-based method was introduced for the systematic derivation of subgrid scale models in the numerical solution of partial differential equations. Starting from a discretization of the multiscale differential operator, the discrete operator is represented in a wavelets space and projected onto a coarser subspace. The coarse (homogenized) operator is then replaced by a sparse approximation to increase the efficiency of the resulting algorithm.

In this work we show how to improve the efficiency of this numerical homogenization method by choosing a different compact representation of the homogenized operator. In two dimensions our approach for obtaining a sparse representation is significantly simpler than the alternative sparse representations. L^∞ error estimates are derived for a sample elliptic problem. An additional improvement we propose is a natural fine-scales correction that can be implemented in the final homogenization step. This modification of the scheme improves the resolution of the approximation without any significant increase in the computational cost. We apply our method to a variety of test problems including one- and two-dimensional elliptic models as well as wave propagation problems in materials with subgrid inhomogeneities.

2.5 Transmission traveltime tomography by Hamilton-Jacobi equations

Speaker: Jianliang Qian, Department of Mathematics and Statistics, Wichita State University

Traditional transmission travel-time tomography hinges on ray tracing techniques. We propose a PDE-based Eulerian approach to travel-time tomography so that we can avoid using the cumbersome ray-tracing technique. We start from the eikonal equation, define a mismatching functional and derive the gradient of the nonlinear functional by an adjoint state method. The resulting forward and adjoint problems can be efficiently solved by using the fast sweeping method; a limited memory BFGS method is used to drive the mismatching functional to zero with quadratic convergence. 2-D and 3-D numerical results as well as Marmousi synthetic velocity model demonstrate the robustness and the accuracy of the method.

2.6 Fast Sweeping Method For Convex Hamilton-Jacobi Equations

Speaker: Hongkai Zho

We will discuss about an efficient iterative method, the fast sweeping method, for convex Hamilton-Jacobi equations. In paticular, both ordering strategies, which can capture all directions of characteristics in a parallel way, and a fixed stencil local solver, which satisfies consistency and causality condition, are designed for unstructured meshes. The key mechanism behind the success of fast sweeping method for hyperbolic problems is due to the causality of the underlying PDEs. This is different from the convergence mechanism for elliptic problems. Extension to high order schemes and other hyperbolic problems and some open issues will also be discussed.

Attached is an example that shows the number of iterations for our fast sweeping method is indpendent of the anisotropy.

2.7 An adaptive spectral/DG method for a phase-space based level set approach to geometrical optics on curved element

Bernardo Cockburn, Chiu-Yen Kao, and Fernando Reitich

In this talk we introduce a numerical procedure for simulations in geometrical optics that, based on the recent development of Eulerian phase-space formulations of the model, can deliver very accurate solutions which can be made to converge with arbitrarily high orders in general geometrical configurations, including curved boundaries. Following previous treatments, the scheme is based on the evolution of a wavefront in phase-space, defined as the intersection of two level sets satisfying the relevant Liouville equation. In order to incorporate the full regularity of solutions that results from the unfolding of singularities, our method is based on their spectral representation and then a discontinuous Galerkin finite element method for the solution of the resulting system of equations. The procedure is complemented with the use of a recently derived SSP-RK scheme for the time integration that, as we demonstrate, allows for overall approximations that are rapidly convergent. Since we are only interested in the intersection of two level sets, we use p-adaptivity (higher order on the elements which are near the wavefront) to accelerate the speed. In Figure 1, we show one result for a point source wavefront propagate with constant speed in a parabola domain. Since the point source is on

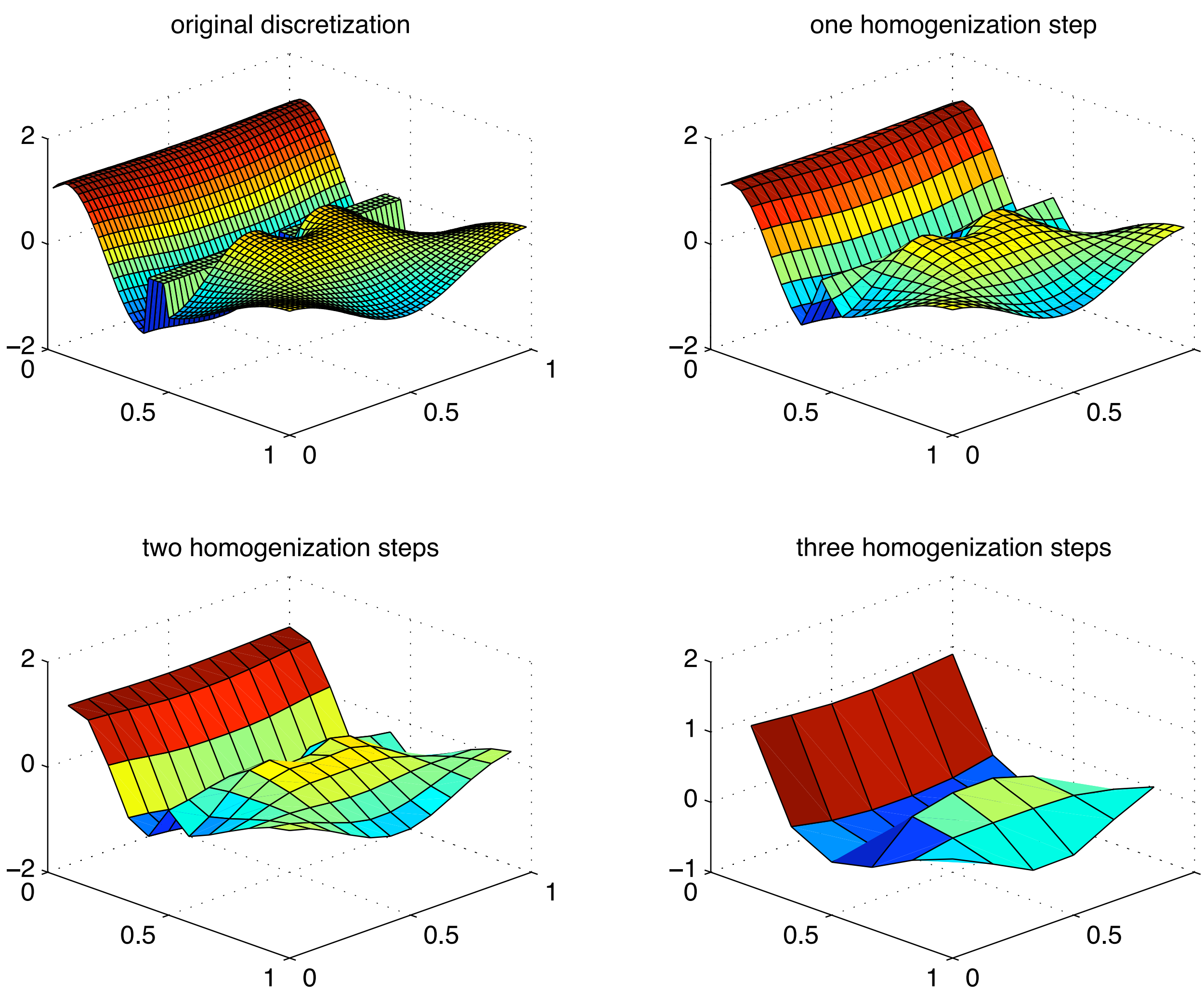

Figure 4: Solution of the 2-D Helmholtz equation after 1,2, and 3 homgenization steps

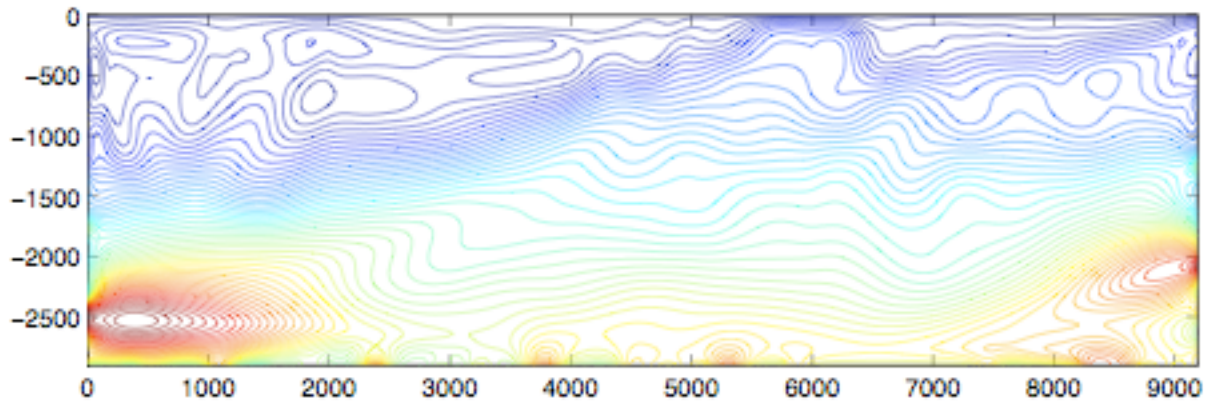

Figure 5: Tomography results by fast sweeping methods for the Marmousi model.

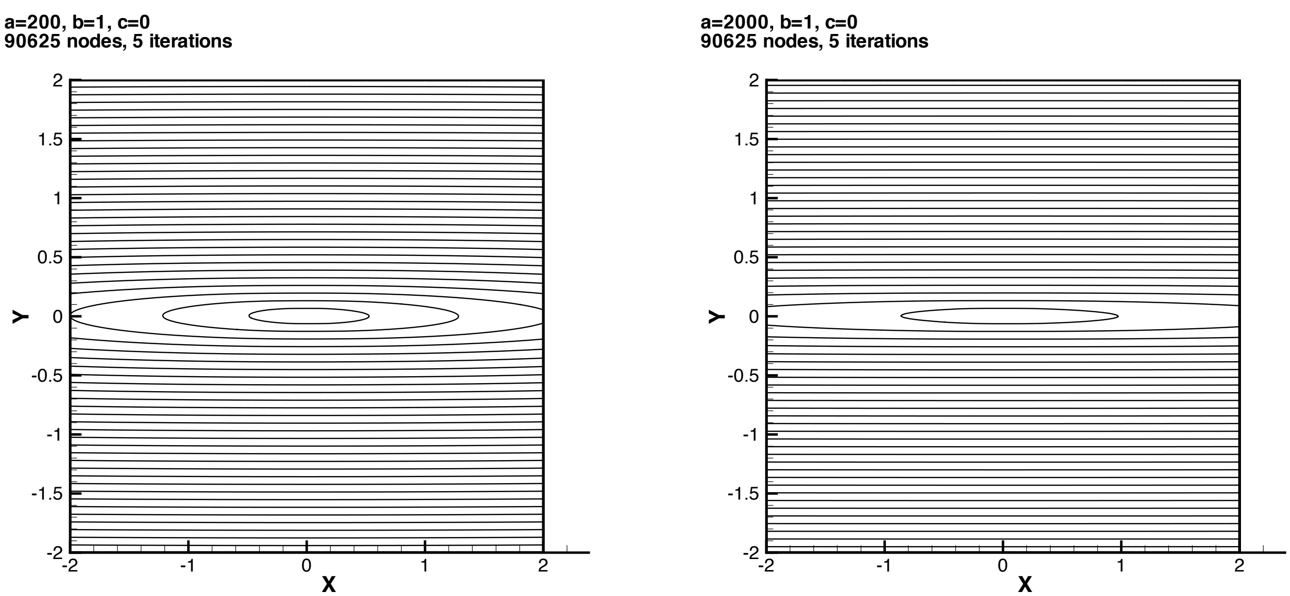

Figure 6: Refined mesh. (a): $a = 200, b = 1, c = 0$, $\eta = \sqrt{200}$; convergence after 5 sweepings; (b): $a = 2000, b = 1, c = 0$, $\eta = \sqrt{2000}$; convergence after 5 sweepings.

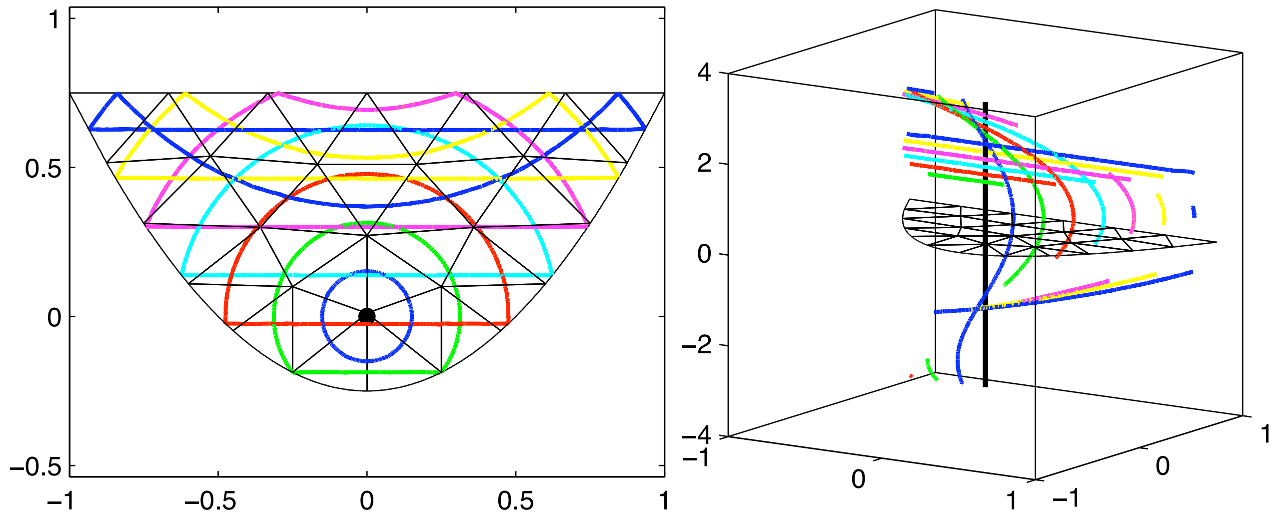

Figure 7: The wave fronts generated by an initial point source at $(0, 0)$. We plot the wavefront for time interval $\Delta t = 0.1625$. There are 36-element spatial mesh in the background. (b) The corresponding wave fronts in phase space at different time.

the focus of the parabola, the reflected wave fronts at different time become straight lines. For more results with inhomogeneous speed functions and irregular domains, please refer to our upcoming paper.

2.8 Recent advances on large time-step schemes for degenerate second-order equations and Mean Curvature Motion

Speaker: Roberto Ferretti

Some recent developments of semi-Lagrangian schemes for level-set equations of second order are presented. We start discussing a suitable technique to treat degenerate second-order operators in large time-step numerical schemes, then apply this approach to the equation describing codimension-1 Mean Curvature Motion in level set models. Several extensions of this scheme are presented: a codimension-2 scheme based on the Ambrosio-Soner formulation of MCM, a time-adaptive scheme, and a stationary version which applies to the (degenerate elliptic) MCM equation for convex curves. We discuss consistency and monotonicity of the various schemes and present a set of numerical tests based on classical benchmarks for such model equation.

In the workshop, this approach has been compared to the current state-of-art in the topic, focusing in particular on the monotonic treatment of the second-order operator, on the non-oscillatory implementation of the scheme and on the techniques for deriving fast (possibly, single-pass) algorithms.

2.9 A fast sweeping method based on discontinuous Galerkin methods for Eikonal equations

Speaker:Fengyan Li

Authors: Fengyan Li, Rensselaer Polytechnic Institute
Chi-Wang Shu, Brown University
Yong-Tao Zhang, University of Notre Dame
Hong-Kai Zhao, University of California, Irvine

We will report some recent work on combining the discontinuous Galerkin methods, a properly chosen numerical Hamiltonian, and Gauss-Seidel iterations with alternating-direction sweepings to design a high order fast sweeping method for Eikonal equations. The main new feature of the method is to explore the advantages of the local compactness of the discontinuous Galerkin methods which fits well in the fast sweeping framework. Numerical results are presented to demonstrate the performance of the method.

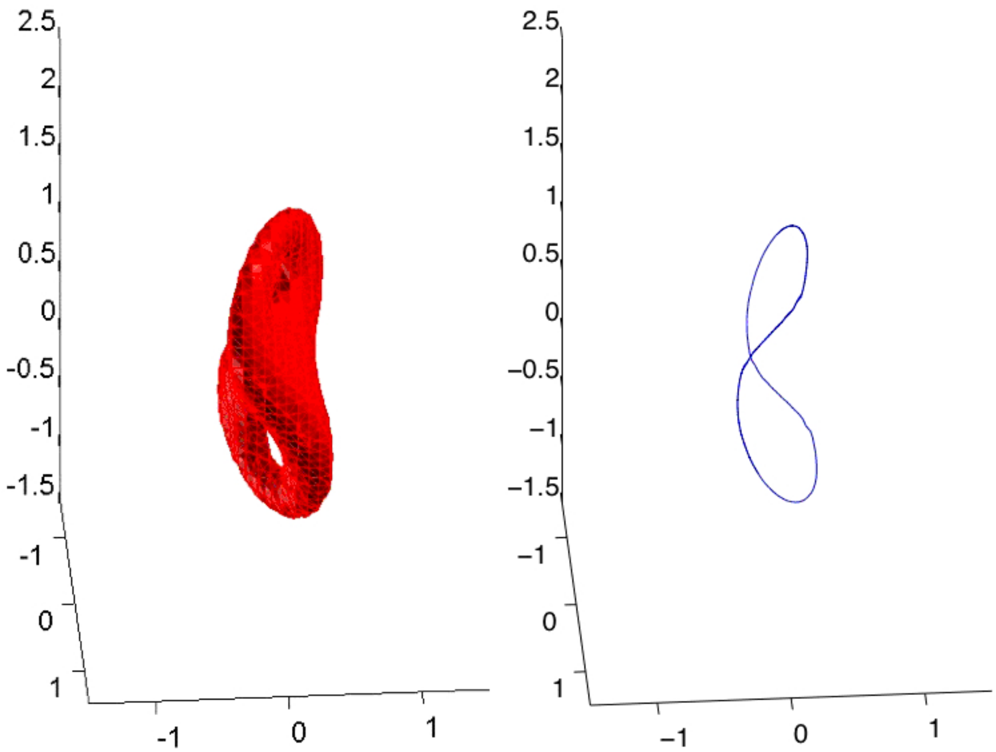

Figure 8: The Ambrosio-Soner type, generalized Mean Curvature evolution of two linked circles in three dimensions, after the topology change.

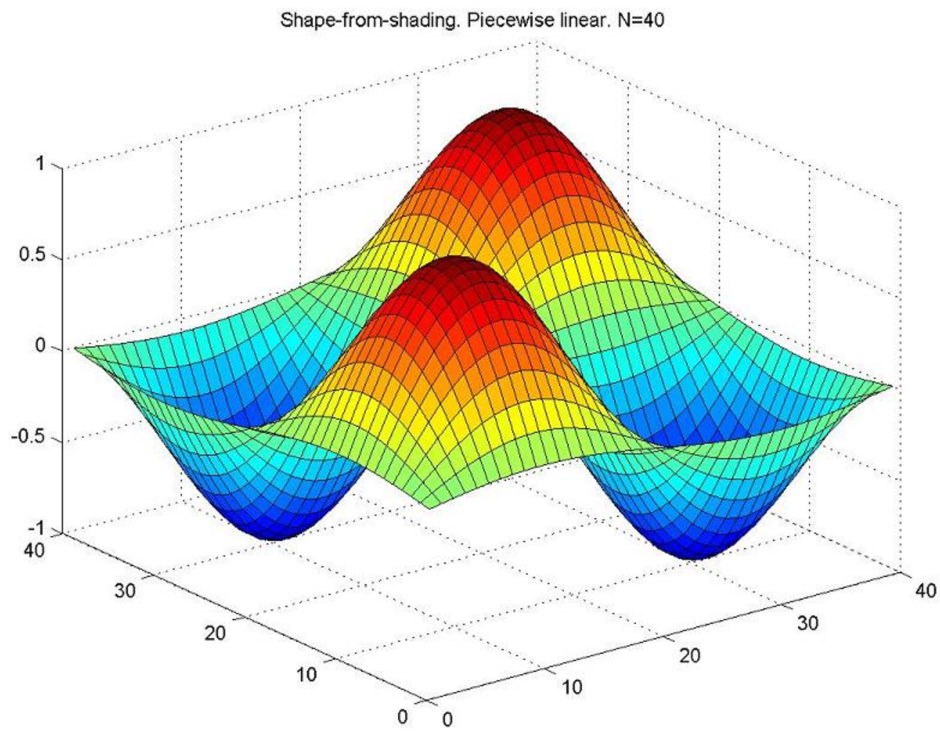

Figure 9: Shape-from-shading, piecewise linear. N=40.