Analytic and Geometric Theories of Holomorphic and CR Mappings

John Bland (University of Toronto),
Herve Gaussier (Université de Provence),
Kang-Tae Kim (Pohang University of Science and Technology),
Steven G. Krantz (Washington University),
Finnur Larusson (University of Western Ontario),
Junjiro Noguchi (University of Tokyo)

April 29, 2006–May 4, 2006

1 Overview of the Field

Several complex variables is a modern and dynamic part of mathematics. Spawned in the early twentieth century by William Fogg Osgood (United States), Henri Poincaré (France), Fritz Hartogs (Germany), and others, the subject has always had an international flavor. Certainly our workshop reflected that flavor, as we had participants from Korea, Japan, Australia, Russia, Iceland, the U.S.A., France, and India. All of the participants already knew each other—at least professionally—but it was a special pleasure to meet face-to-face and to exchange ideas spontaneously and in real time. Many of the participants are young mathematicians on the cutting edge of current research. It is especially important for a broad cross-section of workers in the field to be able to interact with these important researchers. That is what our workshop achieved.

The two seminal results proved at the inception of this subject are these:

Theorem (Poincaré): *Let*

$$B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

be the unit ball in $\mathbb{C}^2$ *and let*

$$D^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$$

be the bidisc. Then there is no biholomorphic mapping

$$\Phi : B \to D^2 .$$

This theorem tells us that any obvious guess at a Riemann mapping theorem in several complex variables will fail—just because the two most obvious candidates for the “default” domain are not equivalent.

Theorem (Hartogs): *Let*

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 2\} \setminus \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \le 1, |z_2| \le 1\} .$$

Let f *be a holomorphic function on* Ω*. Then there exists a holomorphic function* F *on*

$$D^2 = \{(z_1, z_2) : |z_1| < 2, |z_2| < 2\}$$

such that $F\big|_\Omega = f$.

The result of Hartogs says that Ω is *not* the natural domain of definition of any holomorphic function; for any holomorphic function on Ω analytically continues to a strictly larger domain. This theorem should be contrasted with the situation in $\mathbb{C}^1$: For *any* domain $\Omega \subseteq \mathbb{C}^1$, there is a holomorphic function f on Ω that cannot be analytically continued to a larger domain. A domain in any $\mathbb{C}^n$ which is the natural domain of definition of some holomorphic function is called a *domain of holomorphy*.

Thus the focus of several complex variables for the past century has been on holomorphic mappings, and on domains of holomorphy. A vast array of techniques has been developed for coming to grips with these two circles of ideas. Today several complex variables interacts profitably with harmonic analysis, partial differential equations, differential geometry, commutative algebra, algebraic geometry, one complex variable, real variable theory, and even formal logic.

Among the methodologies that have been developed to tackle the questions that have been described are

(i) Sheaf theory (J. Leray and K. Oka);

(ii) Partial differential equations, notably the $\overline{\partial}$ problem (J. J. Kohn [13] and L. Hörmander [12]);

(iii) Methods of differentiable geometry, notably Kähler manifold theory and Stein manifold theory (S. S. Chern, H. Grauert, R. Narasimhan, H. Remmert).

These powerful pieces of mathematical machinery have proved to be useful in a variety of mathematical contexts. They continue to be developed today. Furthermore, more recondite techniques such as the Monge-Ampère equation, dynamical systems, Banach algebras, function algebras, operator techniques, and many other widespread ideas continue to develop alongside the principal streams of thought. All these branches of the subject interact profitably, and produce a colorful and productive melange of mathematical work.

2 Recent Developments and Open Problems

In the past twenty years, the subject of several complex variables has blossomed in a variety of new directions. Among these are

- **The theory of dynamical systems in several complex variables.** Of course dynamical systems found their genesis in the work of H. Poincaré on questions of celestial mechanics. But in fact it was quite early in the twentieth century that Fatou and Julia saw the relevance of these new ideas to complex function theory (in *one complex variable*). In the hands of Hubbard, Douady, Mandelbrot, and many others, one-variable dynamical systems has blossomed into a vigorous part of modern mathematics. The growth of dynamical systems in the several-variable setting is considerably newer.

  And the questions are quite a lot harder. Deep ideas from pluripotential theory, differential geometry, partial differential equations, and many other parts of mathematics must be brought to bear in order to achieve any progress at all. Major workers in the area include Bedford (present at the workshop), Buzzard, Fornaess, Sibony, Lyubich, and Douady.

- **The theory of automorphism groups of domains.** It has already been noted that, because of Poincaré's theorem, we can expect no version of the Riemann mapping theorem in several complex variables. In fact Poincaré's stunning result has been enhanced and refined in the ensuing years. Work of Chern-Moser [7], Burns-Shnider-Wells [6], and Greene-Krantz (organizer of this conference) [10], [11] has shown that the phenomenon that Poincaré discovered is in fact generic in a variety of senses.

  But Poincaré himself gave us some guidance as to how to come to grips with the phenomenon. He proved his result by analyzing the automorphism groups of the two domains D^2 and B. Here, if $\Omega \subseteq \mathbb{C}^n$ is a domain, then the automorphism group $\mathrm{Aut}(\Omega)$ is the collection of biholomorphic self-maps of Ω. This set forms a group under the binary operation of composition of mappings. And in fact, if we equip the automorphism group with the topology of uniform convergence on compact sets (equivalently, the compact-open topology), then (at least for a bounded domain Ω) the automorphism groups turns out to be a real Lie group. Thus powerful bodies of machinery may be brought to bear on the study of automorphism groups.

For a given domain Ω, one may study algebraic, topological, and analytic properties of $\mathrm{Aut}(\Omega)$ and determine how they reflect the complex geometry of Ω (and vice versa). This study, which has been particularly vigorous in the past twenty-five years, has become an *ersatz* for the Riemann mapping theorem: it gives us a device for differentiating and comparing domains. Major workers in the field include Bedford (present at the workshop), Kim (present at the workshop, and an organizer), Krantz (present at the workshop, and an organizer), Pinchuk, Greene, Berteloot (present at the workshop), and Kodama (present at the workshop).

- *CR* **geometry, analysis of hypersurfaces, and normal forms.** Poincaré advocated a program of studying the equivalence and inequivalence of domains in $\mathbb{C}^n$ by **(i)** first proving that any biholomorphic mapping of domains will extend smoothly to the boundaries and **(ii)** then constructing differential invariants on the boundaries. It turned out that the mathematical techniques were not yet available to carry out step **(i)**, and it awaited Charles Fefferman to (in 1974 [8]) prove that a biholomorphic mapping of strongly pseudoconvex domains continues smoothly to the boundaries. His work was quickly followed by work of Chern and Moser on constructing the anticipated differential boundary invariants. Fefferman later used these invariants to develop a classification theory for strongly pseudoconvex domains [9]. Bell and Ligocka [], [] were able to extend and simplify Fefferman's theorem, and to turn it into a powerful and versatile tool for our subject. Today the study of biholomorphic mappings, and the cognate idea known as Condition R, is a vital part of several complex variables.

  Work continues on developing these ideas for broader classes of domains. The methods involve commutative algebra, differential geometry, and partial differential equations. Principal workers in the field today include Gong, Hayashimoto (present at the workshop), Isaev (present at the workshop), Ezhov, and Schmalz (present at the workshop).

- **Holomorphic functions and mappings in infinite dimensions.** The idea of studying holomorphic functions and mappings on Banach spaces is more than fifty years old. Certainly the functional calculus and other natural questions of operator theory begged for such a development. But it must be said that little substantial progress was made in the subject until relatively recently. The difficulty, it seems, is that people were trying to study holomorphic functions and mappings on *any* Banach space. What we have learned in the past ten years—thanks to work of Lempert and others—is that considerable progress can be made if we restrict attention to *particular* Banach spaces. There are now a theory of domains of holomorphic, a theory of normal families, a theory of the $\overline{\partial}$ problem, and many other important cornerstones of function theory in the infinite-dimensional setting. Among the experts in this subject area are Cima, Graham (present at the workshop), Kim (present at the workshop), Krantz (present at the workshop), and Lempert.

- *CR* **functions and mappings** In the 1960s, Kohn and Rossi introduced the concept of a *CR* function. This is a function on a real hypersurface in $\mathbb{C}^n$ that is in effect the "trace" of a holomorphic function. One desires a definition of such an object that is intrinsic, and makes no reference to holomorphic functions; this is achieved by way of partial differential equations. Thus are defined the *CR* functions. Also *CR* mappings are defined similarly. In the intervening forty years, CR functions and mappings have assumed a prominent role in the subject. They are useful for studying the $\overline{\partial}$ and $\overline{\partial}_b$ problems, for studying holomorphic mappings, and for developing function theory. Certainly this subject area was one of the focuses of our conference. Several of the talks were about CR functions and mappings, and they generated vigorous discussions and collaborations.

3 Presentation Highlights

There were a total of twenty presentations made at our workshop. Each one was 45 minutes in duration, followed by a lively discussion period. A sketch of the presentations follows:

Eric Bedford, Indiana University Currents in Complex Dynamics

Abstract: This will concern some results where the study of complex dynamics leads to interesting currents.

Rational mappings are of the form

$$f = \left[\frac{p_0}{q_0} : \ldots : \frac{p_k}{q_k}\right] : \mathbb{P}^k - - \to \mathbb{P}^k$$

where $p_0, \ldots, p_k; q_0, \ldots, q_k$ are polynomials. Such a map is defined in a Zariski open subset of $\mathbb{P}^k$. Birational maps are the rational maps such as f above that admit another rational mapping $g : \mathbb{P}^k - - \to \mathbb{P}^k$ such that $f \circ g = id$ and $g \circ f = id$ except for Zariski closed sets. This work studies the dynamics of the iterates $f^m = f \circ \ldots \circ f$ of a birational map f.

The first important invariant to study is the degree of f, which usually explains the complexity of the dynamics. While the topological degree for the birational maps is generically 1, one should look for the validity of the definition for the dynamic degree

$$\delta(f) = \lim_{n\to\infty} (\deg(f^n))^{1/n}$$

which is not in general well-defined here.

It turns out that this can be made sense if the birational map f has a finite period. In fact, by resolving the singularities by birational blow-up, one can turn this situation into a holomorphic dynamics.

$$\begin{array}{ccc} Y & \xrightarrow{\widetilde{f}} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow[f]{} & X \end{array}$$

$$\begin{array}{ccc} V_1 \xrightarrow{\widetilde{f}} E_1 \xrightarrow{\widetilde{f}} E_2 \xrightarrow{\widetilde{f}} V_2 & \subset & Y \\ & & \downarrow \pi \\ V_1 \xrightarrow{f} \cdot \xrightarrow{f} \cdot \xrightarrow{f} V_2 & \subset & X \end{array}$$

This eventually leads to a dynamics in the Picard group $H^{1,1}(\mathbb{P}^k)$ by the linear mapping $f^* : H^{1,1}(\mathbb{P}^k) \to H^{1,1}(\mathbb{P}^k)$, the set of hyperplanes in $\mathbb{P}^k$. It has turned out that, in such a case

$$\delta(f) = \text{Spectral radius of } f^*.$$

This work is related to some researches in Theoretical Physics.

François Berteloot, Université Paul Sabatier Are proper holomorphic self-maps of smoothly bounded domains automorphisms?

Abstract: The title question has been a long standing question in Complex Analysis, which produced several prominent results in the past.

Besides positive answers (strictly pseudoconvex domains, pseudoconvex domains with real analytic boundaries, Reinhardt or circular domains...) the speaker has described a class of circular domains in $\mathbb{C}^2$, whose boundaries are spherical outside a finite number of circles and which do admit branching proper holomorphic self-maps. He has discussed a new approach which reduces the problem to some special cases and, for complete circular domains, leads to a positive answer.

This approach relates two facts. One is a quantified dilation property for *CR* mappings on strictly pseudoconvex hypersurfaces. The other is a contrast between the dynamics of the map (its topological entropy is zero) and the dynamics of the induced map on the boundary (its topological entropy is greater than the log of its degree). This new viewpoint seem rather promising toward a better understanding of the proper mappings in general.

Alexander Brudnyi, University of Calgary L_2-holomorphic functions on coverings of strongly pseudoconvex manifolds

Abstract: I will talk about L_2-holomorphic functions on coverings of strongly pseudoconvex manifolds and show how to solve some problems posed by Gromov, Henkin and Shubin (including some Hartogs type theorem for *CR*-functions). Techniques will include partial differential equations, especially generalizations of the important technique of Hörmander.

Michael Eastwood, University of Adelaide Complex methods in real integral geometry

Abstract: Here, real integral geometry means the Radon transform, the X-ray transform, and variants thereof. I shall indicate how some standard machinery of complex analysis (cohomology, direct images, and so on) can be used to figure out the range and kernel of the real transforms. This is joint work with Toby Bailey and Robin Graham. The work has far-reaching applications to the study of complex differential invariants, with applications to biholomorphic mappings.

Peter Ebenfelt, University of California at San Diego Rigidity and complexity of *CR* structures and their mappings

Abstract: Among strictly pseudoconvex *CR* manifolds, the simplest one is (arguably?) the sphere. *CR* mappings of the sphere into itself are well understood. On the other hand, mappings of a sphere into a higher dimensional sphere can be very wild, but, in some sense, the complexity of such mappings are controlled by the codimension of the mapping (i.e. the difference between the *CR* dimensions of the spheres). For instance, if the codimension is strictly less than the *CR* dimension of the source sphere, then the mapping is linear (up to automorphisms of the target sphere). If M is a strictly pseudoconvex manifold whose "complexity" is suitably close to that of the sphere, then analogous rigidity properties hold for sphere embeddings of suitably low codimension. In this talk, I will make this more precise and discuss some recent results along these lines.

Buma Fridman, Wichita State University Discrete fixed point sets for holomorphic maps

Abstract: The talk will examine the cardinality and configuration of isolated fixed point sets of holomorphic self-maps of complex manifolds. As a consequence of one of our observations (time permitting) a construction of a bounded domain in $\mathbb{C}^n$ with no non-trivial holomorphic retractions will be presented, and some open questions posed. This is joint work with Daowei Ma, and relates to other joint work with Daowei Ma, Kang-Tae Kim, and Steven G. Krantz.

Ian Graham, University of Toronto The Cartan-Caratheodory-Kaup-Wu theorem in an infinite-dimensional Hilbert space.

Abstract: By using a notion of normality appropriate to the study of geometric problems in infinite-dimensional but separable spaces, as well as a triangularizability assumption, we show how the theorem in the title can be generalized to Hilbert spaces.

Theorem: Let Ω be a bounded convex domain in a separable complex Hilbert space. If a holomorphic mapping $f : \Omega \to \Omega$ satisfies the following three conditions:

(i) $f(p) = p$ for some $p \in \Omega$,

(ii) $f'(p)$ is triangularizable, and

(iii) the spectrum $\sigma(f'(p))$ is contained in the unit circle,

then f is a biholomorphism.

In addition to the fact that this is a generalization of a well-known pivotal theorem in several complex variables to an infinite dimensional space, the methods used in its proof demonstrate a new aspect toward the analysis of dynamics/iterations of holomorphic mappings in the infinite dimensions, which is a territory that needs to be explored in the future.

C.-K. Han, Seoul National University Symmetry algebra for even number of vector fields

Abstract: In the 2004 September Conference on *CR* geometry, Levico, A. Koranyi proposed the problem of determining the dimension of infinitesimal automorphism of the multi-contact structure given

by two independent vector fields in $\mathbb{R}^3$ whose bracket is transversal. This presentation features a complete answer to this question for even number $(2n)$ of vector fields and discuss the existence for the case $n = 1$. This talk is a classic example of PDE presentation of Cartan's prolongation method that has been made explicit.

Adam Harris, University of New England Asymptotic behaviour of J-holomorphic curves near a Reeb orbit of elliptic type

Abstract: The idea of introducing pseudoholomorphic maps into a contact manifold cross the real line was originally used by Hofer as a tool for addressing the Weinstein conjecture, concerning the existence of periodic orbits of the Reeb flow. A more detailed study of the asymptotics of these mappings was undertaken in the late 90s by Hofer, Wysocki and Zehnder. While the existence of these mappings is guaranteed under quite mild conditions, relatively little is known explicitly about them (eg., how to write them down in specific cases). I will discuss some recent joint work with Wysocki in this direction, giving conditions under which they may be represented as a straightforward generalisation of the parametrization of plane algebroid curves.

Atsushi Hayashimoto, Nagano National College of Technology Normal forms for a class of finitely nondegenerate hypersurfaces in $\mathbb{C}^4$

Abstract: Consider real analytic hypersurfaces in $\mathbb{C}^4$ through the origin. Assume that they are finitely nondegenerate and the diagonal components of the Levi forms are -1, 1, 0 at the origin. For such a class of hypersurfaces, we construct normal forms of their defining functions. The normal form is an outgrowth of the founding ideas of Poincaré about biholomorphic mappings and the classification of domains up to biholomorphic equivalence. The method of constraction is an analogy of that of P. Ebenfelt papers which appeared in *Indiana Univ.Math. J.* (1998) and *J. Diff. Geom.* (2001).

Alexander Isaev, Australian National University Proper Holomorphic Maps between Reinhardt Domains

Abstract: Most proper holomorphic maps between bounded Reinhardt domains are elementary algebraic. In dimension 2, we identify all pairs of domain for which there exists a map which is not elementary algebraic, and obtain a complete description of all proper holomorphic maps. The work is joint with N. Kruzhilin.

Akio Kodama, Kanazawa University A characterization of complex manifolds admitting effective actions of the direct product of unitary groups by biholomorphic automorphisms

Abstract: The presentation was designed to lead the audience to some "analytic" natures of complex manifolds M under some "topological" plus "Steinness" conditions on M. The speaker took the problem of characterizing Stein manifold M whose holomorphic automorphism group is topologically isomorphic to that of the product of k-dimensional ball and the ℓ-dimensional complex Euclidean space $\mathbb{C}^k$.

The upshot of such an example is that it admits a unitary action. The speaker made a very clever use of centralizers and normalizers of the torus actions in the automorphism group (techniques developed by S. Shimizu in his study of tube domains) and showed in the end that indeed M is biholomorphic to the product domain specified above.

Loredana Lanzani, University of Arkansas A Real Analysis approach to the d-bar problem

Abstract: In the first part of this talk the speaker gave an overview of joint work with E. M. Stein concerning L^r-estimates of the Hodge system for forms in $\mathbb{R}^N$. In the second part she discussed the following question: can these results be used to obtain new estimates of the d-bar problem for (p, q)-forms in $\mathbb{C}^n$?

The motivation came from the following theorem

Theorem (Bourgain-Brezis 2004, Van Shaftingen 2004/05) If $f, g \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ are vector fields with compact support satisfying the equation

$$\begin{cases} Curl\ Z = f \\ Div\ Z = 0 \end{cases}$$

then it holds that

$$\|Z\|^{L^{n/(n-1)}}(\mathbb{R}^n) \le C\|f\|_{L^1(\mathbb{R}^n)}.$$

In comparison of the earlier theorem by Gagliado-Nirenberg, the author, in a collaboration with E.M. Stein, has put this theorem in a perspective, by reformulating the problem in the context of the Hodge-de Rham complex:

$$0 \to \Lambda_0 \to \Lambda_1 \to \cdots \to \Lambda_n \to 0$$

where Λ_p is the L^2 completion of the set of all compactly supported smooth L^2 forms of degree p. Then the above PDE can be translated to

$$\begin{cases} d_\ell Z = f \\ d_\ell^* Z = 0 \end{cases}$$

for instance (the above theorem concerns the case $\ell = 0$.

More generally, the speaker considers

$$\begin{cases} d_\ell Z = f \\ d_\ell^* Z = g \end{cases}$$

and then obtains, in a collaboration with Stein, the following theorem:

Theorem: For $n \ge 2$, for every ℓ with $2 \le \ell \le n-2$, the solutions for the preceding equation satisfy the estimate

$$\|Z\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|f\|_{L^1} + \|g\|_{L^1}\right).$$

Moreover,

$$\begin{aligned} \ell = 0: &\qquad \|Z\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\|f\|_{L^1} \\ \ell = n: &\qquad \|Z\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\|g\|_{L^1} \\ \ell = 1: &\qquad \|Z\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|f\|_{L^1} + \|g\|_{H^1}\right) \\ \ell + n - 1: &\qquad \|Z\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\left(\|f\|_{H^1} + \|g\|_{L^1}\right), \end{aligned}$$

where

$$\|g\|^{H^1} = \|(Pg)^*\|_{L^1(\mathbb{R}^n)}.$$

Here $(Pg)^*$ denotes the non-tangential maximal function of the harmonic extension of g to the upper half space of $\mathbb{R}^{n+1}$.

The speaker has also mentioned that a similar result should hold for the $\bar{\partial}$ complex (Cauchy-Riemann complex), and expects to obtains definite results soon.

Steven Lu, Université de Quebec á Montreal Brody curves in logarithmic varieties with surjective log-albanese map.

Abstract: We give conditions for the algebraic degeneracies of Brody curves in logarithmic varieties with surjective logarithmic Albanese map and hence conditions for such varieties to be hyperbolic. These questions relate to fundamental questions of complex algebraic geometry.

Alip Muhammed, York University On the Riemann-Hilbert-Poincaré Problem for the inhomogeneous Cauchy-Riemann equation on $\mathbb{C}$

Abstract: The inhomogeneous Riemann-Hilbert-Poincaré problem with general coefficient for the inhomogeneous Cauchy-Riemann equation on the unit disc is studied using Fourier analysis. It is shown that this problem is well posed only when the coefficient is holomorphic. In the other cases poles or essential singularities have to be dealt with and hence only the Robin boundary condition is well posed for the inhomogeneous Cauchy-Riemann equation.

Stefan Nemirovski Domains and their coverings

Abstract: This talk discussed the relationship between the global geometry of the universal covering of a bounded domain and the local geometry of its boundary. The typical theorems are:

Theorem 1. Let D, D' be Stein, bounded strongly pseudoconvex domains with a real analytic (C^ω) boundary in a complex manifold. Let Y, Y' be their universal coverings. Then, Y is biholomorphic to Y', if and only if ∂D is locally biholomorphic to $\partial D'$ at some points $p \in \partial D$ and $p' \in \partial D'$.

There has been several impressive results by Poincaré, Alexander, Chern-Ji, Burns and others about the domains with spherical boundaries. In this regard the following theorem has also been presented.

Theorem 2. Let D be a Stein, bounded strongly pseudoconvex C^2-smooth boundary. Then, $D \simeq B/\Gamma$ (a quotient of the ball) if and only if ∂D is spherical (everywhere locally CR diffeomorphic to the standard sphere).

The speaker also points out that this puts the following two conjectures on perspective:

Conjecture 1 (Ramadanov Conjecture): If the Fefferman's asympototic expansion formula of the Bergman kernel for a strongly pseudoconvex domain

$$B_D(z) = \frac{\varphi(z)}{(\rho(z))^{n+1}} + \psi(z) \log \rho(z)$$

has the coefficient $\psi(z)$ of the logarithmic term vanishing to the infinite order, then the boundary of the domain is spherical.

Conjecture 2 (Cheng's Conjecture): For a strongly pseudoconvex domain, if its Bergman metric is also Kähler-Eistein, then the domain is biholomorphic to the ball.

The speaker demonstrated that in complex dimension 2 the first conjecture implies the second.

Gerd Schmalz, University of New England Cartan connection for Engel *CR*-manifold

Abstract: Engel *CR* manifolds are 4-dimensional manifolds with a 1-dimensional *CR* distribution. Their structure algebra is not semi-simple, therefore the standard methods do not work. The speaker, in a joint work with Beloshapka and Ezhov, presented a simple explicit construction which demonstrates the basic ideas of Cartan connections. Of particular interest, the speaker mentioned the result that they were able to distinguish the most essential 4 components among 30, in the curvature of the Cartan connection for Engel 4-manifold that determines the obstruction entirely.

Rasul Shafikov, University of Western Ontario Extension of holomorphic maps between real hypersurfaces of different dimensions

Abstract: It is shown that a germ of a holomorphic map from a real analytic hypersurface M in $\mathbb{C}^n$ into a strictly pseudoconvex compact real algebraic hypersurface M' in $\mathbb{C}^N, 1 < n < N$ extends holomorphically along any path on M. This result has important applications to analytic continuation of, and uniqueness of holomorphic mappings. It intersects with a number of the other talks presented at this workshop.

Berit Stensones, University of Michigan Plurisubharmonic Polynomials

Abstract: We shall study some properties of plurisubharmonic polynomials in two variables. The questions we address is motivated by the so called peak point problem on pseudoconvex domains with real analytic boundaries. A first step towards constructing peak functions is to show that these domains can be bumped to a desired order. We shall show some positive results in that directions. Peak points have important applications in functions algebras, in the study of holomorphic mappings, and particularly in the study of invariant metrics. They are a technical but incisive part of the function theory of several complex variables.

Kaushal Verma, Indian Institute of Science Smooth isometries of invariant metrics

Abstract: It is known that biholomorphisms are isometries of invariant metrics such as the Kobayashi and Caratheodory metrics. One can ask the converse question: are all smooth isometries of these

metrics biholomorphic (or anti-biholomorphic)? This talk will report on some work in this direction along with some applications. Our results give a new way to formulate the fundamental theorem of Bun Wong and Rosay. This result in turn has been foundational for the theory of automorphism groups and our understanding of the geometric analysis of domains in complex space.

4 Scientific Progress Made

One of the delights of a scientific meeting such as this one is the discovery of mutual interests with mathematicians with whom one has never previously communicated. At our workshop, Kim and Krantz found mutual interests with Verma. Lina Lee found mutual interests with Berit Stensones. Buma Fridman found mutual interests with Stefan Nemirovski. Alexander Isaev found common interests with Hayashimoto. There are many other examples.

We know already that some of our discussions are leading to new results and new papers. We anticipate that communications among us will continue, and new directions in the geometric function theory of several complex variables will be charted as a result.

In fact it seems natural that more meetings will be a natural outgrowth of this one. One of us (Krantz) was already approached by several participants with the idea that we should have further and regular meetings. This type of recurrent activity is frequently the basis for major mathematical progress and also for profound effects on the infrastructure of the subject. We might hope to conduct some of our future meetings in Banff.

5 Outcome of the Meeting

The meeting was a convivial and productive one. We were fortunate that there were attendees from the Steklov Institute (Nemirovski), from Australia (Harris, Ezhov, Eastwood), from Japan (Noguchi, Hayashimoto, Kodama), and others whom we in the West do not frequently encounter. Many new ideas were exchanged, and some new collaborations initiated.

Fridman, Kim, and Krantz came away from the meeting with the start of two new papers. Krantz's graduate student Lina Lee was fortunate to be able to meet with several experts in her subject area (notably Bedford, Berteloot, and Stensones) and come away with many ideas that will be useful in her thesis. Fridman, Isaev, and Nemirovski (all Russians) had several useful discussions.

Today more than half of all published mathematics papers are collaborative. This is in stark contrast to the situation one hundred years ago—when virtually no mathematical work was collaborative. The change is a product of the development of the subject—results are now more difficult to come by. It is also a product of considerable cross-pollination among fields. Finally, it is the product of the proliferation of mathematics institutes and productive meetings like the one that we just completed at BIRS.

Of course the venue of Banff—nestled in the Canadian Rockies—is a very special one. It served as a great attraction for several of our participants to travel a great distance (some traveled for 24 hours!) so that they could participate. The staff at BIRS is a delight, the food is very good, and the accommodations are very appealing. We took an afternoon off so that we could enjoy the natural surroundings; some of us hiked up tunnel mountain and others visited Lake Louise. In all, this was a delightful event for all concerned and we cannot wait to return.

References

[1] H. Alexander, Proper holomorphic mappings in $\mathbb{C}^n$, *Indiana Univ. Math. J.* 26(1977), 137–146.

[2] E. Bedford and S. Bell, Proper self-maps of weakly pseudoconvex domains, *Math. Ann.* 261(1982), 47–49.

[3] S. Bell, Biholomorphic mappings and the $\overline{\partial}$ problem, *Ann. Math.*, 114(1981), 103-113.

[4] S. Bell and E. Ligocka, A simplification and extension of Fefferman's theorem on biholomorphic mappings, *Invent. Math.* 57(1980), 283-289.

[5] J. Bourgain and H. Brezis, New estimates for the Laplacian, the div-curl, and related Hodge systems, *C. R. Math. Acad. Sci. Paris* 338(2004), 539–543.

[6] D. Burns, S. Shnider, and R. O. Wells, On deformations of strictly pseudoconvex domains, *Invent. Math.* 46(1978), 237-253.

[7] S. S. Chern and J. Moser, Real hypersurfaces in complex manifolds, *Acta Math.* 133(1974), 219-271.

[8] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.* 26(1974), 1-65.

[9] C. Fefferman, Parabolic invariant theory in complex analysis, *Adv. Math.* 31(1979), 131-262.

[10] R. E. Greene and S. G. Krantz, Stability properties of the Bergman kernel and curvature properties of bounded domains, *Recent Progress in Several Complex Variables*, Princeton University Press, Princeton, 1982.

[11] R. E. Greene and S. G. Krantz, Deformation of complex structures, estimates for the $\overline{\partial}$ equation, and stability of the Bergman kernel, *Adv. Math.* 43(1982), 1-86.

[12] L. Hörmander, L^2 estimates and existence theorems for the $\overline{\partial}$ operator, *Acta Math.* 113(1965), 89-152.

[13] J. J. Kohn, Harmonic integrals on strongly pseudoconvex manifolds I, *Ann. Math.* 78(1963), 112-148; II, ibid. 79(1964), 450-472.