Curvature and instability of flows of ideal incompressible

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1 Overview of the Field

Serious interaction between the dynamics of an ideal incompressible fluid and the (infinite-dimensional) differential geometry began since the now classical works of V.Arnold who first regarded the flows of an ideal incompressible fluid in a bounded domain M as geodesics on the group $SDiff(M)$ of volume preserving diffeomorphisms of M. This group is an infinite-dimensional manifold with a right-invariant Riemannian metric defined by the kinetic energy. V.Arnold [1] computed the curvature of $SDiff(M)$ in case M is the 2-dimansional torus, and found that it can assume either sign, depending on direction. In this connection he asked whether the negative sign of curvature is connected with instability of steady flows. Further, D.Ebin in his important work with J. Marsden [3] has systematically built the differential geometry of $SDiff(M)$ as a part of global analysis. In further works of V.Yudovich, B.Khesin and other people [2] the curvature of $SDiff(M)$ was found via external geometry of M, which was a great simplification. G.Misiolek [6] investigated the role of positive curvature and established the existence of conjugate points on some geodesics in $SDiff(M)$. Further, D.Ebin, G.Misiolek and S.Preston [4] proved that in 2-dimensional case the exponential map on $SDiff(M)$ is Fredholm, while in 3-d case this is not true (which is again connected with positive curvature). This work included the deep analysis of functional-analytic properties of the curvature operator. On another hand, A.Shnirelman [7] considered the global geometric properties of the group $SDiff(M)$. He proved that it has finite diameter in the dimension of M is greater than 3 (while in 2-dimensional case the diameter is infinite, which was proved by Y. Eliashberg and T.Ratiu [5]). Using his results, A.Shnirelman shows that not all pairs of fluid configurations can be connected by the shortest geodesic. For the exponential map on $SDiff(M)$ in the 2-d case he proved that this map has a rigid geometric structure: it is a Fredholm Quasiruled map [8].

This domain has a good potential for growth which is ensured by the existence of open problems. Some of them are described below.

2 Recent Developments and Open Problems

The stability theory of flows of ideal incompressible fluid is a testbed of all new ideas and methods of mathematical fluid dynamics. One of this approaches which has lead already to deep results is the study of fluid flows from the viewpoint of infinite-dimensional differential geometry. Ideal incompressible fluid inside a bounded domain M is an example of an infinite-dimensional Lagrangian system. Without external forces it moves along geodesics on the group $SDiff(M)$ of volume-preserving diffeomorphisms of M, which is an infinite-dimensional Riemannian manifold. Hence, the stability of the flows should be connected with the Riemannian curvature of $SDiff(M)$, as it was pointed out by V.Arnold. However, the question is far from certain. For example, it was found by V.Yudovich that for any parallel flow in a channel the curvature in any 2-d direction containing velocity itself is negative (this result was considerably extended by A.M.Lukatsky, G. Misiolek and S. Preston). However, stability properties of such flows depend on the velocity profile and are quite different for different profiles.

The differential geometry of $SDiff(M)$ is complicated enough. The curvature can assume either signs and be zero, depending on the direction. The space $SDiff(M)$ is homogeneous, but it is not symmetric. To the contrary, it is extremely far from symmetry. Therefore our "symmetric" intuition may be misleading, and in some situations negative curvature can stabilize the flow, if it varies in certain way. Likewise, positive curvature can be destabilizing. Directions of zero curvature and asymptotically flat geodesic subspaces should play important role in the stability and in the long time behavior of the flows.

The question of interconnection between the stability of fluid flows and the curvature and, possibly, other differential-geometric properties of the space SDiff(M) is far from clear. The curvature, being typically negative in all directions containing the velocity field, is not bounded off from zero. The solution of the Jacoby equations (the linearized Euler-Lagrange equations) can tend to the asymptotically zero-curvature direction, thus neutralizing the effect of negative curvature on the perturbation growth. This phenomenon lies beyond the sheer negativity of curvature, it is possibly associated with the fact that $SDiff(M)$ is not a symmetric space. However, little progress is done in this direction.

Still another problem is the study of the structure of the Jacoby equations themselves. The typical solution is growing (or decreasing) algebraically, rather than exponentially (if there is no unstable eigenvalues). Such behavior may be associated with the presence of Jordan cells in the linearized operator (it is "made of" a continuum of low-dimensional Jordan cells). This idea is far from realization.

Another problem is the structure of singularities of the exponential map on the group SDiff(M). We expect that the Fredholm structure of this map enables to use topological methods (like the degree theory) in the study of the Euler equations. The interesting question is, whether the variant of condition C of Palais-Smale holds for the action functional on $SDiff$ in the 2D case. If true, this could lead to a Morse theory of geodesics in this context.

3 Presentation Highlights

Our group worked in the permanent exchange of ideas, and there were no formal presentations. I'd highlight the impressive demonstration of the use of Maple for bulky asymptotic computations by Steve Preston.

4 Scientific Progress Made

During the meeting D. Ebin and S. Preston worked on an interesting idea of a finite-codimensional approximation to the Euler equations. This work is in its beginning and may bring some interesting results.

5 Outcome of the Meeting

The main outcome of the meeting is the intensive exchange of ideas between the participants. It helped to find and root out some misconceptions, and to formulate some questions. One of the results is the beginning of systematic common work of S. Preston and A. Shnirelman on the problems of the motion of an ideal inextensible thread, which is peculiarly close to the fluid, while appear simpler. Here the combination of differential geometry, functional analysis, dynamics, and even computer simulations works even better than for the fluid (however some features of this system are almost opposite to the fluid).

References

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