Analytic and Algebraic Methods in Complex and CR Geometry

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This workshop focused on both complex analysis and algebraic geometry. Its primary purpose was to foster interactions among researchers in these areas. This report will describe analytic, algebraic, and geometric perspectives and how they blend.

Both the lectures and the informal conversations held in the workshop developed these connections. It is natural to place (most of) the discussions into one or more of three categories: those with a flavor from Partial Differential Equations, those motivated by CR Geometry, and those concerning Algebraic Geometry. Nearly all the lectures made at least some connections among these areas.

We begin by discussing the Cauchy-Riemann operator $\overline{\partial}$ and its impact on complex analysis. The study of $\overline{\partial}$ as a partial differential operator leads to the basic questions of existence and regularity. These basic questions from partial differential equations naturally lead to theorems relating the geometry of the boundary of a domain to the behavior of $\overline{\partial}$ on the domain. Since the 1960's so-called L^2 methods and their applications have played a major role. We recall some of these developments.

Many important developments in complex analysis in the twentieth century arose from the solution of the *Levi Problem* identifying domains of holomorphy with pseudoconvex domains. Pseudoconvexity is a local geometric property of the boundary, whereas the notion of domain of holomorphy belongs to the function theory on the domain itself. The solution of the Levi Problem includes an existence and regularity result for $\overline{\partial}$. A domain Ω in $\mathbf{C}^n$ is a domain of holomorphy if and only if the following statement holds: For each nonnegative integer q and each smooth $(p, q+1)$ form α on Ω such that $\overline{\partial}\alpha = 0$ on Ω, there is a smooth (p, q) form u on Ω such that $\overline{\partial}u = \alpha$.

The so-called $\overline{\partial}$-Neumann problem extends the above idea by considering the Cauchy-Riemann equations on $\Omega \cup b\Omega$. Suppose that the boundary $b\Omega$ is smooth and consider differential forms with L^2 coefficients on the closed domain. Given a $\overline{\partial}$-closed form α, orthogonal to the harmonic space, the $\overline{\partial}$-Neumann problem constructs the N-operator and the solution $\overline{\partial}^* N\alpha$ to the equation $\overline{\partial}u = \alpha$. Spencer first posed this problem in the 1950's in order to extend Hodge Theory to manifolds with boundary, but many analytic difficulties arose before Kohn solved the problem in 1962 using the method of L^2 estimates.

Local regularity holds when $\overline{\partial}^* N\alpha$ must be smooth wherever α is smooth; local regularity follows from subelliptic estimates, which imply that N is a *pseudo-local* (but not a pseudodifferential) operator. *Global regularity* for the $\overline{\partial}$-Neumann problem holds when $\overline{\partial}^* N\alpha$ is smooth everywhere on the closed domain assuming that α is itself everywhere smooth. Several years after solving the $\overline{\partial}$-Neumann problem, Kohn established a global regularity result using weighted L^2 techniques. A smooth solution to $\overline{\partial}u = \alpha$ exists when α is everywhere smooth on the closed smoothly bounded domain. For a long time it was not known however whether the $\overline{\partial}$-Neumann solution was always smooth. When subelliptic estimates (described below) hold, of course, the $\overline{\partial}$-Neumann solution is smooth. In 1996 Christ proved that global regularity of the $\overline{\partial}$-Neumann solution fails for some worm domains. Boas and Straube showed that global regularity for the $\overline{\partial}$-Neumann solution holds for domains with a defining function that is plurisubharmonic on the boundary. They also verified global regularity when the set of points of infinite type satisfies certain topological conditions, but the problem of global regularity is not yet completely understood. A related open problem concerns finding necessary and sufficient conditions for compactness estimates.

Results about global regularity often produce geometric applications. The smooth extension to the boundary of biholomorphic mappings between certain weakly pseudoconvex smoothly bounded domains provides a striking example. Siu's work on the nonexistence of smooth Levi-flat hypersurfaces in the complex projective plane $\mathbf{P^2}$ gives a second example. In the workshop Ohsawa spoke further about the use of L^2 methods to study Levi flat objects. Siu has also applied techniques of L^2 estimates to establish the invariance of plurigenera first for the case of general type and later when the manifold is not necessarily of general type. Thus L^2 estimates for $\overline{\partial}$ have provided a deep link between analysis and algebraic geometry.

Perhaps the major advance at this workshop was Siu's talk on the famous question of the finite generation of the canonical ring of a compact algebraic manifold X of complex dimension n of general type. Siu described the techniques he introduced from L^2 estimates for $\bar{\partial}$ to handle the obstacles of this problem.

He introduced the infinite sum Φ over all m of the m-th root of the sum of the absolute-value squares of elements of a basis of m-canonical sections. By adapting Skoda's L^2 estimates of $\bar{\partial}$ for the generation of ideals, he first reduced the problem to proving that Φ and one of its finite partial sums are each dominated by a constant multiple of the other. His method involves as intermediate steps the proofs of the rationality of the vanishing orders of Φ and the finiteness of the number of irreducible components of the super level sets of the Lelong number of $\sqrt{-1}\partial\bar{\partial}\log\Phi$. For such proofs he used algebraic geometric techniques which are adapted from and motivated by the following two analytic techniques of the complex Monge-Ampère equation for $\left(\sqrt{-1}\partial\bar{\partial}\log\Phi\right)^n$:

(i) an observation of Demailly that $\frac{1}{\Phi}$ is equivalent to the metric $e^{-\varphi}$ of the canonical line bundle K_X of X with φ maximum among all plurisubharmonic φ subject to the normalization of the supremum of $\varphi - \psi$ being 0 for some fixed background metric $e^{-\psi}$ of K_X, and

(ii) a result of Bedford and Taylor that the complex Monge-Ampère equation is the Euler-Lagrange equation for maximizing a function among plurisubharmonic functions.

Notice that two analytic techniques, developed in the study of the complex Monge-Ampère equation, have algebraic applications here. First, Fefferman's work (Annals 1976) on the asymptotic order of the solution of the complex Monge-Ampère equation on a strongly pseudoconvex domain motivates the algebraic geometric technique to prove the rationality of vanishing orders of Φ. Second, Yau's regularity results (Comm. Pure and Applied Math. 1978) for the complex Monge-Ampere equation when the right-hand side has complex analytic singularities motivates the algebraic-geometric techniques for proving the finiteness of the number of irreducible components of the super level sets of the Lelong number of $\sqrt{-1}\partial\bar{\partial}\log\Phi$. By incorporating the techniques developed for the Fujita conjecture type problems and the techniques of Shokurov's nonvanishing theorem, Siu's method translated the analytic techniques to the algebraic geometric settings so that when either some vanishing order of Φ is irrational or there are infinite number of super level sets of the Lelong number of $\sqrt{-1}\partial\bar{\partial}\log\Phi$, some new pluricanonical sections can be produced by L^2 estimates of $\bar{\partial}$ to give a contradiction to the definition of Φ.

We return to the $\overline{\partial}$-Neumann problem on a smoothly bounded domain Ω. The geometry of the boundary enters because of the $\overline{\partial}$-Neumann boundary condition. For a $(0,1)$ form ϕ this condition is the same as saying that the $(1,0)$ vector dual to ϕ is tangent to $b\Omega$. This condition therefore leads to the notion of a CR manifold. CR manifolds are real manifolds whose tangent spaces behave like those of real submanifolds in complex manifolds. The special case of a real hypersurface in complex Euclidean space arises of course as the (smooth) boundary of a domain. The $\overline{\partial}$-Neumann problem therefore provides a deep link between the CR geometry of the boundary of Ω and the function theory on Ω.

The ideas in the proofs of existence and regularity results for $\overline{\partial}$ have led to the development of CR geometry, the calculus of pseudo-differential operators, and subelliptic multiplier ideal sheaves. All three of these topics have evolved considerably, and each played a major role in the workshop.

We next discuss subellipticity and related ideas. After much preliminary work, in 1978 Kohn introduced subelliptic multipliers as a technique for proving subelliptic estimates in the $\overline{\partial}$-Neumann problem. Subelliptic estimates imply local regularity results for the $\overline{\partial}$ operator. In the 1970's Skoda introduced the use of L^2 methods in algebraic geometry. Currently the algebraic geometry community has become actively involved in the study and use of multiplier ideal sheaves. The work of Siu, Nadel, Demailly and others have demonstrated convincingly the power of such analytic methods in algebraic geometric problems. Nearly all participants in the workshop have used either analytic or algebraic aspects of these ideas in their work, and if not, have worked on closely connected problems.

The solution of the $\overline{\partial}$-Neumann problem on strongly pseudoconvex domains can be understood by thinking of the determinant of the Levi form $\det(\lambda)$ as a subelliptic multiplier; for any 1-form ϕ in the domain of $\overline{\partial}^*$, one can control the Sobolev $\frac{1}{2}$-norm of $\det(\lambda)\phi$ in terms of the usual Dirichlet form. When this determinant vanishes things become quite difficult. Kohn posed the problem of determining necessary and sufficient conditions for subelliptic estimates for $\overline{\partial}$. D'Angelo introduced a finite type condition that, through deep work of Catlin, turned out to be necessary and sufficient for subelliptic estimates on $(0, 1)$ forms on pseudoconvex domains. A similar result holds for forms of higher degree. Catlin's proof does not use subelliptic multipliers; instead he constructs bounded plurisubharmonic functions with large Hessians. The precise relationship between the two approaches to subelliptic estimates is not yet understood. Because they apply in the smooth category, Catlin's techniques have significant unrealized potential in subelliptic multiplier theory.

All these ideas are closely related to singularity theory. D'Angelo has discussed a precise analogy: strongly pseudoconvex points correspond to the maximal ideal in the ring of germs of holomorphic functions at a point, and finite type corresponds to ideals primary to the maximal ideal. Thus the problem of subelliptic estimates helped establish a basic connection between hard analysis (PDE estimates) and singularity theory. Developing this connection was one of the reasons for holding this workshop.

The workshop itself succeeded in forging new connections on precisely this topic. For example Lazarsfeld spoke about the *type* of a punctual ideal, a concept invented in algebra for several reasons, and independently in analysis for the purpose of understanding the relationship between finite type and subelliptic estimates. The type of a punctual ideal in the ring of germs of holomorphic functions is finite if and only if the ideal is primary to the maximal ideal, and it provides an interesting numerical measurement (always a rational number) of the singularity. The lecture of Lazarsfeld showed how ideas in algebra such as the integral closure of an ideal, normalized blow-ups, and the Briancon-Skoda Theorem impact the study of the type of a punctual ideal. The theory of finite type shows how to reduce the type of an ideal in the ring of germs of smooth functions to the types of a family of punctual ideals. Closely related to these ideas is an algebraic version of Kohn's theory of subelliptic multipliers in the (simpler) holomorphic setting, a topic which was discussed by many of the participants in the informal discussion held throughout the workshop. Lazarsfeld also gave a simple treatment and extension of a result of McNeal-Nemethi showing how a supremum over all holomorphic arcs can be replaced by a maximum over a finite list of well-chosen holomorphic arcs, thus rendering evident the rationality of the type. This material illustrates well the sort of connections forged by the workshop.

Hwang spoke about the relationship between the Arnold multiplicity and the usual notion of multiplicity connected with orders of vanishing. The Arnold multiplicity is a local invariant of an effective divisor on a complex manifold. It is the infimum of the set of m for which a certain integral is finite; if f is a local equation for the divisor, the Arnold multiplicity is the infimum of the set of m for which $|f|^{\frac{-2}{m}}$ is locally integrable. Hwang established a decisive estimate for the Arnold multiplicity when the base manifold is the quotient of a complex semi-simple Lie Group by a maximal parabolic subgroup. To do so he proved a product theorem concerning the behavior of the Arnold multiplicity for divisors on the product of two manifolds. Again we observe a powerful connection between analysis and algebraic geometry. Hwang discussed upper-semi continuity properties of these multiplicities, making a nice connection with other issues. For example, semi-continuity fails for the type of a family of punctual ideals depending nicely on a parameter, and this result has impacted subelliptic estimates. On the other hand, inequalities relating the type to the co-length, which behaves better under change of parameter, play a role in work on finite type.

D'Angelo spoke on a monotonicity result for holomorphic volumes. At first glance this result is not obviously related to the theme we have discussed so far; on the other hand volumes involve integrals of squared norms of Jacobians, and the results are thereby connected with both complex geometry and L^2 ideas. Note that the determinant of the complex Hessian of the squared norm of a holomorphic mapping is precisely equal to the sum of the squared moduli of all possible Jacobians of the components of the mapping. The ideas are thus connected with properties of the integral of the determinant of the Levi form. The monotonicity result leads to a corollary with a nice algebraic-geometric flavor. Let p be a proper polynomial mapping between balls, of degree d. Then the volume of the image of the ball under p is at most $\frac{\pi^n d^n}{n!}$, with equality if and only if the mapping is homogeneous. For balls and eggs one proves the monotonicity result for volumes of holomorphic images by carefully studying the L^2 norms of monomials. A result for more general pseudoconvex domains can be proved using Stokes's theorem, in case the map has some regularity at the boundary. Again we see how L^2 methods are closely related to complex geometry.

Several other talks in the meeting nicely illustrated L^2 methods. McNeal spoke about a generalization (due to McNeal-Varolin) of the celebrated Ohsawa-Takegoshi Theorem. Suppose first that D is a pseudoconvex domain in $\mathbf{C}^n$ and that H is a complex hyperplane. Let f be holomorphic on $H \cap D$, and in L^2 with respect to some weight. Ohsawa-Takegoshi proved that f can be extended to a function F holomorphic in D whose L^2 norm with respect to the same weight is controlled by the L^2-norm of f. McNeal-Varolin showed how to gain strength in this estimate by manipulating the weights. During the talk Siu observed a parallel with these ideas and his use of the L^2 extension result in order to establish the invariance of plurigenera.

A natural problem in complex analysis asks to express a nonnegative Hermitian symmetric polynomial as a squared norm of a holomorphic mapping, or more generally as a quotient of squared norms of holomorphic mappings. D'Angelo has asked, as a complex variable analogue of Hilbert's 17th problem, for a characterization of quotients of squared norms of holomorphic polynomial mappings. Work of Catlin-D'Angelo relating isometric imbedding of holomorphic bundles to squared norms and quotients of squared norms of holomorphic mappings provides a general framework for such questions. Their result assumes a nondegeneracy condition analogous to strong pseudoconvexity; the degenerate case is quite subtle, because the class of quotients of squared norms is not closed under limits. In his talk at the workshop Varolin announced a complete solution to this question. His proof uses L^2 techniques and a form of the resolution of singularities. Furthermore the setting applies for many bundles, and even the proof in the (simplest) case of powers of the tautological bundle over projective space requires proving the theorem for more general spaces. Varolin's condition states that the real Hermitian polynomial R, which can always be written as $||F||^2 - ||G||^2$ for holomorphic mappings F and G, is a quotient of squared norms if and only if the function

$$\frac{||F||^2 + ||G||^2}{||F||^2 - ||G||^2}$$

is bounded. The proof involves the Bergman kernel function in a rather general setting. As in the above work on isometric embedding, the Bergman kernel function appears as an approximate generating function for tensor powers of a metric.

Varolin also discussed other positivity conditions and Siu mentioned the connection with a famous paper of Calabi on isometric imbedding from the early 1950's. Various forms of a non-linear version of the Cauchy-Schwarz inequality play a key role in all the work on isometric embedding. The condition that a bundle metric satisfies the non-linear Cauchy-Schwarz inequality involves curvature, but it is distinct from the usual curvature conditions. It could therefore could play a role in developing new connections between analysis and complex geometry.

Next we turn to some connections between PDE and CR geometry. Perhaps the most basic example of a CR manifold is the unit sphere. Because the unit ball is biholomorphically equivalent with the Siegel generalized upper half-plane, its boundary (the sphere) is CR equivalent with the Heisenberg group. This connection between several complex variables and harmonic analysis has been especially fruitful in studying the strongly pseudoconvex case, but new ideas are needed in general.

Several talks considered issues centering around differential and pseudo-differential operators on CR manifolds, typically motivated by the Heisenberg group. Melrose began the workshop with a general and abstract treatment of a calculus of pseudo-differential operators that takes into account the anisotropic behavior of the tangent spaces on strongly pseudoconvex boundaries. The anisotropic behavior there has one *parabolic direction*. He showed that operators in a very general class behave properly under composition. Precise descriptions of the kernels of these operators of course epitomizes the theme of the workshop; the relationship between the geometry of the boundary of a domain and analysis on the domain. His general results apply in some cases admitting multiple parabolic directions and also apply to other applied boundary problems.

A central problem of local CR geometry is the embeddability question. Is an abstract CR manifold (of hypersurface type) locally CR-embeddable in $\mathbf{C}^{\mathbf{N}}$? Kuranishi solved the problem for strongly pseudoconvex CR manifolds of dimension at least nine using L^2-estimates. Akahori and later Webster proved the result in dimension seven. Akahori used L^2-methods, whereas Webster used integral formulas for solving the $\overline{\partial}$-equation. Catlin has generalized these results when appropriate finite-type conditions replace strong pseudoconvexity. It has been long known that the result fails in three dimensions, but the case of dimension five remains open.

In the problem session Greiner proposed an approach to prove local embeddability for CR manifolds of dimensions at least five. This approach relies on Greiner's program of constructing fundamental solutions explicitly. Previous approaches construct the embedding by an iterative procedure. In each step one solves an approximate $\overline{\partial}_b$ equation for $(0,1)$-forms on an embedded CR manifold. The solution is obtained by solving a precise $\overline{\partial}_b$ equation on $(0,2)$ forms. Use of this secondary equation requires the dimension to exceed five. Greiner's approach, by contrast, constructs an embedding in one step, by finding

CR functions with prescribed differential at one point. To do so he solves a $\overline{\partial}_b$ equation on (0,1) forms, using explicit kernels. To make this approach work, one needs to extend Greiner's explicit results on fundamental solutions from one PDE to systems of PDE.

Studying which three-dimensional CR manifolds can be embedded is a challenging part of the general problem. Various partial results have opened new avenues for investigating the relationships between function theory for pseudoconcave manifolds and CR deformation theory for the boundary.

Epstein discussed the embedding problem for abstract three-dimensional CR manifolds. He related this question to the Dirac operator $\overline{\partial} + \overline{\partial}^*$. He considered the collection of embeddable CR structures near a given embeddable one, and gave a necessary and sufficient condition; namely, that the restriction of the Szegö projection be Fredholm. Epstein began by describing an extension of the $\overline{\partial}$-Neumann problem to a class of $Spin_C$ manifolds. He used it to study the relative index between two generalized Szegö projectors on a contact manifold. For example, suppose that a three-dimensional contact manifold bounds two strongly pseudoconvex complex surfaces. Then the relative index can be expressed in terms of the differences of their Euler characteristics, their signatures, and the dimensions of their cohomology groups $H^{0,1}$. In certain cases it follows that the relative index assumes only finitely many values among embeddable deformations close to a given embeddable structure. In these cases the set of embeddable CR-structures is closed in the C^∞-topology.

The talks by Greiner and Tie considered sub-Riemannian geometry, motivated again by the Heisenberg group. Greiner's talk provided many explicit relationships between CR geometry and geodesics. He considered second order partial differential operators given as sums of squares of vector fields; these operators arise for example as the Kohn Laplacian in the case of three-dimensional CR manifolds, and information about them is therefore useful for complex analysis. Greiner built explicit formulas for fundamental solutions from geometric invariants. A new phenomenon in this sub-Riemannian geometry is the notion of the "characteristic submanifold" attached to every point p: the locus of points connected to p by an infinite number of geodesics.

Tie's talk also evolved from generalizing some of the basic ideas from CR geometry. For example, we have seen that the anisotropic behavior of the CR geometry of a strongly pseudoconvex manifolds leads to harmonic analysis on the Heisenberg group, which has a nilpotent Lie algebra. For certain 3-dimensional CR manifolds of finite type, Lie algebras of higher step arise. Tie discussed a specific example of step 3 and its relations to Hamilton's equations and the Heisenberg group.

Polarization techniques play a key role whenever real-analytic functions arise, e. g., as defining equations of domains or as metrics on holomorphic line bundles. The ability to vary z and $\overline{z}$ separately lies at the foundation of complex analysis. Segre introduced the varieties which have been used extensively by Webster and others in diverse problems. More recently Baouendi-Ebenfelt-Rothschild developed an iterative procedure to generate additional Segre sets. These ideas have had many uses. In particular Ebenfelt and Rothschild proved a CR transversality result for generic real-analytic CR submanifolds of finite commutator type. The result says that the germ of a finite holomorphic mapping between two such manifolds is necessarily CR transverse. In other words, in codimension d, one obtains a result guaranteeing that a certain derivative mapping has rank d. The codimension one version of this result is a version of the Hopf lemma. The technique of Segre sets also provides a characterization of finite commutator type, due to Baouendi-Ebenfelt-Rothschild; for a generic CR manifold M of codimension d, the Segre set $S_{2d}(p)$ contains an open neighborhood of p if and only if M is of finite commutator type at p. Thus an issue about iterated commutators of vector fields (a part of *complex* control theory) has a description in terms of Segre sets and polarization.

Segre sets also arose in the talk of Christ, revealing quite an interesting connection. Christ considered L^p estimates for generalized Radon transforms. Generalized Radon transforms are defined by integration over families of submanifolds of an ambient space and associated with a certain geometric structure. A basic and fascinating problem here is to relate the analysis to the underlying geometry. Part of Christ's talk considered this idea as a problem in continuum combinatorics. The relationship between geometry and analysis described here meshed especially well with the several talks on pseudodifferential operators and sub-Riemannian geometry.

The recent striking work of Kohn on hypoellipticity despite loss of derivatives was mentioned in the original proposal for this workshop. One talk directly considered this topic. Tartakoff discussed his work with Derridj and Bove showing that Kohn's example of a C^∞ hypoelliptic operator P_k is also locally analytic hypoelliptic. The proof yields a simplification of Kohn's proof. The second order operator P_k has the simple expression

$$P_k = LL^* + (\overline{z}^k L)^* (\overline{z}^k L),$$

where L is a Lewy operator of the form

$$L = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}.$$

Tartakoff also provided a generalization $P_{k,m}$ which is hypoelliptic in both senses but loses $\frac{k-1}{m}$ derivatives. The techniques of proof involve complicated estimations which evoke earlier work by Tartakoff and Treves on global analytic hypoellipticity for operators such as the $\overline{\partial}$-Neumann operator.

A major advance (1981) in CR geometry was the Baouendi-Treves approximation Theorem: A CR function on a CR submanifold of $\mathbf{C^n}$ can be locally uniformly approximated by entire holomorphic functions. The proof uses convolution with a complex Gaussian kernel. A CR function is of course a solution to the homogeneous tangential Cauchy-Riemann equations. Boggess spoke about global and semi-global versions of the Baouendi-Treves result. In particular Boggess and Dwilewicz proved such a result for real hypersurfaces in $\mathbf{C^n}$ that are graphs over a linear space of codimension one.

An important idea in CR geometry concerns the tangential version of the inhomogeneous Cauchy-Riemann equations. As in the case of holomorphic functions, one obtains information about the solutions of the homogeneous equation by studying the inhomogeneous equation as a system of PDE. In the smooth category many such results have been worked out. Shaw spoke about estimates for the tangential Cauchy-Riemann equations on CR manifolds with minimal smoothness. The main point is to prove Hölder and L^p regularity for the tangential Cauchy-Riemann equations on CR manifolds of class C^2. One application of these estimates is to prove the embedding theorem of Boutet de Monvel for strongly pseudoconvex CR manifolds of real dimension at least five and of class C^2.

Stolovitch considered a basic question about CR singularities. Consider a real-analytic $(n+r)$-dimensional submanifold of $\mathbf{C^n}$ having a CR-singularity at the origin. Let us restrict to quadrics for which one can define generalized Bishop invariants. Such a quadric intersects the complex linear manifold $z_{p+1} = \cdots = z_n = 0$ along some real linear set $\mathcal{L}$. Stolovitch discussed what happens to this intersection under perturbation of the quadric. In some cases, if such a submanifold is formally equivalent to its associated quadric, then it is holomorphically equivalent to it.

We next discuss some of the connections with algebraic geometry and complex differential geometry.

Mabuchi considered three notions of stability; K-stability, Chow-Mumford stability, and Hilbert-Mumford stability, and clarified their asymptotic relationships. He showed that asymptotic Chow-Mumford-Veronese stability coincides with asymptotic Hilbert-Mumford stability and that K-stability implies asymptotic Chow-Mumford-Veronese stability. For a polarized projective algebraic manifold with vanishing Futaki character, Mabuchi showed that asymptotic Chow-Mumford stability relative to an algebraic torus implies K-semi-stability.

de Oliveira considered symmetric differentials and the hyperbolicity of hypersurfaces in $\mathbf{P}^3$ with appropriate nodal singularities. The existence of symmetric differentials on an algebraic surface X has a strong impact on the algebraic and transcendental hyperbolicity of X. Unfortunately, smooth hypersurfaces in $\mathbf{P}^3$ have no symmetric differentials. It turns out that there are smooth families whose general member is a smooth hypersurface of degree $d \geq 6$ in $\mathbf{P}^3$, but whose special member which is singular has many symmetric differentials. By using a resolution of singularities in his argument, he showed that the special singular member with appropriate nodal singularities has sufficient independent symmetric differentials to make it quasi-algebraically hyperbolic. This situation exhibits jumping of the cotangent plurigenera along a family.

Miyaoka provided some new examples of stable and semistable Higgs bundles. Higgs bundles arise from representations of the fundamental group of complex or algebraic manifolds, and are part of the active subject of noncommutative Hodge theory. They have played an important role in gauge theory and the geometrization of mathematical physics

Yeung discussed integrality and arithmeticity of lattices in quotients of the ball. The main result is that a co-compact lattice in a complex two ball is integral. He also discussed related geometric and arithmetic problems. Although arithmetic geometry was not the primary focus of this meeting, Yeung's results indicate intriguing connections between algebraic, analytic, and arithmetic geometry.

Nearly every good conference has at least one excellent talk that, at first glance, seems a bit removed from the other talks. Often such talks profoundly impact future developments in the subject, because they provide fresh ideas. Larusson gave such a talk at this meeting, on the subject of model categories and homotopical algebra, a subject invented by Quillen. Model categories provide an abstract setting for developing analogues of the homotopy theory of topological spaces for various other sorts of objects, and they have found important applications not only within homotopy theory itself but also in algebra and algebraic geometry. Recently they have appeared in complex analysis and provided a natural conceptual framework for the Oka Principle. Of course the Oka Principle intimately connects the Cauchy-Riemann equations with topology; one expects, on a Stein manifold, to be able to do with holomorphic functions what one can do with continuous functions.

Many important parts of complex analysis were not explicitly mentioned at the workshop, but the subject remains finely woven, and many such topics made at least a spiritual appearance. We mention in particular the possibilities associated with extending the ideas of the workshop to infinite dimensional holomorphy, an area thriving due to deep work of Lempert.

The workshop ended with a discussion of open problems in complex analysis and algebraic geometry and their connections.

There is no doubt that the workshop forged significant connections between complex analysis and algebraic geometry. The lectures, discussions, and the session on open problems enabled a diverse group of mathematicians with common interests to see first-hand how techniques from other parts of mathematics can be used in their own research specialties. Furthermore the amenities of the BIRS helped create a lively and stimulating environment. Research in both complex analyis and algebraic geometry has advanced as a result of this meeting.