Rigidity, Dynamics, and Group Actions

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Rigidity theory has its roots in classical theorems of Selberg, Weil, Mostow, Margulis and Furstenberg. It extends into diverse areas such as complex and differential geometry, group theory and representation theory, ergodic theory, dynamics and group actions. Our conference "Rigidity, Dynamics and Group Actions" concentrated on the rapid recent progress in these areas. The study of "large" groups (such as lattices in semisimple groups or higher rank abelian groups) and their actions was the focal point of the conference, with particular attention given to the following four closely related topics:

- local and global rigidity of actions,
- low-dimensional actions of large groups,
- orbit-equivalence rigidity, and
- invariant measures for actions on homogeneous spaces.

We had many exciting talks on these and other topics on large groups. Exciting recent progress more generally in dynamics, geometry and geometric group theory was also discussed and presented in talks. Many exciting new connections between dynamics of group actions and other areas, including number theory, geometry, and operator algebras, were discussed.

The organizers have established a resource page for the workshop[1]. There are also plans to expand an existing problem list from an earlier workshop to reflect the problem session at this workshop.

1 Classification of Group Actions

Let $G = SL(n, R)$ and $\Gamma = SL(n, Z)$, with $n \geq 3$. More generally, we can consider any simple Lie group G of real rank at least two, and a lattice Γ in G. For any natural number ℓ, the classical theory of roots and weights determines all of the homomorphisms from G into $GL(\ell, C)$. Roughly speaking, Margulis' Superrigidity Theorem (1975) shows that roots and weights of groups closely related to G determine all of the homomorphisms from Γ into $GL(\ell, C)$.

These two results classify the linear actions of G or Γ on (complex) vector spaces. Zimmer's non-linear generalization of Margulis' superrigidity theorem opened the way to classifying "non-linear represention" of

[1] `http://people.uleth.ca/~dave.morris/banff-rigidity/`

these groups. One such non-linear variant is to study ergodic group actions of G or Γ up to orbit equivalence. For higher rank groups and their lattices, orbit equivalence is now fairly well understood, due primarily to work of Zimmer and Furman. Recent progress has focused on other types of groups, see the recent survey by Shalom [29].

A more difficult nonlinear, problem is to classify the smooth (C^∞) actions of G or Γ on compact, smooth manifolds. This work is closely connected to understanding the structure of known algebraic actions, and also to several questions in pure ergodic theory.

Very few volume-preserving actions of Γ (on a compact manifold) are known. One example is the standard action of $\Gamma = SL(n, Z)$ on the n-torus. Certain examples similar to this are called *affine algebraic actions*; they arise from purely algebraic (group-theoretic) constructions.

In 1996, Katok and Lewis produced the first examples of non-algebraic actions. However, the actions were constructed by making minor topological modifications of algebraic ones. It may be the case that every volume-preserving action is isomorphic to an algebraic action, after certain sets of measure zero are deleted.

1.1 Local rigidity [7, 9]

A smooth action ρ of Γ is said to be *locally rigid* if every "nearby" smooth action is smoothly conjugate to ρ. Building upon many authors' results of the last 15 years, Fisher and Margulis established local rigidity for all affine algebraic actions [9]. Thus, perturbing an affine algebraic action will not result in a non-algebraic action.

Fisher recently pushed through another approach to local rigidity, generalizing some of Weil's ideas for proving local rigidity of lattices in Lie groups. It is often easier to prove infinitesimal rigidity of a subgroup or action. Weil for subgroups and now Fisher for actions showed how to go from infinitesimal to local rigidity. For actions this is a highly non-trivial problem due to the difficulty of suitable inverse function theorems. This approach has many novel applications to groups not covered by any previous local rigidity results. Fisher reported on this in his talk at the workshop. He also discussed work in progress with T.J.Hitchman which would produce further applications of this result.

1.2 Dynamics and global rigidity [6]

Margulis and Qian proved a global rigidity result for actions of Γ on tori under some further assumptions. Goetze and Spatzier completely classified the much more restricted class of "Cartan" actions on arbitrary compact manifolds.

These proofs use the study of "hyperbolic" actions of higher rank abelian groups by Katok, Spatzier and others. As for lattices, all irreducible actions of this type are conjectured to be "algebraic."

The cross fertilization between these areas has been crucial. For example, local rigidity of the higher rank abelian actions led to the proof of local rigidity of projective actions of higher rank cocompact lattices. This is also closely related to work of Katok-Spatzier discussed below in (2.1).

More recent developments concerning global rigidity of group actions have introduced a plethora of new techniques and ideas into the field and some of these were reported on at the meeting, but are discussed below in the section on low dimensional actions. In low dimensions, the classification problem simplifies to showing that no examples exist!

1.2.1 Arithmetic Quotients [10]

Recent work of Lubotzky, Zimmer, and Fisher constructs a measurable map from any volume-preserving action of G or Γ to some algebraic example. Fisher and Whyte gave conditions under which the map is continuous. Under additional assumptions, the algebraic action is "close to" the original action. More recent results of Schmidt have drawn closer connections between global rigidity and the study of arithmetic quotients.

1.2.2 Low-Dimensional Actions of Large Groups [2, 11, 13, 19, 23]

Zimmer conjectured that Γ cannot act (faithfully) on any compact manifold M whose dimension is much smaller than the size of G. This has not been proved in complete generality even when $dim(M) = 1$, although much progress was made in a sequence of works by Witte, Ghys, Burger and Monod, Navas, and Lifschitz and Morris. More recently, there has been dramatic progress when $dim(M) = 2$ as well, assuming the action is volume-preserving, and that G/Γ is not compact. Under these assumptions, Polterovich eliminated all the surfaces of genus at least 1, by using techniques from symplectic topology. Franks and Handel were able to eliminate the other surfaces, under a stronger assumption on Γ, by using a completely different approach based on low-dimensional dynamics, including a structure theory for area preserving diffeomorphisms of surfaces. Franks discussed some of these results in his talk. In connection with this work, M. Handel explained his joint work with Franks on fixed points for actions of higher rank abelian groups on R^2 and S^2. Higher rank abelian actions have been prominent in recent years, due to the discovery of many rigidity properties. The work of Franks and Handel again shows that such actions are very special.

Vanishing of bounded cohomology groups is an obstruction to non-trivial actions on the circle. Recent work of Ghys-Gambaudo and Polterovich indicate that bounded cohomology may also be relevant to studying actions on surfaces.

One can interpret elements of the second bounded cohomology of a group Γ as quasi-morphisms. Polterovich gave a brief overview over of quasi-morphisms and how they arise for groups of Hamiltonian diffeomorphisms at the workshop. This very inspiring lecture will serve as an excellent departure point for future work in the area.

In the complex analytic setting, S. Cantat recently established a version of Zimmer's conjecture. This combines holomorphic dynamics with arguments from algebraic geometry, and is the first result of this kind for actions preserving a non-rigid geometric structure (the complex structure).

1.2.3 Cocycle Superrigidity [5, 8, 30]

A fundamental tool, in the analysis of actions of large groups is Zimmer's extension of Margulis superrigidity theorem for cocycles for higher rank semisimple Lie groups without compact factors. As reported by Hitchman, he and Fisher extended these cocycle superrigidity results to actions of the Kazhdan rank 1 groups and their lattices using the harmonic maps approach to superrigidity. This builds on earlier work of Korevaar-Schoen and Corlette-Zimmer and also gives new proofs of the known cases of superrigidity. This will allow Fisher and Hitchman to prove many results for actions of these groups which had so far only been available for higher rank groups.

In recent years various superrigidity results were obtained for lattices in products of groups and even simply for products of finitely generated groups by many authors, particularly Shalom and Monod. Furman reported on work with Monod in which they generalized Zimmer's superrigidity theorem for (certain) cocycle over actions of such groups, and applied it to the study of their smooth actions.

Another extension of superrigidity for cocycles was announced at the conference by Popa. His result applies to a wide class of groups, but requires that the cocycle be over an action which is Bernouilli. This is related to Popa's recent work on orbit equivalence, which is discussed in the next subsection.

1.3 Orbit Equivalence Rigidity [12, 24, 29]

Two actions on measure spaces are said to be *orbit equivalent* if there is a bi-measurable map that takes orbits to orbits. This notion is central in ergodic theory. For discrete amenable groups, essentially all actions preserving a finite measure are orbit equivalent. At the other extreme, Zimmer's superrigidity theorem implies that non-isomorphic actions of G are never orbit equivalent. The situation for actions of a lattice is more subtle and was recently resolved by Furman. Furman was inspired by the classification of lattices up to quasi-isometry in geometric group theory. Furman's work has been used by logicians to solve longstanding problems on Borel equivalence relations.

Several authors have recently proven orbit equivalence results for more general groups. Gaboriau showed that ℓ^2-Betti numbers of groups are invariant under measure equivalence. This allowed him to distinguish free groups and their products under measure equivalence. Monod and Shalom used techniques from bounded cohomology to prove measure-equivalence rigidity of products of groups acting on CAT(-1)-spaces. More recently, Gaboriau and Popa have used techniques from operator algebras in conjunction with ideas from rigidity theory to produce uncountably many non-orbit equivalent actions of the free group.

During the workshop Popa reported on his recent work on the strong rigidity of II_1-factors of rigid groups, and in particular of Bernoulli actions of groups which have relative property (T). He also sketched some of the ideas in his more recent work, which yields orbit equivalence “super-rigidity” theorems for remarkably broad classes of groups. His lecture provided a good bridge to the world of operator algebras from the more classical areas of rigidity theory.

2 Flows on homogeneous spaces and related topics

In the previous section we described attempts to classify actions of large groups. Another major theme of research has been the study of the properties of concrete group actions. A basic class of such actions is the following: Let G be a locally compact group (usually either a Lie group or an S-arithmetic algebraic linear group), $\Gamma < G$ a discrete subgroup, and $H < G$ some other closed subgroup G. Then one may study the action of H on G/Γ. These actions are fascinating for their own sake and arise naturally in many contexts particularly in number theory, and also in the study of the rigidity questions discussed in the other sections of this summary.

A basic question considered about these actions in the classification of H-invariant measures on G/Γ and of H-invariant closed subsets. A major landmark in this direction has been Margulis' resolution of the long-standing Oppenheim conjecture regarding values of indefinite quadratic forms by classifying closures of $SO(2,1)$-orbits in $SL(3,R)/SL(3,Z)$.

This classification result is a very special case of much more general theorems proved a few years later by Ratner [26, 27] on invariant measures and orbit closure for actions of groups generated by unipotents (such as the Lie group $SO(2,1)$).

2.1 Invariant Measures For Actions on Homogeneous Spaces and Applications to Number Theory [3, 14, 16, 18, 20, 21, 26, 28]

Ratner's work (even her work on orbit closures) is based on the classification of measures invariant under groups generated by unipotents, and the many applications of this work are too numerous to be listed here! In the workshop H. Oh explained her work with Gorodnik and Shah on equidistribution of rational points in affine spaces refining earlier work of Eskin and McMullen on the growth of the number of such points, a key ingredient of which was Ratner's theorems.

Ratner's measure classification results apply only to finite invariant measures. If one considers flows on a quotient space G/Γ of infinite volume the situation is much less is understood. O. Sarig explained his work with Ledrappier on the horocycle flow on infinite normal covers of surfaces. Amazingly, even in this infinite geometric setting it is possible to classify invariant measures. Furthermore, Sarig reported that only one of these invariant measures satisfies a generalized law of large numbers.

Another type of actions that often arise in applications is the action of multidimensional abelian subgroups which are Ad-diagonalizable over R. At first sight it seems rather unlikely that anything useful can be said about invariant measures for such actions, since the action of a single hyperbolic diffeomorphism has many invariant measures and complicated orbit closures. But in fact, for an abelian group generated by several such diffeomorphisms, it seems that the invariant measures again are scarce. In the early $60's$ Furstenberg conjectured that ergodic measures invariant under both $\times 2$ and $\times 3$ on the unit interval are either supported on periodic orbits or are Lebesgue measure. Rudolph has proven the conjecture provided the entropy of at least one transformation is positive. Katok and Spatzier studied general affine algebraic actions of higher rank abelian groups, and proved algebraicity of the measures under a positive entropy condition and other strong ergodicity assumptions. Two new measure classification methods have been introduced that do not require these ergodicity assumptions — one by Einsiedler and Katok which deals with measures with “high” entropy and a second by Lindenstrauss dealing with measures of “low” entropy. These have been combined in [3] to classify all the measures on $SL(n,R)/SL(n,Z)$ ergodic and invariant under the action of the full diagonal group with positive entropy, which gives a partial result towards Littlewood's conjecture on simultaneous diophantine approximation. Lindenstrauss used the low entropy methods, in conjunction with his work with Bourgain in number theory, to show quantum unique ergodicity for certain arithmetic surfaces.

In the workshop E. Lindenstrauss discussed his work with M. Einsiedler, P. Michel and A. Venkatesh which gives another application of the results of [3] which gives information regarding the distribution of compact orbits of on homogeneous spaces indexed by discriminant.

Another talk which deals with the same type of action was given by Tomanov who presented generalization of his previous work with Weiss regarding the classification of closed orbits, and presented an application regarding the set of values attained by a product of k linear forms in $n \geq k$ variables at integer points.

2.2 Other related topics[15, 22]

Using ideas developed to study unipotents flows, and in particular their behavior near the cusp in the space $SL(n, R)/SL(n, Z)$, Dani, Kleinbock, Margulis and others have proven many results regarding diophantine approximations. During the workshop, D. Kleinbock discussed his recent work on quantitative divergence estimates for unipotent flows and how they give precise formulas for Diophantine exponents of affine subspaces of R^n, and Weiss explained how similar techniques work in the Teichmuller space setting.

Classical ergodic theory concerns itself with ergodicity and equidistribution problems for actions of "small" groups such as the reals and integers, and, more generally, amenable groups. It was only in the 1990's that ergodic theorems for actions of semisimple groups were established by Nevo, Stein and Margulis. They proved both strong maximal inequalities and pointwise ergodic theorems for avergaes over Riemannian balls in the group bi-invariant under a maximal compact subgroup. A. Gorodnik and A. Nevo recently generalized such theorems to a more general class of increasing compact sets. As a consequence, they obtained strong maximal inqualities, mean ergodic theorems and pointwise ergodic theorems for actions of lattices in semisimple groups, as was reported by Gorodnik.

3 GEOMETRY [4, 17]

A common theme of rigidity in geometry is the characterization of locally symmetric metrics in simple geometric or topological terms. The prime example is the Strong Rigidity Theorem of Mostow, Margulis and Prasad. Later examples are the rank rigidity theorems by Ballmann and Burns-Spatzier, and the characterization by Besson, Courtois and Gallot of real hyperbolic space by minimal volume and the other negatively curved symmetric spaces by minimal entropy. A related topic of interest is the study of similar rigidity properties for homogeneous spaces which are not locally symmetric, see work of Connell, Eberlein and Heber.

Minimal volume is closely related to Gromov's simplicial volume. The vanishing of the latter has important consequences for the topology and geometry of the space. Thurston had shown non-vanishing of the simplicial volume for closed real hyperbolic spaces. More generally it is known for closed manifolds of negative curvature. B. Schmidt reported on his recent work with J. Lafont that the simplicial volume of closed higher rank locally symmetric spaces of nonpositive curvature and no Euclidean facfors is not 0. This is based on a non-trivial extension of a Jacobian estimate of Besson, Courtois and Gallot to the higher rank situation by C. Connell and B. Farb.

Another approach to charaterize locally symmetric spaces is by symmetry: assume that the universal cover of a closed manifold has a non-discrete group of isometries. If it is also assumed that the sectional curvature is non-positive, then the metric is automatically locally symmetric, as was proved by P. Eberlein in the 80's. B. Farb reported on his beautiful work with S. Weinberger that achieves essentially the same conclusion without the curvature assumption. This work has recently been extended to other Lorentz and other pseudo-Riemannian metrics by K. Melnick. This will prove important in the context of group actions preserving such structures.

Mostow's use of quasi-isometries in establishing strong rigidity led to many outstanding problems in geometric group theory. Gromov in particular asked for the quasi-isometric classification of groups. For special groups such as lattices in semisimple groups, this was established in the early 1990's in a remarkable series of works by Casson, Chow, Drutu, Eskin, Farb, Gabai, Gromov, Jungreis, Kleiner, Koranyi-Riemann, Leeb, Pansu, Schwartz, Sullivan, and Tukia. One obtains both quasi-isometric rigidity and classification. Thus, any group quasi-isometric to such a lattice is isomorphic to one on a subgroup of finite index. There is one quasi-isometry class of cocompact lattices for each semisimple group G. Further, there is one quasi-isometry class for each commensurability class of irreducible non-cocompact lattices, except for $G = SL(2, R)$ where there is precisely one quasi-isometry class of non-cocompact lattices.

The case of nilpotent groups is still open even though Pansu showed that the associated graded group of two quasi-isomorphic nilpotent groups have to agree. Shalom recently found further invariants for quasi-isometry which distinguish some nilpotent groups with isomorphic graded group. These invariants have been further refined by R.Sauer.

The case of solvable groups however was wide open until our workshop when A. Eskin announced his recent joint work with Fisher and Whyte on Sol and other more general solvable groups. Again they establish quasi-isometric rigidity. Interestingly, the proof borrows techniques more commonly seen in ergodic theory.

Marked length spectrum rigiidity is yet another sought after characterization of a negatively curved Riemannian manifold. Much progress has been achieved in the last two decades. U. Bader in collaboration with R.Muchnik connected marked length spectrum rigidity to a natural representation of the fundamental group coming from the canonical action on the sphere at infinity.

3.1 Lattices [1, 25]

Boundaries have played a central role in rigidity theory. Yet we still do not understand boundaries completely. H. Furstenberg's lecture on problems in boundary theory will be made available as a video on the BIRS website, and is suitable for an introduction to the field for a more general audience.

The fine theory of lattices is still making major advances as exemplified by E. Breuillard's talk on his work with Gelander on the uniform Tits' alternative. Tits' famous result says that a finitely generated linear group either has a subgroup of finite index or contains a free group. This new work gives an estimate how close to the identity one can find two generators for a free group. This improves earlier work of Eskin, Mozes and Oh for free semigroups. They also obtained uniform Kazhdan L^2 constants and uniform Cayley graph Cheeger constants.

Raghunathan gave an introductory survey lecture on the congruence subgroup problem. While this question has been resolved in many cases, the general result seems to require significant new ideas and Raghunathan gave an excellent survey of known methods, their applicability and their limitations.

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