05w5025 Critical Scaling for Polymers and Percolation

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1 Background

Equilibrium statistical mechanics is the mathematical framework created by Gibbs for predicting macroscopic properties of matter from a microscopic description. Within this framework thermodynamic functions of state such as *temperature* and *entropy* are defined in such a way as to satisfy the laws of thermodynamics. In principle the formalism determines the model specific relation called the equation of state. The first example of such a prediction is the famous ideal gas law $P = (n/V)RT$ relating pressure P to the number of atoms/volume and temperature T. This is the equation of state for an assembly of non-interacting particles. This equation of state is notable for having no singularities: in the physical domain $T > 0$ the pressure is a smooth function of temperature. However interacting systems will not in general relate thermodynamic variables in a smooth way and therefore the equation of state has singularities which reflect phase transitions. For example the density of water changes discontinuously as a function of temperature at the boiling point. In practice the determination of the complete equation of state is not realistic for systems with interactions, but the nature of the singularities, the exponents of power law divergences at these singularities, are more accessible. Thus these singularities have been the focus of research.

The basic setup for equilibrium statistics is a probability space Ω whose points $\omega \in \Omega$ are possible configurations of the physical system and a probability measure called the *Gibbs* measure which has the form

$$\frac{1}{Z} e^{-\beta H(\omega)} \times \text{Uniform Measure}$$

where β is proportional to the inverse temperature and H is the energy of the system in configuration ω. Z normalises the Gibbs measure so that it is a probability measure. Z is called the partition function (of β and parameters in H).

In the context of polymer physics we are interested in the thermodynamic properties of large molecules called *polymers* formed by atoms or smaller molecules arranged in a chain or other topologies such as trees (*branched polymers*) or rings. The subunits from which the polymer is formed are called *monomers*.

The goal of the physical chemist is to predict the properties of the polymer starting with knowledge of the forces acting between the monomers. The desired properties include the average size of the polymer as a function of the number of monomers in the polymer, when the number is large. Size is measured by diameter or, in the case of chains, end-to-end distance. This size can exhibit discontinuities when parameters such as temperature are varied. In other words a long chain molecule such as DNA can suddenly change its conformation and therefore its size. These discontinuities are called *theta phase transitions*.

The medium in which the polymer is immersed is called the *solvent*. Instead of starting with detailed information on the solvent molecules and the forces that act between them and the monomer it is common

to leave out any explicit description of the solvent and suppose that the forces experienced by the monomers have been adjusted to take into account the effect of the solvent. In physics this is expressed by the words "the interactions are equivalent to a simplified model with an *effective interaction*". The idea is that a complicated model may have a large scale structure that is the same as a simpler model and we may hope to classify all models into equivalence classes labeled by these simpler models. For example, consider a polymer modeled as a simple random walk: there is a first monomer at $X_0 = 0$ and then the next one in the chain occupies a position X_1 randomly chosen near X_0 according to some probability density p and then the one after that X_2 is chosen independently according to the same density but centred on X_1, ..., i.e. we have a Markov chain $X_0, X_1, \ldots$ of random variables whose law satisfies $P(X_{i+1} = x | X_i = y) = p(x - y)$. The density $p(x - y)$ is likely to be very complicated, being determined by microscopic chemistry. However the theorem of Donsker [16] tells us that if we scale the random variables so as to see only the large scale structure by looking at the chain from far away,

$$Y_t = \lim_{T \to \infty} T^{-\nu} X_{\lfloor Tt \rfloor}, \quad \nu = \frac{1}{2},$$

then the form of p is not important. All that will matter is the matrix of second moments of p. This is called taking the *scaling limit*. No matter how we choose p the scaling limit will be a continuous random path $Y(t)$ called Brownian motion. The probability law of Brownian motion is completely determined by the matrix of second moments. All different choices of p with the same second moments give rise to the same scaling limit. One can take this a step further and show that there is a direction-dependent scaling such that the scaling limit is isotropic, i.e., standard Brownian motion. In this model we see a good example of the notion of a *critical exponent*, namely ν. The existence of the scaling limit for $\nu = 1/2$ implies that the typical random walk with n monomers will have an end-to-end distance of $O(n^{1/2})$.

The Donsker theorem is an ultimate version of the Central Limit Theorem, but the theory of scaling limits starts where the central limit theorem ends. For example, modeling a polymer by simple random walk is rather optimistic: we know that different monomers cannot occupy the same position so we ought at least to consider that, and adopt as a better model, a *self avoiding walk*. More generally we should consider attractive and repulsive interactions. The simple random walk model is the analogue of the ideal gas mentioned in the first paragraph. Once we introduce interactions the existence of a scaling limit for some ν is still largely Terra Incognita. We think the scaling limits exist because we can prove they do in a small number of cases and because this belief now permeates theoretical physics. Almost all theoretical physicists work on models which are vast simplifications of reality. They do so because they think their models classify the large scale structures of reality. The long term health of their enterprise will be improved if we succeed in adding to the list of cases where this can be proved to be true. As with any hard problem, the struggle is spinning off new developments in mathematics and has formed a nice community of researchers with very different backgrounds.

Self-avoiding walk is the archetypical problem that embodies a combinatorial aspect of polymer physics. In this model we generally start with a simple cubic lattice $\mathbb{Z}^d$, hoping that the scaling limit will make the lattice invisible. The points in $\mathbb{Z}^d$ are called sites and they represent the possible positions of a monomer. A long chain molecule consisting of n monomers is represented as a sequence $\omega = (\omega_0, \ldots, \omega_n)$. We define the energy $H(\omega)$ of the polymer ω to be infinite if ω has a self-intersection and zero otherwise. This reduces the role of β in the Gibbs measure to cases $\beta = 0$, in which case the Gibbs measure is the standard measure on simple random walk and $\beta > 0$, in which case the Gibbs measure is the uniform measure on the subset of Ω consisting of self-avoiding walks (SAW). These are the sequences $\omega = (\omega_0, \ldots, \omega_n)$ consisting of *distinct* nearest neighbour sites. Thus the temperature appears in a trivial way in this model. The fundamental question for this model is what is the right scaling: does there exist ν such that the scaling limit exists and what is that limit? How does the end-to-end distance of self-avoiding walk grow as a function of n?

Percolation is the archetypical model for a phase transition. We again start with the simple cubic lattice $\mathbb{Z}^d$. An unordered pair $\{x, y\}$ of nearest neighbour sites is called an edge. Each edge can be either *open* or *closed*. Think of open as meaning that fluid can pass from x to y and closed as meaning that it cannot. A configuration ω of percolation is a possible choice of open/ closed for every edge. We make this choice independently for each edge; edge $\{x, y\}$ is open with probability p and closed with probability $1 - p$. In this model the connection to Gibbs measures is not apparent but there is a connection called the $Fortuin - Kasteleyn$ representation which will not be discussed here. p plays the role of temperature as follows: the

sites of the lattice fall into clusters connected by open edges. One can ask whether there is an infinitely large cluster. For $d > 1$ there is a critical probability $p_c(d)$ such that for $p > p_c$ there is an infinitely large cluster whereas for $p < p_c$ all clusters are finite. Thus the probability that the origin lies in an infinite cluster is zero for an interval $p \in [0, p_c)$ and non-zero for $p \in (p_c, 1]$.[1] The outstanding open questions concern the existence and values of critical exponents. An example of a critical exponent is the expected size of the cluster containing the origin as a function $(p - p_c)^\beta$ for $p > p_c$. In $d = 3$ dimensions the existence of ν and β has not been proved.

One may wonder what this model has to do with polymers. At the outset it was a separate subject but now both models are slowly becoming united in a larger framework of random geometry and there is a commonality of concepts and techniques.

2 Scaling limits in two dimensions

The 1984 paper by Belavin et al. [8] theory started two decades of progress by theoretical physicists in two dimensional statistical mechanics based on conformal field theory (CFT) and more recently also string theory and quantum gravity. In addition there are exact but non-rigorous solutions to lattice models based on the Bethe Ansatz and Yang Baxter equations. We conflate these subjects under the initials CFT.

The book [15] provides a good review: the range of statistical mechanical models for which critical exponents can be calculated (in advance of knowing if they exist!) is remarkable. Mathematicians have yet to find their Euclid for CFT and so they have to regard these calculations as conjectures. There are partial axiomatic programs, for example [18].

The methods of conformal field theory give information on correlations but less directly on random geometry. The family of stochastic processes SLE_κ studied in the work of Lawler Schramm and Werner over the last five years describes the geometric objects within statistical mechanical models. For example, as reviewed in the lecture by Lawler and in [34], the distribution on simple random curves prescribed by SLE(8/3) is the only possible scaling limit for SAW if the scaling limit exists and is conformally invariant. SLE_κ is effective for calculations and some scaling exponents such as ν have been verified to be equal to the values provided by Nienhuis [39] by CFT (actually by first mapping to Solid on Solid models). Thus there is still a very hard open question to show that the discrete process approaches a conformally invariant limit. Another fundamental question is to give the "correct" parametrization of the path which would correspond to the limit of the natural discrete parametrization. This was discussed at this meeting by Lawler and in detail by Kennedy who has examined it by Monte Carlo simulations. Here is a brief summary of Kennedy's lecture.

"$\mathrm{SLE}_{8/3}$ is believed to describe the scaling limit of the two-dimensional self-avoiding walk. However, these two processes have different natural parameterizations. SLE is parameterized by the capacity of the curve, and the length of the SAW leads to a natural parameterization in the scaling limit. One can reparameterize the SAW using its capacity. Monte Carlo simulations were presented which indicate that with this parameterization the SAW process agrees with the SLE process. A more interesting question is to find the parameterization of the SLE that would make it agree with the SAW with its natural parameterization. Lawler gave several properties such a parameterization should have - local dependence on the curve, additivity and an appropriate transformation property under conformal maps which reflects the Hausdorff dimension of the curves. A possible candidate is what probabilists call the "pth variation" where p is taken to be $1/\nu$. Monte Carlo simulations presented showed that while this parameterization makes the SLE agree with the SAW for one random variable, for another random variable there is roughly 6% discrepancy. Understanding the source of this difference is an important open problem for future simulations. Several lines of attack have been developed as a result of conversations with other participants at the conference."

For percolation the geometrical object is the boundary of a percolation cluster for the critical model. Recalling that the critical clusters in percolation are known to be finite in dimension two one needs a method to constrain them to be as large as the scaling limit scale L as the scaling limit $L \to \infty$ is taken. This can be achieved by by using boundary conditions that force a percolation boundary to pass from one side a region of scale L to the other. The scaling limit of the resulting boundary curve has been identified with SLE_6.

[1] In dimension $d = 2$ and in very large dimensions it is known that there is no infi nite cluster at p_c.

Remarkably, existence and conformality in the scaling limit was established by Smirnov, not for the whole model, but for expectations of a specific crossing probability [45].

In this conference Camia gave a report on his work with Newman in which the full scaling limit and conformality of percolation has been established. The crucial point is that this work considers the set of all interfaces as opposed to one forced by boundary conditions. This is fundamental progress because one wants to make contact with the methods of CFT which are based on random fields and one wants to identify the interfaces with contours of the random field. Here is a brief summary of the talk by Camia.

"In my talk, I discussed some aspects of the convergence of critical percolation interfaces to their continuum scaling limits, following a recent joint paper with Charles M. Newman [14]. More specifically, I looked at the "percolation exploration path," conjectured by Oded Schramm to converge to SLE(6) and used by Stanislav Smirnov and Wendelin Werner to rigorously obtain various critical exponents for percolation, and at the set of all percolation interfaces. Percolation is so far the only model for which one can go beyond a single interface and prove the scaling limit of the set of all interfaces. This gives rise to a "full" scaling limit in terms of fractal, continuous loops in the plane. Similar objects should arise when taking the full scaling limit of other models, like Ising and Potts models, and should be described by conformal loop ensembles (CLE) as described by Scott Sheffield and Wendelin Werner in their talks. Such relations, for models other than percolation, are still conjectural. In the case of percolation, the full scaling limit was first constructed in a joint paper with C.M. Newman [13].

The field is rapidly moving forward, and various talks at the meeting showed that the understanding of the continuum counterpart of various discrete models is reaching its maturity. There is hope that this will lead in the future to proofs that the beautiful continuum objects described by Sheffield and Wendelin are indeed the continuum scaling limits of discrete models, extending the results known for percolation to Ising, Potts and O(N) models."

The importance of making complete contact with CFT is illustrated by "Duplantier duality". Using CFT Duplantier noticed that, in the scaling limit, for a spin model or percolation, there must be a relation between the SLE that describes the boundary of a cluster and the SLE that describes the outer boundary of a cluster, namely Duplantier duality $\kappa \to 16/\kappa$. The continuing inspiration coming from these lines of thought is evident in this summary of his lecture.

"I presented a unified heuristic point of view on the Stochastic Loewner Evolution (SLE). It consisted in relating critical exponents for conformally invariant random paths in the plane to similar ones on a random surface with fluctuating metric. The key ingredient was the so-called Knizhnik, Polyakov, and Zamolodchikov (KPZ) relation between these exponents. The status of this relation is to be considered as true in theoretical physics, but conjectural from the point of view of rigorous mathematics. The machinery it provides, however, is strikingly efficient. Any exponent from SLE can be predicted this way. This was illustrated in several instances:

The duality $\kappa \to 16/\kappa$ which is believed to map hulls of SLEs to their external boundaries is reflected by a similar duality built in the two analytical determinations of the inverse KPZ map.

The pressure effect on an SLE path coming from a drift term of strength ρ in the Brownian source of the Loewner equation of that path (the $SLE(\kappa, \rho)$), can be analyzed through the KPZ relation in terms of an equivalent number of multiple SLEs, or of a certain equivalent number of Brownian paths.

The "shadow exponents" describing the probability that some Brownian paths screen some others from the exterior can also be calculated systematically in terms of these Quantum Gravity equivalences.

The multifractal harmonic and rotational spectra of the SLE curves have been obtained from that approach.

A challenge is now to establish the proper rigorous form of this fundamental tool coming from conformal field theory."

And here is a summary of Cardy's lecture which also underlines the need for a complete understanding of CFT.

"There has been a very fruitful interdisciplinary connection formed between the study of critical behaviour by theoretical physicists and the approach to random spatial processes of probabilists. In recent years this has been brought to the fore by the spectacular progress by mathematicians using ideas such as SLE, which have reinterpreted and put on a more systematic basis the earlier results of the physicists in conformal field

theory, inspired by ideas which were born in string theory. Now this subject is reaching a point where there is a mutual flow back and forth between the SLE ideas and CFT. My talk gave a very simple example of this interdisciplinary thinking, concentrating on a small result which, however, both illuminates what should be the correct extension of SLE to many random curves, and also on the physics side relates to potentially measurable phenomena in the quantum Hall effect."

Conformal Field Theory is the study of correlation functions for a random field on a Riemann surface. The simplest example is called the (massless) Gaussian field on the complex plane $\mathbb{C}$. This may also be the fundamental example since a method called the Coulomb gas representation is used to write correlations of other CFT's in terms of the Gaussian field. One immediately discovers that the Gaussian free field ϕ is not actually a random field, but instead is a generalised random function. This means that for each test function f on the plane there is a random variable $\phi(f)$ which would be given by $\phi(f) = \int \phi(x)f(x)\,dx$ except that $\phi(x)$ does not exist because there is too much oscillation at arbitrary small scales. This would seem to be a major obstacle to making a connection between SLE and the free field based on the idea mentioned above: that scaling limits of interfaces should be contours of a conformal field. Thus we were excited by the lectures of Sheffield and Werner in which contours of the free field were defined and related to variants of SLE$_\kappa$. Here is summary of the lecture given by Werner who gave a talk based on joint work with Scott Sheffield.

"Motivated by identifying and understanding better the possible conformally invariant scaling limits of various 2-dimensional models such as O(N) models or the Ising model, we define a natural property that these continuous limits should satisfy:

We are considering random collections of disjoint simple non-nested loops in a domain D. A sample is therefore a collection of loops $(\gamma_j, j \in J)$. We assume conformal invariance so that one can define for any simply connected domain such a law P_D (in a conformally invariant way).

Suppose now that $D' \subset D$. Then, one can define two sets of loops: Those that stay in D' (for which we say that $j \in \tilde{J}$) and those that exit D' (for which $j \in I$). One defines $\tilde{D} = D' \setminus \cup_{j\in I}\gamma_j$. Roughly speaking the condition is that conditionally on $\tilde{D}$, the law of $(\gamma_j, j \in \tilde{J})$ is $P_{\tilde{D}}$.

If this is true for all D', then we say that (γ_j) is a conformal loop ensemble.

We study various properties of these loop-ensembles. In particular, we show that

1) The outer boundaries of loop-soup clusters (related to the Brownian loop-soup introduced in joint work with Greg Lawler) are examples of such loop-ensembles. These examples are parametrized by the intensity of the loop-soup c. This works for all $c \leq c_0$ where c_0 is a critical intensity.

2) In fact these are the only conformal loop-ensembles.

3) The level-lines of the Gaussian Free Field studied by Schramm and Sheffield are other examples of such conformal loop-ensembles and they coincide with the c_0 case in 1).

4) One can construct all these conformal loop-ensembles via SLE related processes, the SLE($\kappa, \kappa - 6$) processes and the relation between κ and c is $c = (3\kappa - 8)(6 - \kappa)/2\kappa$ where $\kappa \in (8/3, 4]$".

3 High Dimensions

For dimensions above two nothing is known about the scaling limits of self-avoiding walk and percolation until one gets to the *critical dimension*, which is four in the case of self-avoiding walk and six in the case of percolation. Above the critical dimension there is a method called the *Lace Expansion* which one can read about in the lecture notes of Slade [44]. The range of usefulness of this tool has expanded very greatly from the original application [12] to a variant of self-avoiding walk. In this conference Sakai reported on a first time application to the Ising model. The significance of this application is that it provides a way to prove "universality" which is the conjecture that drives much of theoretical physics: that the scaling limit is independent of details of the local interactions: refer to the discussion at the beginning of Donsker's theorem. Sakai's success should encourage us to look for ways to extend it to other spin models. Here is a summary of the lecture by Sakai.

"In this talk, I describe a lace expansion for the Ising model, which can be applied to prove Gaussian infrared asymptotics for the critical two-point function for Ising ferromagnets above four dimensions, assuming that the dimension d or the range of the spin-spin coupling is sufficiently large [42, 43]. As a consequence,

the other observables also exhibit the mean-field behavior for $d > 4$ [1, 3, 4, 5]. The main point is that the proof of these results does not require the reflection positivity [17].

For reflection-positive models, it is known that the two-point function obeys a Gaussian infrared bound for $d > 2$. Although the nearest-neighbor model satisfies the reflection positivity, other finite-range models (e.g., the next-nearest-neighbor model) do not. Since local details of the models should not affect the critical behavior (i.e., universality), all these finite-range models must exhibit the same mean-field behavior for $d > 4$, no matter whether the reflection positivity holds or does not. Therefore, our approach using the lace expansion is more robust.

The lace expansion has been used for stochastic-geometrical models, such as self-avoiding walk (e.g., [12, 21]), lattice trees and lattice animals (e.g., [20]), percolation (e.g., [19]), oriented percolation (e.g., [38]) and the contact process (e.g., [41]), to prove a Gaussian infrared bound or asymptotics of the critical two-point function above the upper-critical dimension. The lace expansion gives rise to a recursion equation similar to the one for the random-walk Green's function, and this is the foundation of the Gaussian behavior for the two-point function. The lace expansion for the Ising model has just been proved for the first time [42]."

Applications of the Lace expansion to percolation are also in the news. The important development is the analysis of a cluster constrained to be very large. The idea is that in critical percolation no cluster is infinite but the slightest increase in the density of occupied bonds will cause the critical clusters to link up into an infinite cluster. In some sense one can therefore see the infinite cluster before it has appeared. Here is a summary of the lecture of van de Hofstad based on joint work with Frank den Hollander, Antal Járai and Gordon Slade.

"The incipient infinite cluster (IIC) describes the infinite cluster which is on the verge of arising in critical percolation models. Kesten [31] first constructed the incipient infinite cluster for two-dimensional percolation. Kesten's IIC describes the infinite cluster which is on the verge of appearing at the critical value, and is constructed by conditioning the origin to be connected to infinity by an appropriate limiting procedure. This construction of the IIC is different in spirit as the one suggested by Aizenman in [2], which is closely related to the scaling limit and the behaviour of all critical clusters in a large cube simultaneously, and is also studied in [22, 23] in the high-dimensional case.

We discuss Kesten's results in two dimensions, as well as the extension by Járai [27, 28] for the two-dimensional case and give some of its properties, such as its dimension and its backbone dimension, which follow from the connection to SLE_6 (see [31, 33]).

We also present the constructions of Kesten's IIC for percolation above the upper critical dimension. We will give 2 different constructions for the IIC for sufficiently spread-out percolation above six dimensions, and 3 different constructions for the sufficiently spread-out oriented percolation IIC above 4+1 dimensions. We will also discuss properties of the IIC, such as its dimension and, in the oriented case, its scaling limit. The high-dimensional results are taken from the papers [24, 26, 25].

One reason to study Kesten's IIC is that it is the natural context for a random walk on a critical cluster. Random walk on a super-critical cluster is expected to converge to Brownian motion, which is recently proved by Berger and Biskup (see the talk by Marek Biskup). Kesten [32] studied the random walk on the two-dimensional IIC, and proved that it is subdiffusive. He also proved that a random walk of n-steps on a branching process cluster scales like $n^{1/3}$, which suggests that a random walk on a critical branching random walk cluster conditioned to survive forever, has displacement of order $n^{1/6}$. Since in high-dimensions, Kesten's IIC has similar scaling properties as critical branching random walk cluster conditioned to survive forever, this suggests that a random walk on the IIC has displacement $n^{1/6}$ after n-steps. The latter problem is still open."

There are at least three further developments one should hope for. (1) The Lace expansion applies to models whose interactions are repulsive. A self-avoiding walk with a small nearest-neighbour attraction is not amenable to the Lace expansion. The best attempt so far is [48]. (2) The critical dimension models are not accessible to the Lace expansion. The Renormalisation Group is likely to be the key to progress on these models. You can see a start in this direction at [10, 11]. (3) Stochastic models such as *True Self-Avoiding Walk* are almost Terra Incognita in dimensions greater than one. A good starting point is to develop more understanding of random walk in random environments. Here is a summary of the lecture by Biskup on this topic.

"Random walk in random environment is a subject of considerable interest in probability community. One particular setting concerns simple random walk on supercritical percolation cluster. In 2003, Sidoravicius and Sznitman proved that, in dimensions four and higher, such walk scales to Brownian motion under the usual diffusive scaling of space and time. Their proof uses heavily the path-transience of simple random walk in high dimensions and, as such, it does not seem to be generalizable to include the "hard" dimensions $d = 2$ and 3. In my talk, I have described the recent result – obtained jointly with Noam Berger – that establishes the invariance principle for such walks in any dimension $d > 1$. The principal idea of our proof is to consider the embedding of the percolation cluster into R^d that makes the random walk an L^2 martingale".

Random walk in random environment gives an opening into the study of walks which have attractive self interaction because integrating over the environment constructs a self interaction. A particularly nice example is provided by the self-reinforced walk discussed in the lecture by Rolles. Here is a summary of her lecture. In her third paragraph she is referring to the fact that this model is equivalent to a random walk over an environment which has been integrated out.

"Linearly edge-reinforced random walk was introduced by Diaconis in 1986. Diaconis asked whether edge-reinforced random walk on $\mathbb{Z}^d$ is recurrent or transient. As usually, the random walk is called recurrent if almost all paths visit all vertices infinitely often. For all dimensions $d \geq 2$, this question is still open.

Recently, progress has been made in studying the edge-reinforced random walk on ladders $\mathbb{Z}\times\{1, 2, \ldots, d\}$ and, more generally, on graphs of the form $\mathbb{Z} \times T$ with a finite tree T: For large constant initial weights, recurrence was proved by Merkl and Rolles in [36] and [40]. A more detailed analysis was obtained in [37]. There, it is shown that the edge-reinforced random walk on infinite ladders has the same distribution as a random walk in a random environment given by spatially decaying random edge weights. Convergence theorems and estimates for the position of the random walker at large times are given.

A crucial tool in the analysis is a representation of the edge-reinforced random walk on finite graphs as a mixture of reversible Markov chains; see e.g. [29]. Transfer operator techniques are used to analyze the random environment."

4 Other models

Fortunately we are not constrained to work on the archetypal problems, which are not always making good progress. The archetypal problems arose by distilling challenges from other subjects to their simplest level. New archetypes and fresh ideas will arise from problems such as the localisation problems discussed by Whittington and den Hollander. Here is a summary of their talks.

"A random copolymer is a copolymer in which the sequence of monomers is determined by a random process and is then quenched. Suppose we have two immiscible liquids, A and B, and suppose that one type of monomer prefers to be in the A phase and the other prefers to be in the B phase. For instance, think of the two monomer types as being hydrophobic and hydrophillic and the two liquids as oil and water. Depending on the chosen parameters (e.g., temperature or relative interaction strength) the polymer can localize at the interface so as to optimize the numbers of monomers in their preferred phases, or delocalise into one of the bulk phases to optimize the entropy of the system. One has a choice as to how the configurational properties of the polymer are modeled (e.g., as directed or undirected self-avoiding walks, with possibly other appropriate restrictions). Similarly there is some choice about the details of the interaction Hamiltonian.

The aim is to establish the existence of a phase transition in the system and to study the properties of the phase transition curve. This can be done either at the thermodynamic level or at the level of path properties. Most work has focused on localization at a single infinite flat interface but there has been recent interest and progress in multi-interface and random interface problems, e.g., as models of polymers in an emulsion.

An early paper on the directed walk model is Bolthausen and den Hollander, Localization transition for a polymer near an interface, [9], and some results on the self-avoiding walk model can be found in Madras and Whittington, Localization of a random copolymer at an interface, [35]. For a recent review see Soteros and Whittington, Statistical mechanics of random copolymers, [46]. There are important recent results on the single interface problem by G. Giacomin, and work in progress on random interfaces by den Hollander and Whittington."

Jarai introduced us to self-organised criticality and the relations it has with models we already study such as the uniform spanning tree. Here is a summary of his lecture.

"In the last 15 years, a lot of attention was devoted in the physics literature to so called self-organized critical (SOC) systems. In these systems critical scaling appears with a somewhat different flavour than in the well-known examples of percolation and the Ising model. Namely, criticality is generated by a highly non-local dynamics that is a result of a separation of scales, rather than a parameter passing through a critical point. The main challenge in the area is to develop rigorous methods to study SOC. For some SOC models there is a close connection with a corresponding classical critical model, which opens up the possibility to study SOC via these connections. For example Chayes, Chayes and Newman have shown the intimate relationship between invasion percolation and critical percolation. Dhar and Majumdar have established a mapping between the Abelian sandpile model and the uniform spanning tree, which is the $q \to 0$ limit of the Potts model. In a rigorous approach to SOC, one of the first problems is to establish the existence of infinite volume limits, which can be difficult due to the non-local interaction present. In this regard, recent progress has been made for the Abelian sandpile model based on the connection with spanning trees in dimensions $d > 4$. Future challenges include extending these to lower dimensions. It is expected that in d = 2 conformal invariance plays a role, and therefore a description using SLE is to be explored".

¿From the world of Biology we have a lecture on Vesicles by Buks van Rensburg.

"Vesicles in the biological world, such as blood cells, are known to posses a number of different phases. For example, red blood cells, which are normally shaped as an indented disk (also called "discocytes") become sickled shaped in individuals with cickle cell anemia. In addition, red blood cells could be burred, pinched, pointed, indented, berry-shaped, etc., in either normal or pathological conditions depending on on factors such as dehydration and membrane properties. Vesicles appear to have a very rich phase diagram, and some of these phases can be examined by building mathematical models of a vesicle.

Perhaps the simplest approach to a vesicle would be a two or three dimensional discrete vesicle in the square or cubical lattices. These lattice vesicles include models such as square, partition and convex polygon vesicles in two dimensions, and cubical, rectangular and plane partition models of vesicles in three dimensions. Curvature and osmotic pressure terms van be build into the models by defining partition and generating functions with activities conjugate to volume, area or perimeter. From a mathematical point of view these models offer considerable challenges in combinatorics and in statistical mechanics, and tricritical scaling appears to be the appropriate physical frame for a mathematical description.

In my talk I will present some results on models of three dimensional cubical and rectangular vesicles in a perimeter-area-volume ensemble. These models have a multicritical point and the values of scaling exponents around this point is determined by the asymptotic analysis of the generating functions in both cases. For example, in the volume-area ensemble, the crossover exponent ϕ has value $2/3$ in both these models, and this transition is an inflation-deflation transition between vesicles which are cubical or square shaped".

Finally there was an introduction to one of the outstanding problems of condensed matter physics: to prove that adding a small density of randomly placed impurities to a conductor does not make it an insulator, at least in dimensions three or more. The first first fundamental issue is whether a Schroedinger operator with a random potential has absolutely continuous spectrum. Here is a summary of the lecture by Aizenman on joint work with Robert Sims and Simone Warzel.

"Spectral and dynamical properties of linear operators with extensive disorder are of interest for condensed matter physics, as well as the broad subject of mathematics of graphs and related structures.

An outstanding challenge in the field is to shed light on the existence of extended states in the presence of disorder. Such states play a basic role in conduction properties of an "electron gas" and in the spreading of vibrations in randomized systems. The converse of conduction is Anderson localization which was proven to occur at high disorder, and is particularly pronounced in low dimensions [47], in particular at d=1 where it precludes extended states even at weak levels of the disorder.

The models discussed start from a well familiar Laplacian, or the incidence matrix on a regular graph, which is then modified by incorporating the effect of disorder, represented by random terms with a homogeneous distribution and a control parameter. Specific examples are provided by:

i the discrete Schrdinger operator with a random potential, and

ii the Laplacian acting on functions supported along the edges of a graph, whose edge length are stretched by random factors.

Other models are not hard to formulate; the obstacles encountered in their analysis are similar. The challenge is to prove that if the dimension is high enough, and the disorder not too strong, then the spectral measures associate with the action of the random operators on local functions have an absolutely continuous (ac) component. Such delocalised states associated with ac spectrum are expected to occur in dimensions $d > 2$. The issues under study resemble a quantum version of percolation, in that it concerns conduction, or the spread of correlations, connectivity, etc., occurring through a local mechanism but measured at a distance. A common experience is that moderate dimensions, like $d = 3$, are out of reach of mathematics. For present case the spectral analysis has been most effective in dimension $d = 1$, where no extended states exist at any strength of the disorder, and at the opposite extreme of the tree graphs, which in a sense represent the case of infinite dimension. In fact, the latter situation is the only for which existence of delocalised states has been established; a proof due to A. Klein [30]. The talk presented a new method for establishing the persistence of ac spectra on tree graphs [6, 7].

The main result is the continuity of the Lebesgue measure of the ac spectrum as the disorder is turned on. The proof makes an essential use of the study of the fluctuations of the relevant Green functions, which are bounded through a non-linear recursion relation which these obey on a tree. Useful input is obtained from the analytical theory of the Lyapunov exponent. The new criterion for the continuity of the ac spectrum has been applied also to the case of radial quasi-periodic potentials. These allow to draw an instructive contrast between the effects of radially correlated disorder versus one which is weakly correlated among different tree branches."

References

[1] M. Aizenman. Geometric analysis of ϕ^4 fields and Ising models. *Commun. Math. Phys.* **86**: 1–48, 1982.

[2] M. Aizenman. On the number of incipient spanning clusters. *Nuclear Phys. B*, **485**: 551–582, 1997.

[3] M. Aizenman, D.J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *J. Statist. Phys.*, **47**:343–374, 1987.

[4] M. Aizenman and R. Fernández. On the critical behaviour of the magnetization in high dimensional Ising models. *J. Statist. Phys.*, **44**:393–454, 1986.

[5] M. Aizenman and R. Graham. On the renormalized coupling constant and the susceptibility in ϕ^4_4 field theory and the Ising model in four dimensions. *Nucl. Phys.*, **B225** [FS9]:261–288, 1983.

[6] M. Aizenman, R. Sims, and S. Warzel. Stability of absolutely continuous spectrum of random Schrdinger operators on tree graphs. http://arxiv.org/abs/math-ph/0502006.

[7] M. Aizenman, R. Sims, and S. Warzel. Absolutely continuous spectra of quantum tree graphs with weak disorder. http://arxiv.org/abs/math-ph/0504039.

[8] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B*, **241**(2):333–380, 1984.

[9] Erwin Bolthausen and Frank den Hollander. Localization transition for a polymer near an interface. *Ann. Probab.*, **25**(3):1334–1366, 1997.

[10] David C. Brydges and John Imbrie. End-to-end distance from the Green's function for a hierarchical self-avoiding walk in four dimensions. *Comm. Math. Phys.*, **239**(3):523–547, 2003.

[11] David C. Brydges and John Z. Imbrie. Green's function for a hierarchical self-avoiding walk in four dimensions. *Comm. Math. Phys.*, **239**(3):549–584, 2003.

[12] D. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Commun. Math. Phys.* **97** : 125–148, 1985.

[13] Federico Camia and Charles M. Newman. Continuum nonsimple loops and 2D critical percolation. *J. Statist. Phys.*, **116**(1-4):157–173, 2004.

[14] F. Camia and C. Newman. The full scaling limit of two-dimensional percolation. arXiv:math.PR/0504036.

[15] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[16] Monroe D. Donsker. An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.*,, **1951**(6):12, 1951.

[17] J. Fröhlich, B. Simon, and T. Spencer. Infrared bounds, phase transitions, and continuous symmetry breaking. *Commun. Math. Phys.*, **50**:79–95, 1976.

[18] Matthias R. Gaberdiel and Peter Goddard. Axiomatic conformal field theory. *Comm. Math. Phys.*, **209**(3):549–594, 2000.

[19] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.*, **128**:333–391, 1990.

[20] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.*, **59**:1469–1510, 1990.

[21] T. Hara and G. Slade. Self-avoiding walk in five or more dimensions. I. The critical behaviour. *Commun. Math. Phys.* **147**: 101–136, 1992.

[22] T. Hara and G. Slade. The scaling limit of the incipient infinite cluster in high-dimensional percolation. I. Critical exponents. *J. Stat. Phys.*, **99**:1075–1168, 2000.

[23] T. Hara and G. Slade. The scaling limit of the incipient infinite cluster in high-dimensional percolation. II. Integrated super-Brownian excursion. *J. Math. Phys.*, **41**:1244–1293, 2000.

[24] R. van der Hofstad. Infinite canonical super-Brownian motion and scaling limits. Preprint 2004.

[25] R. van der Hofstad, F. den Hollander and G. Slade. Construction of the incipient infinite cluster for spread-out oriented percolation above $4+1$ dimensions. *Commun. Math. Phys.* **231** 435–461, 2002.

[26] Remco van der Hofstad and Antal A. Járai. The incipient infinite cluster for high-dimensional unoriented percolation. *J. Statist. Phys.*, **114**(3-4):625–663, 2004.

[27] A.A. Járai. Invasion percolation and the incipient infinite cluster in 2D. *Commun. Math. Phys.* **236**: no. 2, 311–334, 2003.

[28] A.A. Járai. Incipient infinite percolation clusters in 2D. *Ann. Probab.* **31** 444–485, 2003.

[29] M.S. Keane and S.W.W. Rolles. Edge-reinforced random walk on finite graphs. In *Infinite dimensional stochastic analysis (Amsterdam, 1999)*, pages 217–234. R. Neth. Acad. Arts Sci., Amsterdam, 2000.

[30] Abel Klein. Extended states in the Anderson model on the Bethe lattice. *Adv. Math.*, **133**(1):163–184, 1998.

[31] H. Kesten. The incipient infinite cluster in two-dimensional percolation. *Probab. Theory Related Fields*, **73** 369–394, 1986.

[32] H. Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. Henri Poincaré*, **22** 425–487, 1986.

[33] G.F. Lawler, O. Schramm and W. Werner. One-arm exponent for critical 2D percolation. *Electron. J. Probab.*, **7**, 13 pp., electronic. 2002.

[34] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. On the scaling limit of planar self-avoiding walk. In *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2*, volume 72 of *Proc. Sympos. Pure Math.*, pages 339–364. Amer. Math. Soc., Providence, RI, 2004.

[35] N. Madras and S. G. Whittington. Localization of a random copolymer at an interface. *J. Phys. A*, **36**(4):923–938, 2003.

[36] F. Merkl and S.W.W. Rolles. Edge-reinforced random walk on a ladder. To appear in The Annals of Probability, 2005.

[37] F. Merkl and S.W.W. Rolles. Asymptotic behavior of edge-reinforced random walks. Preprint, 2004.

[38] B.G. Nguyen and W-S. Yang. Triangle condition for oriented percolation in high dimensions. *Ann. Probab.*, **21**:1809–1844, 1993.

[39] B. Nienhuis. Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas. *J. Stat. Phys.*, **34**:731–761, 1984.

[40] S.W.W. Rolles. On the recurrence of edge-reinforced random walk on $\mathbb{Z} \times G$. Preprint, 2004.

[41] A. Sakai. Mean-field critical behavior for the contact process. *J. Statist. Phys.* **104**: 111–143, 2001.

[42] A. Sakai. Lace expansion for the Ising model. I. Derivation of the expansion. *Preprint* 2005.

[43] A. Sakai. Lace expansion for the Ising model. II. Bounds on the expansion coefficients. In preparation.

[44] G. Slade. The Lace expansion and its applications. Lecture notes for the 2004 Saint Flour Summer School. Preprint.

[45] S. Smirnov. Critical percolation in the plane. I. Conformal invariance and Cardy's formula. II. Continuum scaling limit. Preprint, 2001.

[46] C. E. Soteros and S. G. Whittington. The statistical mechanics of random copolymers. *J. Phys. A*, **37**(41):R279–R325, 2004.

[47] Peter Stollmann. *Caught by disorder*, volume 20 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2001. Bound states in random media.

[48] Daniel Ueltschi. A self-avoiding walk with attractive interactions. *Probab. Theory Related Fields*, **124**(2):189–203, 2002.