Directions in Combinatorial Matrix Theory

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The Directions in Combinatorial Matrix Theory workshop was held at BIRS May 7–8, 2004, and attracted 29 researchers (10 from Canada, 15 from the U.S. and 4 from abroad) and 7 post-doctoral or graduate students. Talks discussed current developments and open problems in the following emerging themes in Combinatorial Matrix Theory: Spectral properties of families of matrices associated with graphs; Matrix theory and graph theory in the service of Euclidean geometry; Algebraic tools for combinatorial problems; and Spectral properties of classes of matrices. Titles and abstracts of the talks presented can be found at

http://www.pims.math.ca/birs/workshops/2004/04w2525/abstracts.pdf.

Below each of these themes is briefly discussed.

Spectral properties of families of matrices associated with graphs

Numerous connections between the spectrum of the adjacency matrix of a graph G and the graphical and combinatorial properties of the graph have been long, and fruitfully explored. An emerging, promising trend is to study the spectra of an entire class of matrices associated with G. More specifically, let $S(G)$ denote the set of all symmetric matrices whose graph is G (i.e. all symmetric n by n matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if there is an edge in G joining vertex i and j.) Fundamental questions in this area are:

(a) What combinatorial and geometric properties of a graph G can be ascertained from the invariants of $S(G)$?

(b) What constraints are placed upon the spectrum of matrices in $S(G)$ by the graphical properties of G?

Perhaps the first results along these lines are the early papers of Parter [3] and Fiedler [1] that establish some striking results about the spectrum of acyclic matrices, that is, matrices in $S(G)$, where G is a tree. More recently, the Colin de Verdière invariant [4] (which is the maximum of the second smallest eigenvalue of matrices in a special subset of $S(G)$) has been shown to have deep connections with the embedability, and hence the geometry, of G.

Several conference talks presented new results about the minimum rank, or equivalently the maximum multiplicity of an eigenvalue, of a matrix in $S(G)$, and the inverse eigenvalue problem for

$S(G)$ (that is, determine necessary and sufficient conditions for the ordered sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ to be the ordered eigenvalue list of some matrix in $S(G)$).

Matrix theory and graph theory in the service of Euclidean geometry

The Gram-matrix $M = [\langle v_i, v_j \rangle]$ of a collection $v_1, v_2, \ldots, v_n$ of vectors in a Euclidean space stores much information about the geometric arrangement of the vectors, and often matrix theoretic properties of M produce startling corresponding geometric properties of the vectors.

One presentation at the workshop, illustrated the promise of this research theme, by deriving necessary and sufficient conditions for the existence of a simplex whose edges make prescribed (perpendicular, acute, or obtuse) angles. Future directions include extending this characterization to other families of polytopes.

Symmetric factorizations $A = BB^T$ (B (entry wise) nonnegative) of a nonnegative matrix A correspond geometrically to a system of nonnegative vectors with prescribed inner products. Such factorizations arise in image processing, physics, and statistical applications. Two fundamental questions are:

(a) If A has a symmetric nonnegative factorization $A = BB^T$ what is the smallest possible number k of columns in such a B? What is the largest k can be for a fixed dimension of A?

(b) If A is an integer matrix, how can one determine whether or not A has a factorization $A = BB^T$ where B is a $(0, 1)$-matrix?

Algebraic tools for tackling combinatorial problems

The Yin of Combinatorial Matrix Theory (CMT) is the use of combinatorial ideas and theorems to more closely analyze problems in matrix theory. The Yang of CMT is to use algebraic concepts and results to tackle combinatorial problems. Perhaps the prime example of the Yang in CMT is the Witsenhausen-Graham-Pollak theorem [2], that asserts that every biclique partition of the complete graph K_n has at least $n-1$ bicliques, and whose only known proofs are all based on linear algebra.

The Yang of CMT was represented at the conference through talks on a number of subjects. These include: results on the behavior of the inertia of a matrix when perturbed, and their use in the study of minimum biclique partitions of various families of graphs; a survey of the wide range of matroid theoretic, and graph coloring problems arising in Combinatorial Scientific Computing; a survey of the major questions and a proposed uniform theory for the various generalizations (e.g. Hadamard matrices, weighing matrices, symmetric designs, Type II matrices, etc.) of orthogonal matrices; and a talk using number theory to obtain new results on the long-standing problem of classifying the graphs with integral spectrum.

Spectral properties of classes of matrices

Several promising Yin directions in CMT were discussed.

New families of spectrally arbitrary n by n sign patterns (that is, a sign pattern with the property that every conjugate closed multi-set of n complex numbers is the spectrum of some matrix with the given sign pattern) were presented, and a field theoretic argument was presented to show that if A is an irreducible, spectrally arbitrary sign pattern, then A has at least $2n-1$ nonzero entries. This leaves the intriguing open problem:

> Is there an n by n, irreducible, spectrally arbitrary sign pattern with $2n-1$ nonzero entries?

A talk concerning the possible Jordan Canonical Forms of a nonnegative matrix with prescribed eigenvalues of largest modulus illustrates that while the Perron-Frobenius theory for nonnegative matrices began over 100 years ago, there are still many interesting open questions about nonnegative matrices to be resolved.

Several results and problems concerning the relationship between the sign patterns of commuting matrices were discussed. For example, an interesting open problem is to determine necessary and sufficient conditions for a pair of sign patterns to allow a pair of commuting matrices.

The Workshop's Open Problem sessions were highly successful, and a list of problems presented will be posted at

http://www.pims.math.ca/birs/workshops/2004/04w2525/openprobs.pdf .

New collaborative efforts resulting from the workshop are already noticable, especially among the post-docs and graduate students. Results presented at the conference will be disseminated through a special 2005 issue of the *Electronic Journal of Linear Algebra*.

In summary, the future directions for research in Combinatorial Matrix Theory are abundant, promising, and central to mathematics.

References

[1] M. Fiedler, Eigenvectors of acyclic matrices. Czechoslovak Math. J. 25 (1975), 607–618.

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[3] S. Parter, On the eigenvalues and eigenvectors of a class of matrices. J. Soc. Indust. Appl. Math. 8 (1960) 376–388.

[4] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, 137–147.