Report of 04rit553

R.V. Gurjar, K. Masuda, M. Miyanishi, P. Russell

The four members of our team met at BIRS from July 24 to August 14, 2004. We started with several questions related to our proposed project of study of (proper or étale) self-maps of smooth affine surfaces. It soon became clear that the study of different $\mathbf{A}^1$-fibrations and existence of affine lines on such a surface which are not fibre components of a given fibration is quite important for our purpose. Since our main aim is the study of affine rational surfaces without non-trivial regular invertible functions, the base of an $\mathbf{A}^1$-fibration for such a surface is either the affine line $\mathbf{A}^1$ or the projective line $\mathbf{P}^1$. We realised that the base of the fibration makes a lot of difference to the properties of our surfaces. We defined several classes of smooth affine rational surfaces, depending on how many different $\mathbf{A}^1$-fibrations the surface has and what the base of such a fibration is. The study of interrelations between these classes formed an important part of our work. In the description below, by a surface we mean a smooth complex, affine surface with no non-trivial regular invertible functions.

A surface X is called an ML_0-surface if X has at least two transverse $\mathbf{A}^1$-fibrations with base $\mathbf{A}^1$. It is called an ML_1-surface if it has exactly one such fibration with base $\mathbf{A}^1$ and an ML_2-surface if it has no $\mathbf{A}^1$-fibration over $\mathbf{A}^1$.

The main results which we have proved so far are as follows. The assumption about torsionness of the Picard group of X in some statements below is quite natural in our context since we intend to study surfaces which are as close to the affine plane $\mathbf{C}^2$ as possible, since the famous unsolved Jacobian Problem is always at the back of our mind.

(1) (i) Let X be an ML_0 surface with torsion Picard group. Then any affine line $C \subset X$ is a fiber of an $\mathbf{A}^1$-fibration over X with base $\mathbf{A}^1$.

(ii) If X is an ML_1 surface with torsion Picard group then any affine line contained in X is a fiber of the unique $\mathbf{A}^1$-fibration on X.

These results generalize the famous Abhyankar-Moh-Suzuki result about uniqueness of embedding of an affine line (upto automorphism of $\mathbf{C}^2$) in

$\mathbf{C}^2$. The proof uses full force of the classification theory of non-complete surfaces.
If the rank of the Picard group of X is > 0 then we found counterexamples to this result. The constructions use special types of surfaces.

(2) Let X be an ML_0 surface with torsion Picard group. If $f : X \to Y$ is either an étale finite or a finite Galois map onto the surface Y then Y is an ML_0 surface. If f is assumed to be only proper then Y is at least ML_1.
The question whether the result is true only assuming properness of f is a tantalizing one and is closely connected to our project of proper self-maps.

(3) We found examples of ML_0 and ML_1 surfaces X such that in some cases C^2 is an open subset of X and in some cases it is not. Our understanding of this phenomenon is far from being complete.

(4) A somewhat unexpected result proved by us says that if X is an ML_0 surface such that the Picard group of X has rank > 0 then X always has an $\mathbf{A}^1$-fibration with base $\mathbf{P}^1$.
For the proof we had to study the set of all possible $\mathbf{A}^1$-fibrations on X with base $\mathbf{A}^1$ and an analysis of the ring generated by all the base parameters for these fibrations.

These are some of the main results we have proved during our stay at BIRS. We also raised some new and interesting questions, the answers to which will be quite important for the study of self-maps. Although the members of our team have known each other for a long time, collaborated in pairs, met each other on several occasions for short durations and followed each other's work closely, this is the first time we got together for intense discussions about problems of interests to all of us. The three week period has been mathematically very exhilarating for each of us and has given us much to think about when we go back to our respective places of work and to continue our collaboration by e-mail.

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