Centers of mass of convex bodies

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The research efforts of our team were dedicated to investigation of questions regarding the mutual positions of various centers of mass in a convex body. First studies of centers of mass (also known as centers of gravity or centroids) of solids go back to Archimedes, yet many related open problems in various disciplines (engineering, physics, astronomy, to name a few) fuel the ongoing work. Let K be a convex body in the n-dimensional Euclidean space $\mathbb{R}^n$, that is, K is a compact convex set with non-empty interior. The centroid of K is the point

$$c(K) = \frac{1}{|K|}\int_K x\,dx,$$

where $|K|$ denotes the volume of K and the integration is with respect to the standard Lebesgue measure.

An inequality of Grünbaum [3] (see also [6]) gives a lower bound for the volume of a portion of a convex body lying on one side of hyperplane that slices the convex body through its centroid. Namely, Grünbaum's inequality states that any cut through the centroid of a convex body K divides K into two parts K_+ and K_- that are not too small,

$$\frac{|K_+|}{|K|} \geq \left(\frac{n}{n+1}\right)^n \quad \text{and} \quad \frac{|K_-|}{|K|} \geq \left(\frac{n}{n+1}\right)^n. \tag{1}$$

In the past few years, there has been a significant progress in extending Grünbaum's inequality, such as generalizations of Grünbaum's inequality for projections and sections [2, 5, 7, 13].

Since the centroid of a section of K through $c(K)$ does not necessary coincide with $c(K)$, this gives rise to numerous questions regarding their relative positions. The analogous observation also holds for projections. Naturally, one may wonder how far apart these points can be located.

Recently in [8], we found an optimal upper bound on the distance between a projection of the centroid of a convex body K and the centroid of the projection of K. The analogue of the latter question for sections stated by Stephen in [12] and motivated by Grünbaum-type inequalities is currently open. More precisely, what is the smallest constant $C_n > 0$ such that for every convex body $K \subset \mathbb{R}^n$ and every hyperplane H through $c(K)$, we have

$$|c(K) - c(K \cap H)| \leq C_n\, d_K, \tag{2}$$

where d_K is the diameter of K? During our time in BIRS, we worked on this problem and considered several ways to adjust our argument in [8] to the case of sections. We also investigated a variation of the above question for a length of the chord that contains the centroid of the body and the centroid of the section of K instead of diameter d_K.

In addition to the questions described above, we also investigated closely related problems. For instance, the following question was asked by Grünbaum and Loewner (see, for example, [1, A8]):

Is it true that the centroid $c(K)$ of a convex body $K \subset \mathbb{R}^n$ is also the centroid of at least $n+1$ different hyperplane sections of K through $c(K)$?

In $\mathbb{R}^2$, this problem has the affirmative answer. The centroid of a triangle bisects three chords that are parallel to the sides. In [9], using Fourier analytic tools, we recently showed that this problem has a negative answer for $n \geq 5$. However, this question remains open in dimensions $n = 3$ and $n = 4$. At BIRS, we identified possible approaches to tackle the problem via the Fourier transform techniques.

In [4], Grünbaum also claimed that there exists a point in the interior of K which is the centroid of at least $n + 1$ different hyperplane sections of K (the previous question is more particular, and asks whether the centroid $c(K)$ belongs to the set of such points). However, Patáková, Tancer and Wagner [11] discovered that one of the auxiliary statements in Grünbaum's argument is incorrect. Additionally they proved that every convex body in $\mathbb{R}^n, n \geq 3$ contains a point p that is the centroid of at least four hyperplane sections through p. This leaves the question about existence of such a point open in dimensions $n \geq 4$. We worked on this problem during our stay at BIRS as well.

References

[1] H. Croft, K. Falconer, R. K. Guy, UNSOLVED PROBLEMS IN GEOMETRY, Probl. Books in Math., Unsolved Probl. in Intuitive Math., II Springer-Verlag, New York, 1991. xvi+198 pp., ISBN:0-387-97506-3

[2] M. Fradelizi, M. Meyer, V. Yaskin, ON THE VOLUME OF SECTIONS OF A CONVEX BODY BY CONES, Proc. Amer. Math. Soc. 145 (2017), 3153–3164.

[3] B. Grünbaum, PARTITIONS OF MASS-DISTRIBUTIONS AND OF CONVEX BODIES BY HYPERPLANES, Pacific J. Math. 10 (1960), 1257–1261.

[4] B. Grünbaum, MEASURES OF SYMMETRY FOR CONVEX SETS, Proc. Sympos. Pure Math, vol. 7 (1963), 233–270.

[5] M. Meyer, F. Nazarov, D. Ryabogin, V. Yaskin, GRÜNBAUM-TYPE INEQUALITY FOR LOG-CONCAVE FUNCTIONS, Bull. Lond. Math. Soc. 50 (2018), no. 4, 745–752.

[6] B. S. Mityagin, TWO INEQUALITIES FOR VOLUMES OF CONVEX BODIES (Russian), Mat. Zametki 5, (1969), 99–106.

[7] S. Myroshnychenko, M. Stephen, N. Zhang, GRÜNBAUM'S INEQUALITY FOR SECTIONS, Journal of Functional Analysis (275) 2018, 2516–2537.

[8] S. Myroshnychenko, K. Tatarko, V. Yaskin, HOW FAR APART CAN THE PROJECTION OF THE CENTROID OF A CONVEX BODY AND THE CENTROID OF ITS PROJECTION BE?, Math. Ann. (2024).

[9] S. Myroshnychenko, K. Tatarko, V. Yaskin, ANSWERS TO QUESTIONS OF GRÜNBAUM AND LOEWNER, preprint.

[10] F. Nazarov, D. Ryabogin, V. Yaskin, ON THE MAXIMAL DISTANCE BETWEEN THE CENTERS OF MASS OF A PLANAR CONVEX BODY AND ITS BOUNDARY, Discrete & Computational Geometry (2024).

[11] Z. Patáková, M. Tancer, U. Wagner, BARYCENTRIC CUTS THROUGH A CONVEX BODY, Discrete & Computational Geometry 68 (2022), 1133–1154.

[12] M. Stephen, SOME PROBLEMS FROM CONVEX GEOMETRY AND GEOMETRIC TOMOGRAPHY, University of Alberta, 2018.

[13] M. Stephen, N. Zhang, GRÜNBAUM'S INEQUALITY FOR PROJECTIONS, Journal of Functional Analysis 272 (2017), 2628–2640.