Stochastic lattice differential equations and applications

Xiaoying Han (Auburn University),
Hakima Bessaih (Department of Mathematics, University of Wyoming),
Maria J. Garrido-Atienza (University of Sevilla),
Bjorn Schmalfuss (Institut for Mathematics)

September 10 - 17, 2017

1 Introduction

Lattice dynamical systems arise naturally in a wide range of applications where the spatial structure has a discrete character, such as image processing, pattern recognition, and chemical reaction theory. For some cases, lattice dynamical systems arise as discretization of partial differential equations, while they can be interpreted as ordinary differential equations in Banach spaces which are often simpler to analyze.

In particular, lattice systems have been used in biological systems to describe the dynamics of pulses in myelinated axons where the membrane is excitable only at spatially discrete sites. For example, dynamics of excitable cells can be described by the following lattice system:

$$\frac{\partial v_n}{\partial t} = d(v_{n+1} - 2v_n + v_{n-1}) + f(u_n, v_n), \tag{1}$$

$$\frac{\partial u_n}{\partial t} = g(u_n, v_n), \tag{2}$$

where v represents the membrane potential of the cell, u comprises additional variables such as gating variables, chemical concentrations, necessary to the model, the subscript n indicates the nth cell in a string of cells, and d is the coupling coefficient $d = 1/R$, where R is the inter-cellular resistance. Typical models that can be fit in the scheme of (1) - (2) include Beeler-Reuter, Hodgkin-Huxley, FitzHugh-Nagumo, and many other models.

Lattice systems have also been used in fluid dynamics to describe the fluid turbulence in shell models. For example, the GOY and Sabra models:

$$\left(\frac{\partial}{\partial t} + \nu k_n^2\right) u_n = i(a_n u_{n+1} u_{n+2} + b_n u_{n-1} u_{n+1} + c_n u_{n-1} u_{n-2})^* + f_n, \tag{3}$$

where u represents the complex modes, n denotes the "shell index" that runs from 1 to ∞, and $*$ stands for complex conjugation.

Most systems in natural sciences and engineering are affected by uncertainty. From modeling point of view, to take into account the uncertainty, random effects have to be included. These random effects are considered not only as compensations for the defects in some deterministic models, but rather essential phenomena. Stochastic lattice systems come into play to describe systems with discrete spatial structure and random spatio-temporal forcing, i.e., noise. More specially, the term $f(u_n, v_n)$ ca be described as follows:

$$f(u_n, v_n) = \sigma(u_n, u_n) dB_n^H(t) \tag{4}$$

where $B_n^H(t)$ is Brownian (or fractional) Brownian motion of Hurst parameter H.

2 Summary of activities held at the BIRS station

Our research team was scheduled to meet on September 11-18, 2016 but that date got re-scheduled because of time conflicts with some members of the team. The meeting was held a year later: September 10-17, 2017. Hence, part of the research proposed initially was already finished before the meeting, see [1].

The team had lively discussion during their stay at the BIRS station and brainstorming ended up very positive and we have already a few new projects lined up. We will inform BIRS when manuscripts are ready and will acknowledge BIRS support. In particular, we have 3 defined projects outlined below:

Problem I. Synchronization of lattice equations

The synchronization of systems in physics and other sciences is an important issue. For an overview we refer to A. Pikovsky et al. First time that synchronization was considered from the point of view of physics goes back to C. Huygens. He observed in 1665 that two pendulum clocks which hang at the same wall or another structure oscillate synchronized after some time. Similar phenomena can be observed in other systems. To forecast synchronization in physical systems it is necessary to formulate mathematical models and to find mathematical tools which allow to interpret these phenomena in the sense of dynamical systems. An appropriate model for our physical system should contain random parameter. Hence the theory of random dynamical systems seems to provide the right tools to deal with the systems mentioned above.

The subgroup of the Focused Research Team by Bessaih, Garrido-Atienza and Schmalfuß dealt to formulate a research plan for the following system. We consider the following lattice system

$$du_i(t) = (\nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f_i(u_i))\,dt + \sigma_i h_i(u_i) d\omega_i(t),\ i \in \mathbb{Z}. \tag{5}$$

We consider any of these equation as a nonlinear model of a physical grid. Any grid is coupled with the neighbor elements of the grid by a linear operator. The subsystems of the grid is perturbed by a noise $d\omega_i$ (either Brownian or fractional Brownian motion).

Consider now two or a finite number of parallel grid. The mathematical model is a system of equations like (5) where maybe the nonlinear parts can vary form equation to equation. These models also contain an operator which is responsible for the coupling of the systems of the different grids. In the case of two systems we consider

$$\begin{aligned} du =& (\nu A u - \Lambda u - \kappa K_1(u - v) + F(u))dt + d\omega \\ dv =& (\nu A v - \Lambda v - \kappa K_2(v - u) + G(v))dt + d\hat{\omega} \end{aligned} \tag{6}$$

which is now a system of stochastic evolution equations in the space of square integrable functions such that $u,\, v \in l_2$. Here A denotes a (generalized) linear operator responsible for the interaction of neighbored elements in each of the grids. The coupling between different grids is given by the operators K_i.

The first step we had to overcome is to ensure that the system of equations generates a random dynamical system ϕ^κ which depends on the intensity κ of the coupling.

One of our goals is to proof the following theorem:

Theorem 1 *Assume appropriate assumptions such that* (6) *generates a random dynamical system on $l_2 \times l_2$. Assume that K_j is appropriate.*

- *Then for $\kappa > \kappa_0$ the random dynamical system has a random attractor $\mathcal{A}^\kappa$.*

- *We have*
$$\lim_{\kappa\to\infty} d_{l_2\times l_2}(\mathcal{A}^\kappa, \mathcal{A}^\infty) = 0$$
where $\mathcal{A}^\infty = \mathcal{A}^0 \times \mathcal{A}^0$ *to be the attractor of the (random) dynamical system generated by*
$$du = (\nu Au - \Lambda u + \frac{1}{2}(F(u) + G(v))dt + \frac{1}{2}d(\omega - \hat{\omega}).$$

- *For* $\kappa > \kappa_0$ *and the additional assumption* $F = G$ *we have synchronization on the level of trajectories, namely*
$$\lim_{t\to\infty} \|u(t) - v(t)\|_{l_2} = 0.$$

To interpret this theorem we note at first that a (random) attractor is a small often finite dimensional set in a infinite dimensional phase space which is in our case $l_2 \times l_2$. This set contains essential information about the long term behavior of a system. The second statement of the above theorem allows us to conclude that for large intensity parameter the motion of the coupled system takes place in a neighborhood of the product of the attractor of an averaged system. The third statement teaches us that the long time states of the subsystems u, v are synchronized. We are convinced that there are many applications for our results.

Problem II. Dynamics of lattices generated by partial differential equations with nonlocal terms

T. Caraballo proposed and discussed with the organizers some details about the idea of studying the asymptotic behaviour of the lattice dynamical system generated by the discretization of this problem:

$$\begin{cases} \dfrac{du}{dt} - a(l(u))\Delta u = f \quad t > 0, x \in \mathbb{R} \\ u(0) = u_0 \in L^2(\mathbb{R}). \end{cases} \tag{7}$$

We consider $l(u) = \int_{\mathbb{R}} u(x)dx$, the function a is continuous and bounded by two positive constants:

$$0 < m \le a(s) \le M \quad \forall s \in \mathbb{R}.$$

A tentative discretized problem could be

$$\begin{cases} \dfrac{du_i}{dt} = a(\tilde{l}(\overrightarrow{u}))(u_{i-1} - 2u_i + u_{i+1}) + f \quad t > 0, x \in \mathbb{R} \\ u_i(0) = (u_0)_i, \end{cases} \tag{8}$$

where, for instance, we could use the trapezium formula, $l(u) \approx \widetilde{l}(\overrightarrow{u})$ to discretize the nonlocal term, although other alternatives could be considered as well.

The main objectives could be:

1. To prove that (8) can be rewritten like a system of ordinary differential equations (lattice)
$$\begin{cases} \dfrac{d\overrightarrow{u}}{dt} = a(\tilde{l}(\overrightarrow{u}))A\overrightarrow{u} + f = F(\overrightarrow{u}) \\ u(0) = u_0 \in l^2, \end{cases} \tag{9}$$
where $\overrightarrow{u} = (u_i)_{i\in\mathbb{Z}}$.

2. To prove existence and (eventually) uniqueness of solution in l^2 for the problem (9). As far as we know, this is a new problem still not analyzed in the literature.

3. To study the existence of attractors.

4. To analyze more general frameworks (considering more general terms like $f(u)$, $N > 1$).

During the discussions, it was emphasized that, before carrying out this program, it would be necessary a deeper study about the PDE model with nonlocal term and all of its dynamical aspects. In fact, there exists an extensive literature already published on this model with interesting applications to some problems of the applied sciences. This means that this is a long term program which cannot be developed in a short stay, but that it is worth being analyzed in future.

Problem III. Neural oscillator networks models

Han proposed a stochastic lattice dynamical system arising from neural oscillator networks. The system reads:

$$\dot{\theta}_i = f_i + z(\theta_i)(\sum_{j \neq i} a_{ji} g(\theta_j) + \epsilon_i I(t)). \tag{10}$$

Here the unknown variables θ_i are the states of the phase oscillators, i.e., angles parametrized by numbers in the interval $[0, 1]$ with periodic boundary conditions. The coupling matrix a_{ji} describes the network structure, and the function $z(\theta)$ measures how big an effect an input has on a neuron in state θ. $I(t)$ in an external input and ϵ_i is the intensity of external forcing current at the ith neuron.

To align with the topic of the focused group research, Han proposed to study the dynamics of the above lattice system where $I(t)$ is a random input, such as, Brownian motion or fractional Brownian motion. The group had a lively discussion on what the exact meaning of the model was, and what useful information could be obtained by studying such a system. In the end the group concluded that deeper understanding is needed to further consider this system.

During the rest of their stay, the subgroup formed by T. Caraballo and X. Han started an extensive literature review on the system (10), and gained an overview of system (10) as well as lattice models for neural networks in general. They initiated a detailed plan to study the stability and attractivity of system (10) and its generalizations. The plan involves T. Caraballo, X. Han and one of X. Han's current Ph.D. student. Two weeks after the closure of this focused research group, T. Caraballo visited X. Han at Auburn University, when the three of them were able to carry out the plan and obtain necessary analysis. Primitive calculations indicate that system (10) is stable and attractive in a proper sequence space under certain assumptions. More solid results are expected next year.

References

[1] H. Bessaih, M.J. Garrido-tienza, X. Han, B. Schmalfuß, *Stochastic Lattice Dynamical Systems with Fractional Noise*, SIAM Journal Anal. Math., 49(2):1495–1518, 2017.