Whittaker Functions: Number Theory, Geometry and Physics (16w5039)

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1 The Unreasonable Ubiquity of Whittaker Functions

Classical Whittaker functions were initially defined by Whittaker as solutions to the confluent hypergeometric differential equation. Like many special functions, they have a group theoretic meaning, and the classical Whittaker functions are an instance of a more general class of special functions on a reductive group G over a local field F. Gradually it has been realized that Whittaker functions functions play a critical role in such varied areas as automorphic forms, algebraic combinatorics, geometry of flag varieties and mathematical physics.

Let G be a split, reductive group over a local field F with maximal unipotent subgroup U. Consider functions f on $G(F)$ satisfying the transformation property:

$$f(ug) = \psi(u)f(g)$$

where ψ is a non-degenerate character of U. Then $G(F)$ acts on such functions by right translation. Any function appearing in an irreducible subspace of this regular representation is called a *Whittaker function*. An irreducible representation (π, V) of $G(F)$ has at most one space of Whittaker functions isomorphic to π as a representation of G. They are intimately related with the representation theory of another group, the Langlands dual group. Whittaker functions exhibit what one might call an "unreasonable ubiquity" as they arise crucially in many different contexts in mathematics and physics.

Two developments in the 1970's led to the current landscape. On the one hand, it was realized that a group-theoretic uniqueness principle explains many of the remarkable properties of Fourier coefficients of automorphic forms; over time this has led to the pervasive role of Whittaker functions in the theory of L-functions in modern number theory. On the other hand, a seemingly separate development of Whittaker functions occurred in mathematical physics, particularly around the classical and quantum Toda lattice.

Eventually these threads of separate origin merged into a common tapestry. After Givental studied stationary phase integrals for Whittaker functions coming from the quantum Toda lattice, the geometric role of Whittaker functions led to important developments in the quantum cohomology of flag varieties and mirror symmetry. The geometry of flag varieties also proved important in the theory of automorphic forms, and since the 1990's the geometric Langlands theory has developed with equal inputs from number theory and physics. Here Whittaker functions appear as sheaves on affine flag varieties. Meanwhile the theory of quantum groups, itself having separate origins in the physics of solvable lattice models and quantum mechanics on the one hand, and geometry on the other, also sheds light on Whittaker functions. In the current picture, the separate origins of Whittaker functions in number theory and physics can be understood as part of a larger picture.

Yet the current picture is in great flux. *Crystal bases* are combinatorial analogs of Lie group representations that appeared mysteriously useful in the theory of p-adic Whittaker functions, particularly on the metaplectic group. Pursuing this phenomenon leads to surprising connections with quantum groups and the Yang-Baxter equation. Also in the p-adic theory, Iwahori Whittaker functions are connected with representations of Hecke algebras, Demazure operators, Kazhdan-Lusztig theory and the geometry of flag varieties. Pursuing these connections leads into deep algebraic combinatorics such as the theory of non-symmetric Macdonald polynomials and the double affine Hecke algebra. *Geometric crystals* are algebraic varieties with extra structure whose tropicalizations are crystal bases; these have appeared in the theory of archimedean Whittaker functions, particularly in connection with the Toda lattice and its geometric connections, and other connections in mathematical physics such as Brownian motion. Many of the same geometric connections appear in both the archimedean and non-archimedean theories and it appears possible to interpolate between the two theories. Both theories relate to the geometric Langlands theory. Recent progress on Eisenstein series on Kac-Moody groups and their Whittaker functions has been motivated by connections to both mathematical physics (string theory, quantum gravity) and number theory, leading to rich interactions between the fields. Meanwhile Zeta functions of prehomogenous vector spaces, which have applications in number theory, are related to Whittaker functions via the theory of multiple Dirichlet series.

The previous BIRS workshops "Whittaker Functions, Crystal Bases, and Quantum Groups" (10w5096, June 2010) and "Whittaker Functions: Number Theory, Geometry and Physics" (13w5154, October 2013) brought together a diverse group of researchers in these different fields. Due in part to collaborations initiated or furthered during these meetings, rapid developments in several interdisciplinary research areas are occurring. This current workshop served to highlight the rapid developments in interdisciplinary research areas due, in part, to collaborations initiated or furthered during the past meetings.

In the 2016 workshop "Whittaker Functions: Number Theory, Geometry and Physics" (16w5039) we continued studies in these directions by bringing together researchers in number theory, mathematical physics, combinatorics and representation theory to explore these different manifestations of Whittaker functions.

2 The 2016 Meeting in Banff

The conference talks highlighed recent developments and open problems concerning Whittaker functions in number theory, geometry, combinatorics, and physics. Most talks were videotaped. In the following we summarize the highlights of each talk. Junior speakers are indicated by a bullet (•). Speakers from under-represented groups are indicated with a star ($\star$).

Benjamin Brubaker (University of Minnesota) spoke on *Hamiltonian interpretation of p-adic Whittaker functions*. Whittaker functions on p-adic groups are expressible as partition functions of six-vertex models on a rectangular lattice; at least, this is known for Cartan types A and B and expected more generally. This lecture explained that in type A, this may alternately be viewed as the discrete time evolution of a one-dimensional system of free fermions. The Hamiltonian dictating the evolution arises from the Lie superalgebra $\mathfrak{gl}(1 \mid 1)$ and the Whittaker function may thus be viewed as a kind of generalized "tau function" in the terminology of the Kyoto school. The work reported on is joint with A. Schultz based on `arXiv:1606.00020`.

Daniel Bump (Stanford University) spoke on *Metaplectic Whittaker functions and the Yang-Baxter equation*. In 2012, Brubaker, Bump, Friedberg, Chinta and Gunnells proposed statistical-mechanical models for p-adic Whittaker functions on the degree n metaplectic cover of $GL(r)$. This lecture concerned recent work of Brubaker, Buciumas and Bump, in which the corresponding Yang-Baxter equations have been found. The corresponding quantum group is identified as a Drinfeld twist of $U_q(\hat{\mathfrak{gl}}(n))$. The effect of the Drinfeld twisting is to introduce Gauss sums into the R-matrix. The scattering matrix of the intertwining operators on the (nonunique) Whittaker models, previously studied by Kazhdan-Patterson and Chinta-Gunnells, is thus reinterpreted as the R-matrix of this quantum group. Moreover, the internal states of these generalized ice-type models (which are not visible to the intertwining operator) are built up by tensoring from $(n+1)$-dimensional supersymmetric modules for the quantum affine Lie superalgebra $U_q(\hat{\mathfrak{gl}}(n \mid 1))$.

Yuanqing Cai (•) (Boston College) spoke on *Fourier coefficients of theta functions on metaplectic groups*. Kazhdan and Patterson constructed generalized theta functions on covers of general linear groups as multi-residues of the Borel Eisenstein series. These representations and their unique models were used by Bump and Ginzburg in the Rankin-Selberg constructions of the symmetric square L-functions for $GL(r)$. This lecture discussed the two other types of models that the theta representations may support. The first are semi-Whittaker models, which generalize the models used in the work of Bump and Ginzburg. Secondly, the lecturer discussed unipotent orbits, how to determine the unipotent orbits attached to theta functions, and how to determined the covers when these models are unique. He concluded by outlining some applications in Rankin-Selberg constructions.

Reda Chhaibi (•) (Institut de Mathématiques de Toulouse) spoke on *A probabilistic approach to the Shintani-Casselman-Shalika formula*. Jacquet's Whittaker function for a group G, in the non-Archimedean case, is essentially proportional to a character of an irreducible representation of the Langlands dual group – a Schur function in the case of GL_n. This statement is known as the Shintani-Casselman-Shalika formula. This talk presented a probabilistic proof of this formula that is the natural generalization of Shintani's. In it, the appearance of the Weyl character formula is explained from a reflection principle for random walks.

YoungJu Choie ($\star$) (POSTECH) spoke on *Periods of modular forms on $\Gamma_0(N)$ and Jacobi theta functions*. Generalizing a result of Zagier for modular forms of level one, she gave a closed formula for the sum of all Hecke eigenforms on $\Gamma_0(N)$ multiplied by their odd period polynomials in two variables, as a single product of Jacobi theta series for any squarefree level N. She also showed that for $N = 2, 3$ and 5 this formula completely determines the Fourier expansions all Hecke eigenforms of all weights on $\Gamma_0(N)$.

Holley Friedlander ($\bullet$, $\star$) (Dickinson College) spoke on *Twisted Weyl group multiple Dirichlet series over the rational function field*. Similar to zeta functions associated to algebraic function fields, Weyl group multiple Dirichlet series associated to algebraic function fields are rational functions in several variables. The denominators of these rational functions are known, but the numerators are not well understood. Like zeta functions, we expect the coefficients of the numerators to encode information about the arithmetic of the defining curve. As a step toward understanding this relationship, this talk described the support of Weyl group multiple Dirichlet series defined over the rational function field $\mathbb{F}_q(t)$. In particular, the speaker showed that up to a variable change, all such series can be expressed as a finite sum of simpler local series, which act analogue to Euler factors in the construction of the global object.

Nadya Gurevich ($\star$) (Ben Gurion University of the Negev) spoke on *The twisted Satake map and the Casselman Shalika formula*. In this lecture, working with an arbitrary split group, she explained how to identify the unramifed Whittaker space with the space of skew-invariant functions on the lattice of coweights and deduce the Casselman-Shalika formula from it.

Angele Hamel ($\star$) (Wilfrid Laurier University) spoke on *Factorial characters and Tokuyama's identity for classical groups*. In this talk she introduced factorial characters for the classical groups and derived a number of central results. Classically, the factorial Schur function plays a fundamental role in traditional symmetric function theory and also in Schubert polynomial theory. She developed a parallel theory for the classical groups, offering combinatorial definitions of the factorial characters for the symplectic and orthogonal groups, and further proving flagged factorial Jacob-Trudi identities and factorial Tokuyama identities. These identities are established by manipulating determinants through the use of certain recurrence relations and by using lattice paths. This work is joint with Ron King.

Axel Kleinschmidt (Max Planck Institute for Gravitational Physics) spoke on *Automorphic forms and lattice sums in exceptional field theory*. This talk concerned the connection between automorphic forms and string theory. Automorphic forms appear in string theory considerations for example in scattering amplitudes. The simplest cases where their form is understood is related to processes that are (partially) controlled by an additional symmetry, called supersymmetry. The lecture explained how one may use the modern language of exceptional field theory to determine parts of these scattering amplitudes that give alternative expressions for Eisenstein series on exceptional groups (attached to small representations). This approach also gives some insight into certain non-Eisenstein series that are expected to arise in string theory.

Kyu-Hwan Lee (University of Connecticut) spoke on *Convergence and holomorphy of Kac-Moody Eisenstein series*. Let G be a Kac-Moody group associated to a nonsingular, symmetrizable generalized Cartan matrix. First, this lecture considered Eisenstein series on G induced from quasi-characters, and explained how to prove the almost-everywhere convergence of Kac-Moody Eisenstein series inside the Tits cone for spectral parameters in the Godement range. For a certain class of Kac-Moody groups satisfying an additional combinatorial property, the lecturer then showed the absolute convergence everywhere in the Tits cone for spectral parameters in the Godement range. Next, he considered Eisenstein series on G induced from unramified cusp forms on finite-dimensional Levi subgroups of maximal parabolic subgroups. Under some 'ample' conditions on a maximal parabolic subgroup, he proved that the Eisenstein series are entire on the whole complex plane. This is joint work with L. Carbone, H. Garland, D. Liu and S.D. Miller.

Cristian Lenart (SUNY Albany) spoke on *New results on Kirillov-Reshetikhin modules and Macdonald polynomials*. In a series of papers with S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, the lecturer has developed two uniform combinatorial models for (tensor products of one-column) Kirillov-Reshetikhin (KR) modules of affine Lie algebras; he also showed that their graded characters coincide with the specialization of symmetric Macdonald polynomials at $t = 0$. This lecture presented the latest work in this direction: the extension of the above results corresponding to the non-symmetric Macdonald polynomials. It also explained the connection of this work to the q-Whittaker functions of Braverman-Finkelberg and their results, which extend to quantum K-theory Schubert calculus.

Anthony Licata (Australian National University) spoke on *Partial orders on the Weyl group, monoids in the braid group, and homological algebra*. Let W be a finite Weyl group. Associated to W there are two important partial orders – (weak) Bruhat order, and absolute order. These two partial orders are related to a pair of combinatorial lattices, and these lattices are in some (not completely understood) senses 'dual' to one another. As a result many other structures related to W come in dual pairs; for example, the braid group of W has a pair of 'dual' Garside structures, and a pair of 'dual' positive monoids. This talk explained these themes, and showed how these dual structures appear in the higher representation theory of the braid group. This work is joint with Hoel Queffelec.

Stephen Miller (Rutgers University) spoke on *Do growing Whittaker functions actually occur in automorphic forms?* Whittaker's original work involves both exponentially decaying and growing functions, generalizing the K- and I-Bessel functions that occur in the Fourier expansions of cusp forms on rank-1 groups. Decaying Whittaker functions are the bedrock of Fourier expansions of L^2 cusp forms for general groups. There is a corresponding beautiful theory of non-decaying Whittaker functions, dating back to work of Kostant and Casselman-Zuckerman in the 1970s. However, there are no known examples of automorphic functions which have non-decaying Whittaker functions in their Fourier expansions outside of rank-1 examples. This is consistent with a tantalizing conjecture of Miatello-Wallach, which asserts that all automorphic eigenfunctions on higher rank groups automatically have moderate growth (and hence decaying Whittaker functions). This lecture, based on joint work with Tien Trinh, presented some cases of that conjecture for $SL(3, \mathbb{Z})$, which rule out analogs of the classical modular j-function. (Note: Due to flight cancelations, Prof. Miller was unable to attend the conference, but he prepared a video presentation as a substitute and was available by phone for questions and comments as it was shown.)

Dinakar Muthiah (•) (University of Alberta) spoke on *Double affine Bruhat order and Iwahori-Hecke algebras for p-adic loop groups*. Recently, Braverman, Kazhdan, and Patnaik have constructed Iwahori-Hecke algebras for p-adic loop groups. Unsurprisingly, the resulting algebra is a variation on Cherednik's DAHA. The p-adic construction also comes with a basis (the double-coset basis) consisting of indicator functions of Iwahori double cosets. Braverman, Kazhdan, and Patnaik also proposed a (double affine) Bruhat preorder on the set of double cosets, which they conjectured to be a poset. This lecture described a combinatorial presentation of the double-coset basis and also an alternative way to develop the double affine Bruhat order that is closely related to this combinatorics; from this perspective the order is manifestly a poset. One new feature is a length function that is compatible with the order. It also discussed joint work with Daniel Orr proving that the length function can be specialized to take values in the integers. This proves finiteness of chains in the double-affine Bruhat order, and it gives an expected dimension formula for (yet to be defined) transversal slices in the double affine flag variety. The lecture conclued by discussing how these results are stepping stones for developing Kazhdan-Lusztig theory in this setting and offered a number of open questions.

Maki Nakasuji ($\star$) (Sophia) spoke on *Casselman's basis, Yang-Baxter basis, and Kostant-Kumar's twisted group algebra*. Casselman's basis is the basis of Iwahori fixed vectors of a spherical representation of a connected reductive p-adic group over a non-archimedean local field, which is dual to the intertwining operators at the identity indexed by elements of the Weyl group. The problem of Casselman is to express Casselman's basis in terms of another natural basis, and vice versa. This talk provided one solution to Casselman's problem, using the Yang-Baxter basis of the Hecke algebra and Kostant-Kumar's twisted group algebra. This is joint work with H. Naruse.

Omer Offen (Technion) spoke on *Integrability of matrix coefficients and periods of automorphic forms*. Let G be a p-adic reductive group and H a symmetric subgroup. This lecture presented joint work with Max Gurevich giving a criterion for H-integrability of matrix coefficients of representations of G. This is a generalization of Casselman's criteria for square integrability. Chong Zhang has applied these results to show that for some symmetric subgroups all H-invariant linear forms of square integrable representations emerge as H-integrals of matrix coefficients. In particular, in a global setting, this provides information on the local components of factorizable period integrals of automorphic forms.

Manish Patnaik (•) (University of Alberta) spoke on *Kac-Moody Eisenstein series*. Eisenstein series play an important role in the theory of automorphic forms on reductive groups. They are essential in Langlands program, in particular, spectral decomposition, Arthur trace formula, and Langlands-Shahidi method of studying automorphic L-functions. On the other hand, Kac-Moody groups are infinite dimensional groups, and only recently there have been attempts to define Eisenstein series and automorphic forms on Kac-Moody groups which found an application in string theory in physics. This lecture explained obstacles to doing so and recent positive developments.

Daniel Persson (•) (Chalmers University of Technology) spoke on *Automorphic representations, Whittaker vectors, and black holes*. Automorphic forms on exceptional Lie groups appear naturally in string theory compactifications. They manifest themselves as couplings in higher derivative corrections and in terms of generating functions of black hole microstates. This lecture explained how certain Fourier coefficients attached to the minimal automorphic representations of E_6, E_7, and E_8 are determined by maximally degenerate Whittaker vectors. This fact allows for a simple method for calculating explicit Fourier coefficients which are relevant in string theory. Various recent results, conjectures and open problems were then outlined.

Anne Puskás ($\bullet$, $\star$) (University of Alberta) spoke on *Metaplectic Iwahori-Whittaker functions and Demazure-Lusztig operators*. Metaplectic Demazure-Lusztig operators are built on the Chinta-Gunnells action, and (analogously to their non-metaplectic counterparts) are useful in the study of p-adic (metaplectic) Whittaker functions. This talk presented work, joint with Manish Patnaik (University of Alberta), that relates metaplectic Iwahori-Whittaker functions to these operators directly. This process gives a metaplectic analogue of earlier work of Brubaker-Bump-Licata in the non-metaplectic setting. She also discussed aspects of the extension of relevant formulae to the affine Kac-Moody groups.

Arun Ram (University of Melbourne) spoke on *Alcove walks and the Peterson isomorphism*. He described a labeling of the points of the moduli space of genus 0 curves in the complete complex flag variety using the combinatorial machinery of alcove walks. Following Peterson, this geometric labeling explains the 'quantum equals affine' phenomenon which relates the quantum cohomology of this flag variety to the homology of the affine Grassmannian. This is joint work with Elizabeth Milicevic.

Anne Schilling ($\star$) (University of California, Davis) spoke on *Cohomology of affine Grassmannians, combinatorics, and crystals*. In it, she explained how the theory of crystal bases can be used to understand some of the affine combinatorics that arises in the (co)homology of the affine Grassmannian. In particular, this approach leads to a combinatorial interpretation of the coefficients of a Schur function in the expansion of a Stanley symmetric function (based on joint work with Jennifer Morse). She also presented some new results and conjectures regarding the Schur expansion of the k-Schur functions, which are dual to affine Stanley symmetric functions.

Takashi Taniguchi (Kobe University) spoke on *Orbital exponential sums for prehomogeneous vector spaces*. Exponential sums arise various context in algebraic and analytic number theory. This talk explained how they may appear in the study prehomogeneous vector spaces, and outlined a new method for evaluating them explicitly. As an application, he showed that there are 'many' quartic field discriminants with at most eight prime factors. This is joint work with Frank Thorne.

Nicolas Templier (Cornell University) spoke on *Kloosterman families, quantum cohomology, and geometric Langlands*. In it, he explained his proof, joint with Thomas Lam, of cases of the Rietsch mirror conjecture that the Dubrovin quantum connection for projective homogeneous varieties is isomorphic to the pushforward D-module attached to Berenstein-Kazhdan geometric crystals. The idea is to recognize the quantum connection as Galois and the geometric crystal as automorphic. He explained surprising relations with the works of Frenkel-Gross, Heinloth-Ngo-Yun and Zhu on Kloosterman sheaves. The isomorphism comes from global rigidity results where Hecke eigensheaves are determined by their local ramification. It implies combinatorial identities for the counts of rational curves, the Peterson variety presentation and other consequences.

Ian Whitehead ($\bullet$) (University of Minnesota) spoke on *The Chinta-Gunnells construction for affine groups*. Let W be the Weyl group of a simply-laced affine Kac-Moody Lie group, excepting type A affine root systems of even rank. In this talk, it was explained how to construct a multiple Dirichlet series that is meromorphic in a half-space, satisfying a group W of functional equations, using generalizations of the Chinta-Gunnells construction. This series is analogous to the multiple Dirichlet series for classical Weyl groups constructed by Brubaker-Bump-Friedberg, Chinta-Gunnells, and others. It is completely characterized by four natural axioms concerning its coefficients, axioms which come from the geometry of parameter spaces of hyperelliptic curves. Evidence was presented to suggest that this series appears as a first Fourier-Whittaker coefficient in an Eisenstein series on the twofold metaplectic cover of the relevant Kac-Moody group. The construction is limited to the rational function field, but it also describes the p-part of the multiple Dirichlet series over an arbitrary global field

3 Outcome of the Meeting

The workshop was well-attended by researchers at all points in their careers, and from a wide range of countries: Canada, USA, Australia, China, France, Germany, Japan, Korea, Israel, and Singapore. After the completion of the workshop, the organizers received many positive comments from both junior and senior participants, and in the year since the workshop ended there have been a number of important papers by participants that are directly related to the workshop themes. Moreover, the broad themes, including interaction between mathematicians and physicists related to Whittaker models, have continued in other settings, notably last fall's program at the Simons Center for Geometry and Physics entitled "Automorphic forms, mock modular forms and string theory."

The workshop itself generated a great deal of energy among the participants. In terms of immediate feedback, one participant wrote

> *"The workshop was very fruitful for my research."*

and another wrote

> *"One week at BIRS gave me a year's worth of new ideas and projects to tackle. BIRS workshops are some of some of the most fruitful conferences I've ever attended, thanks to their camaraderie, beautiful atmosphere, and intense mathematical focus.*

This workshop is the third we have organized on the broad theme of Whittaker functions and their appearance in different fields. (The prior two were 10w5096 and 13w5154.) This continuity has led to a strong group of researchers in diverse areas who are aware of and build on these cross-field and even cross-disciplinary connections. We are very appreciative of the opportunity to organize such a workshop periodically, and we hope to continue this string of successes by organizing a week-long workshop at Banff in 2019 or 2020 emphasizing the latest developments. We look forward to continuing this high-impact workshop series.