12w5118 Optimal Transportation and Differential Geometry

Alessio Figalli (The University of Texas at Austin),
Young-Heon Kim (University of British Columbia)

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1 Overview of the Field

Optimal mass transportation can be traced back to Gaspard Monge's famous paper of 1781: 'Mémoire sur la théorie des déblais et des remblais'. The problem there is to minimize the cost of transporting a given distribution of mass from one location to another. Since then, it has become a classical subject in probability theory, economics and optimization.

At the end of the 80's, the seminal work of Brenier [7, 8] paved a way to connect optimal mass transportation to partial differential equations and related areas. On the one hand, his theory was followed by McCann's displacement convexity and Otto's differential geometry of the space of probability measures, making the theory of mass transportation applicable to wide range of problems in differential geometry, geometric and functional inequalities, and nonlinear diffusions. On the other hand, it stimulated Caffarelli, Urbas, and many others, to develop regularity theory of Monge-Ampère equations.

These two related directions have seen unexpected recent progresses: One of the highlights is the breakthrough of Lott and Villani [58] and Sturm [72, 73] who have characterized singular spaces with lower bounded Ricci curvature, solving one of the well-known open problems in Riemannian geometry. Also, Ma, Trudinger and Wang [65] extended the regularity theory for Monge-Ampère to a more general class of Monge-Ampère type equations, which surprisingly led the discovery of a new type of curvature, called Ma-Trudinger-Wang curvature, which is nowadays an object of many investigations in different directions.

2 Recent Developments and Open Problems

2.1 Ricci curvature on metric measure spaces, and geometry of the space of probability measures

Links from optimal transport to geometric analysis, including to the theory of Ricci curvature and Ricci flow, take their origin in the work of Otto and Villani [68] and Cordero-Erausquin, McCann, and Schmuckenschlager [17], and have received even more attention after the recent works of Lott and Villani [58], Sturm [72, 73], McCann and Topping [64], Lott [57] and Topping [74]. The possibility to define useful analogs of such concepts in a metric measure space setting has been a tantalizing goal, only partly realized so far. Still this progress, together with the original contribution due to Otto [67] on the formal Riemannian structure of the Wasserstein space and its applications to PDE's, is having a strong impact on the research community.

For other but closely related aspects of optimal transportation, there are studies of Sturm, Gigli, Ohta, Von Renesse, Ambrosio, Savarè, and others, on the heat flow on metric spaces spaces. Recently, Gigli, Kuwada and Ohta [39] have obtained a fundamental result showing the equivalence between gradient flow of energy functional and that of entropy functional on Alexandrov spaces as the underlying space. Apart from allowing to extend the Otto's work on gradients flow in Wasserstein spaces [67, 2] to Alexandrov spaces for constructing solutions to the heat equations (see also [38]), a simple corollary of this equivalence is that solutions to the heat equations become instantaneously Lipschitz continuous in space (before this result, only Hölder continuity was known).

After this work, this theory has been pushed even further by Ambrosio, Gigli and Savarè in a series of papers [3, 4] where they can study fine properties of Sobolev functions on metric measure spaces, they introduce a notion of Riemannian Ricci curvature for metric spaces, they prove equivalence between horizontal and vertical derivatives, etc.

2.2 Regularity of optimal transportation, fully nonlinear partial differential equations, and Riemannian geometry

The smoothness of optimal transport maps is an important issue in transportation theory since it gives information about qualitative behavior of the map, as well as simplifying computations and algorithms in numerical and theoretical implementations (see [78, 79] for discussions on this issue). Thanks to the results of Brenier [7, 8] and McCann [63], it is well known that the potential function of the map satisfies a Monge-Ampère type equation, an important fully nonlinear second order elliptic PDE arising in differential geometry. In the case of the quadratic cost function in Euclidean space, pioneering papers in this field are due to Delanoë [18], Caffarelli [9, 10, 11, 12], and Urbas [77]. More recently, Ma, Trudinger and Wang [65, 75] discovered a mysterious analytical condition, now called the Ma-Trudinger-Wang condition (or simply MTW condition) to prove regularity estimates for general cost functions [19, 21, 27, 25, 76, 26, 20, 60, 61, 47]. Costs functions which satisfy such a condition are called regular. At this point, Loeper [55, 56] gave a geometric description of this regularity condition, and he proved that the distance squared on the sphere is a uniformly regular cost, giving the first non-trivial example on curved manifolds. Moreover, he also showed that nonnegativity of MTW curvature is necessary for smoothness of optimal transportation maps (or simply optimal maps): more precisely, there are discontinuous optimal maps even between smooth distributions whenever the manifold has negative curvature at one point [55], and moreover the sole positivity of the sectional curvature is not enough for regularity [44, 31].

The Ma-Trudinger-Wang tensor is reinterpreted by Kim-McCann [45, 46] in an intrinsic way, and they show that it can be identified as the sectional curvature tensor on the product manifold equipped with a pseudo-Riemannian metric with signature (n, n). Recently, Kim, McCann and Warren [48] have found a pseudo-metric with respect to which the graphs of optimal maps give volume maximizing space-like Lagrangian submanifolds, thus giving some hope for relating optimal transportation theory to submanifold theory and symplectic geometry.

In addition, recent results of Loeper and Villani [58, 80] and Figalli, Rifford and Villani [30, 31, 32, 33, 34] show that the regularity condition on the square distance of a Riemannian manifold implies geometric results, like the convexity of the cut-loci. These developments show fruitful interactions of analysis and geometry around optimal transportation.

A wide open problem of regularity theory is to understand the nature of discontinuity/singularity set of optimal maps when the MTW condition is not satisfied, e.g., the distance squared cost on negatively curved Riemannian manifolds. As Villani asked in his book [79], does such set have nice geometry or does it show fractal nature? Partial results in this direction have been recently proved in [23, 24], but a complete answer to this problem is still missing.

2.3 Geometric and functional inequalities

A further line of research, which takes its origin in McCann's proof of Brunn-Minkowski inequality via optimal transport [62], is to apply optimal transport to prove geometric and functional inequalities (such as isoperimetric or Sobolev inequalities). For instance, Figalli, Maggi and Pratelli [29] were able to exploit the optimal transport proof of the Wulff inequality (an anisotropic version of the isoperimetric inequality) to prove a sharp stability estimate, solving a long-standing open problem in crystals shape formation (see also [28]).

More recently, Castillon [14] and Chang and Wang [15] have used optimal transport maps to prove some versions of Micheal-Simon and Alexandrov-Fenchel inequalities.

One of the main open problems in this area is to understand whether one can find optimal transport proofs of relevant inequalities also on manifolds (for instance, for proving the isoperimetric inequality on the sphere or the hyperbolic space), which may then lead to new results with wider applications.

3 Presentation Highlights

3.1 Theoretical aspects of optimal transportation

3.1.1 The classical optimal transport problem and distances between measures

A basic question in optimal transport is whether an optimal transport map exists. This depends of course on the cost function, and the Monge problem with “cost=distance” is one of the most difficult and challenging cases.

Stefano Bianchini has described a recent work where he shows that the Monge-Kantorovich problem with any convex cost function has always a solution which is induced by a transport map. This includes in particular the case “cost=distance”, as well as much more degenerate cases. The proof is based on a decomposition of Sudakov’s type.

Another important topic in optimal transport is how to use transportation distance to construct gradient flows in the space of measures.

In this direction, Matthias Erbar has presented a joint work with Karl Theodor Sturm [22], where they introduce a new transportation distance between probability measures that is built from a Lévy jump kernel. It is defined via a non-local variant of the Benamou-Brenier formula. They study geometric and topological properties of this distance, in particular they prove existence of geodesics. For translation invariant jump kernels they can identify the semigroup generated by the associated non-local operator as the gradient flow of the relative entropy w.r.t. the new distance and show that the entropy is convex along geodesics.

Nassif Ghoussoub has presented his recent work on the Monge-Kantorovich problem with symmetry. This kind of problem has strong relations with self-dual Lagrangians, convex analysis, and monotone operators. In particular he can obtain important variants of Brenier’s polar decomposition theorem [8].

Jonathan Korman has presented a joint work with Robert McCann [52] where they consider a variant of the classical Monge-Kantorovich problem by imposing a constraint on the joint measures: find an optimal one among all joint measures with fixed marginals, which are dominated by a given measure. They show uniqueness of the solution, and explicitly compute some examples.

Brendan Pass has formulated and studied the problem of aligning a continuum of marginals as efficiently as possible. In his formulation, he looks for the stochastic process with prescribed single time marginals which minimizes the expectation of a certain functional. This problem is a natural extension of a multi-marginal optimal transportation problem previously studied by Gangbo and Swiech [37], and in his talk he has shown how to obtain existence, uniqueness, and characterization results for the quadratic cost function.

3.1.2 Regularity of optimal transportation and nonlinear PDEs.

A number of speakers discussed topics around Ma, Trudinger and Wang conditions on regularity of optimal transportations and fully nonlinear elliptic Hessian equations. In addition, important extensions of this theory have been presented.

Jun Kitawaga presented a joint work [49] with Micah Warren, where they consider regularity for Monge solutions to the optimal transport problem when the initial and target measures are supported on the embedded sphere, and the cost function is the Euclidean distance squared. Gangbo and McCann [36] have shown that when the initial and target measures are supported on boundaries of strictly convex domains in R^n, there is a unique Kantorovich solution, but it can fail to be a Monge solution. In the case when one deals with the sphere with measures absolutely continuous with respect to surface measure, they could obtain two different types of conditions on the densities of the measures to ensure that the solution given by Gangbo and McCann is indeed a Monge solution, and obtain higher regularity as well.

Alexander Kolesnikov has presented his recent study on the optimal transportation mapping $\nabla\Phi : R^d \mapsto R^d$ pushing forward a probability measure $\mu = e^{-V}\ dx$ onto another probability measure $\nu = e^{-W}\ dx$, which follows a line of research previously investigated in [13, 16]. Following a classical approach of E. Calabi he introduces the Riemannian metric $g = D^2\Phi$ on R^d and studies spectral properties of the metric-measure space $M = (R^d, g, \mu)$. He proves, in particular, that M admits a non-negative Bakry-Emery tensor provided both V and W are convex. If the target measure ν is the Lebesgue measure on a convex set Ω and μ is log-concave, he can show that M is a $CD(K, N)$ space. Applications of these results include some global dimension-free a priori estimates of $\|D^2\Phi\|$. With the help of comparison techniques on Riemannian manifolds he estimates the diameter of M in terms of the dimension and the diameter of Ω.

Jiakun Liu presented his recent work on the study of the general case of the light reflection problem, showing how it is related to a nonlinear optimization problem. This problem involves a fully nonlinear PDE of Monge-Ampere type, subject to a nonlinear boundary condition, and generalizes previous works of Xu-Jia Wang [81, 82] in the special far field case, which are related to the reflector antenna design problem.

Neil Trudinger has shown how to develop the fundamentals of a local regularity theory for prescribed Jacobian equations which extend the corresponding results for optimal transportation equations. In this theory the cost function is extended to a generating function through dependence on an additional scalar variable. In particular he can recover in this generality the local regularity theory for potentials of Ma, Trudinger and Wang, along with the subsequent development of the underlying convexity theory.

3.1.3 Geometry of Wasserstein spaces

Luigi Ambrosio has presented a joint work with Gigli and Savaré [4] where they compare several notion of weak (modulus of) gradient in metric measure spaces and prove their equivalence. This equivalence is part of the ”calculus program” they developed, largely based on tools from optimal transportation theory. In particular, they prove density in energy of Lipschitz maps in Sobolev spaces independently of doubling and Poincaré inequality assumptions on the metric measure space.

Nicola Gigli has shown that every metric measure space (complete, separable endowed with a Borel locally finite measure) has a natural first order differentiable structure: it is possible to speak about differential and gradients of Sobolev functions. As an application, he has introduced a general definition of distributional Laplacian and shown that on spaces with Ricci curvature bounded from below, for the Laplacian of the distance function, the standard comparison estimates hold.

Benoit Kloeckner has illustrated that classical ideas in metric geometry can be used to generalize Hausdorff dimension in a way that distinguishes many Wasserstein spaces [50, 51]. One nice consequence is that the Wasserstein space of a compact manifold can never be Lipschitz embedded in the Wasserstein space of a compact manifold of lower dimension.

Kazumasa Kuwada has presented a joint with Karl Theodor Sturm [53] where new monotonicity in time of a time-dependent transportation cost between distribution of diffusion processes is shown under Bakry-Emery’s curvature-dimension condition on a Riemannian manifold. This result is an analog of the L^p-Wasserstein contraction of heat distributions under lower Ricci curvature bound. The cost function comes from the total variation distance between heat distributions on the space forms. As a corollary, they obtain a comparison theorem for the total variation distance between heat distributions. They also obtain an explicit expression of the cost function. It leads to a time-independent transportation cost which is non-increasing in time for heat distributions even on a negatively curved space. In addition, they show that their monotonicity is stable under the Gromov-Hausdorff convergence of the underlying space under a uniform curvature-dimension and diameter bound.

Paul Lee has presented some recent work [54] on interpolation of measures from a Hamiltonian point of view: he has discussed displacement interpolations from the point of view of Hamiltonian systems and give a unifying approach to various known results.

Shin-Ichi Ohta has discussed the notion of Ricci curvature in Finsler geometry and, as a recent application, shown generalizations of the Cheeger-Gromoll type splitting theorem [66].

Tommaso Pacini has presented a joint work with Wilfrid Gangbo and Hwa-Kil Kim [35], discussing differential forms and symplectic geometry on Wasserstein spaces. More precisely, he has talked about differential calculus on Wasserstein spaces, de Rham cohomology, symplectic structures, and Hamiltonian systems.

Tapio Rajala has talked about some recent work on interpolation measures with bounded density in $CD(K,N)$-spaces [69, 70, 71]. He has explained how using only the convexity-inequality for the critical entropy-functional in a $CD(K,N)$-space one can construct geodesics in the Wasserstein space along which all the measures have bounded density. He also discussed some applications of these "good" geodesics.

Giuseppe Savarè has presented a joint work with Ambrosio and Gigli [3] on the links between the displacement convexity of entropy functionals and the characterizations of their gradient flows in Wasserstein spaces in terms of a family of evolution variational inequalities. In the particular case of the logarithmic entropy the above properties are strictly related to the linearity and the contractivity of the flow. Various properties and applications to metric measure spaces with Ricci curvature bounded from below have also been discussed.

3.2 Riemannian geometry

Jerome Bertrand has illustrated a new proof of Alexandrov's theorem on the Gauss curvature prescription of Euclidean convex body, based on mass transport and the classical theory of convex bodies duality [5]. In particular, this proof does not rely on PDEs method nor convex polyhedra theory. With this approach, it is also possible to treat the case of equivariant convex bodies in the (Lorentzian) Minkowski space.

Simon Brendle has discussed minimal tori in S^3, and has shown how to prove Lawson conjecture on the fact that any embedded minimal torus in S^3 is congruent to the Clifford torus [6].

Yi Wang has reported some recent joint work with Alice Chang in which they generalize Michael-Simon inequality and partially generalize the Aleksandrov-Fenchel inequalities for quermassintegrals from convex domains in the Euclidean space to a class of non-convex domains [15]. In the talk she has also discussed about optimal constants of the inequality in some special cases.

Guofang Wei has presented a joint with Wylie [83] where they extended several comparison results (in particular, the Bishop-Gromov volume comparison) for manifolds with lower Ricci curvature bound to smooth metric measure spaces with Bakry-Emery Ricci tensor bounded from below. She also discussed several applications of this. In particular, Peng Wu uses it to show that the infimum of the potential function of a gradient steady Ricci soliton grows linearly. Jointly with P. Wu, for a large class of gradient steady Ricci solitons, she has also obtained optimal growth estimate for the potential function, and show the volume grows at most like polynomial of degree n.

William Wylie has presented joint works with Chenxu He and Peter Petersen [40, 41, 42, 43] where they show that any algebraic Ricci soliton on a left invariant Lie group can be extended to a homogeneous warped product Einstein space. This extends a result of Lauret for solvable groups. It also provides the existence of many new homogeneous smooth metric measure spaces which are m-Quasi Einstein manifolds. They also give a strong characterization of the geometric structure of such spaces.

Results in other related areas

Beatrice Acciaio has shown how optimal transport is related to robust pricing and trajectories inequalities [1]. Robust pricing basically corresponds to considering extremal pricing rules coming from possible pricing measures which satisfy marginal constraints. This problem is naturally connected to optimal transportation. Mathematically the crucial difference is that in her setting transport plans are required to be martingales. She has discussed the advantages of relating the robust pricing problem to the theory of mass transportation. In particular, she has shown that the duality theorem from optimal transport can be used to establish new robust super-replication results. This dual point of view also provides new insights on classical martingale inequalities, such as a (new) sharp version of the classical Doob maximal inequality.

4 Outcome of the Meeting

The present meeting between experts in optimal transportation and Riemannian geometry has come at a time that saw an explosion of interests in the link between the two fields.

The workshop brought together mathematicians working on optimal transport, Riemannian geometry, metric geometry, geometric flows, and geometric inequalities. The audience included specialists in all these fields.

By bringing together researchers from a range of different fields with common interests, this meeting has showcased some of the recent progresses and set the stage for future developments, while stimulating new collaborations, new questions, and new lines of research. By making these connections, we believe that the meeting has accelerated the rate of progress within these two important areas of mathematics , having a lasting impact through influencing new directions for future research.

In addition to all the directions of research already discussed previously, this workshop has allowed us to find two new very promising and interesting directions to investigate.

The first one has been suggested by Nassif Ghoussoub: in his talk he has shown strong links between convex analysis, self-dual lagrangians, cyclical monotonicity, monotone operators, and idempotent measure preserving maps. It looks natural now to try to understand what are the sharp conditions on the cost function in order to obtain optimal maps, and what are their regularity properties. In addition, as pointed out by Ambrosio, when specialized to the Coulomb interaction cost this new theory of Ghoussoub seems to have very important potential applications in theoretical physics.

The second one has been suggested by Neil Trudinger: in his talk he has shown how, by considering generating functions on R^{2n+1}, one can include in a unifying setting not only Monge-Ampère type equations arising from optimal transport, but also for instance equations coming from reflector antenna problems. A natural question is how to extend the results known up to now for optimal transport maps, to this more general setting, and it seems to give an interesting and challenging line of research.

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