BANFF WORKSHOP REPORT

GENERALIZED GAUSS MAPS AND FAREY STATISTICS

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Abstract. This is a short summary of the work done during our Banff FRG Workshop (September 16-23, 2012). We focus on the issue of the mixing of the BCZ map, as this was our primary subject of discussion.

1. **Background.** The workshop focused on recent developments in homogeneous dynamics, as related to Diophantine approximation. Boca, Cobeli and Zaharescu, in the early 2000s, made a study of the fine statistics of Farey fractions. To a first approximation, the Farey fractions with denominator up to Q are uniformly distributed in the unit interval. The distribution of gaps between successive Farey fractions, however, was found to be far from the exponential distribution that one might naively expect. In fact, with suitable normalization, the distribution is seen to be piecewise analytic. More recently, Athreya and Cheung unified the study of Boca, Cobeli and Zaharescu with the study of horocycle flow on the space of unimodular lattices. The purpose of the workshop was to further develop this circle of ideas, and set out future research directions. The participants had a fairly wide array of research specialties: interval exchange transformations, homogeneous dynamics, C^* algebras, Diophantine approximation, translation surfaces and ergodic theory.

2. **The BCZ map.** Motivated by the study of Farey fractions, Boca-Cobeli-Zaharescu [5] introduced the *BCZ map* $T : \Omega \to \Omega$ on the triangle $\Omega := \{(a, b) : a, b \in (0, 1], a + b > 1)\}$. T acts via

$$T(a, b) = (a, b)A_k^T,$$

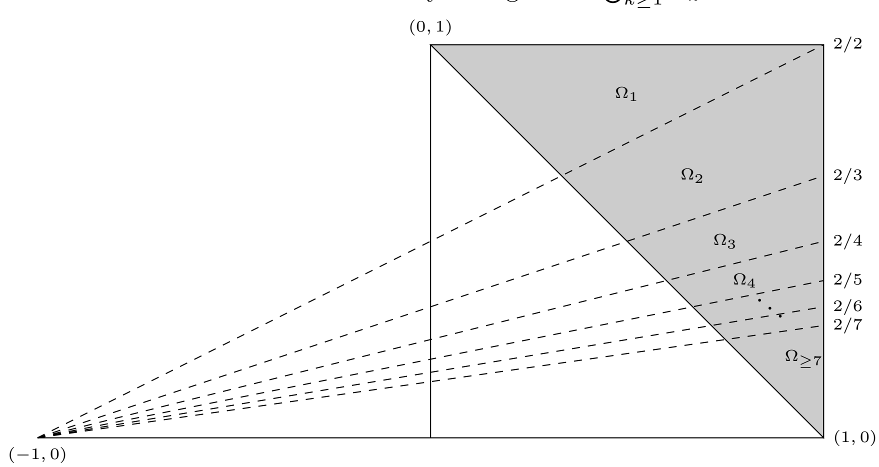
where

$$A_k = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix}$$

on the region $\Omega_k := \{(a, b) \in \Omega : \kappa(a, b) = k\}$, where $\kappa(a, b) = \left\lfloor \frac{1+a}{b} \right\rfloor$. Boca-Zaharescu [4] posed several questions about the ergodic properties of this map:

Question. *Is T ergodic? Mixing? What is the entropy of T?*

Figure 1. The Farey triangle $\Omega = \bigcup_{k\geq 1} \Omega_k$



Several questions were answered by Athreya-Cheung [3], who showed:

Theorem 1. *T is a first return map of the horocycle flow on the space of lattices $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$. Let $h_s = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$, the first return map to the transversal $\tilde{\Omega} := \left\{ \Lambda_{a,b} := \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} SL(2,\mathbb{Z}) : (a,b) \in \Omega \right\}$ is given by T. The return time function is $R(a,b) = \frac{1}{ab}$. That is, $h_{R(a,b)}\Lambda_{a,b} = \Lambda_{T(a,b)}$.*

Using the ergodicity, zero-entropy, and measure rigidity of the horocycle flow, this result shows that the BCZ map is ergodic, zero-entropy, and that in fact Lebesgue measure is the unique ergodic invariant measure not supported on a periodic orbit. However, mixing (and weak mixing) are not properties that pass from a flow to a section. Mixing of the BCZ map would also have some interesting number theoretic applications. Thus, a motivating question of the workshop was:

Question. *Is the BCZ map mixing? Is it weak mixing?*

3. **Gauss Map Dreams.** Another motivating question of the workshop was to place the BCZ map in the general context of naturally ocurring, geometrically meaningful cross-sections to various homogeneous flows. The most classical example of such is the (natural extension) of the Gauss Map, which links geodesic flow on the modular surface $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ to the study of continued fractions. More generally, the study of these maps gives number theoretic and statistical applications. Thus, we had a general goal:

Idea. *The BCZ is a 'Gauss Map' for horocycle flow'. Try and find more Gauss maps explicitly computable transversals and return maps for well-known flows, especially so that number theoretic quantities are computable. That is, can we fill more entries in the table below?*

Interesting Invariants	Farey Statistics	Gauss Map	Renormalization Dynamics	'Resident' Dynamics
	Levy Constant $\frac{\pi^2}{12\log 2}$	$G(x) = \{\frac{1}{x}\}$	geodesic flow on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$	circle rotations
	Levy Constants	Cheung-Chevalier Map	diagonal flow on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$	toral translation
		Rauzy Induction	Teichmüller Flow on Strata of Hodge Bundle Ω_g	IETs/Translation Surfaces
Index	Farey Sequences	BCZ	horocycle flow on $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$	
	Generalized Farey Sequences		Horospherical Flows on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$	

Table 1. Gauss Map Dreams

While there was significant discussion on these topics (particularly following the work of Cheung-Chevallier [6] on higher-order Levy constants and the work of Athreya-Chaika [1] and Athreya-Chaika-Lelievre [2] on translation surfaces, the primary focus of the workshop became the question of mixing of the BCZ map.

4. **Mixing.** There are well-known dynamical arguments to prove and disprove mixing in various examples. As our initial intuitions on the properties of BCZ varied, we decided to discuss two possible approaches; one to disprove mixing, using an argument of Katok [7] on Interval Exchange Transformations (IETs), and one to prove mixing, using an argument of Marcus [8] for horocycle flows.

4.1. *Katok.* J. Chaika, an expert on IETs, described Katok's argument on non-mixing of IETs, which relies on a double-inducing construction and controlling the sizes of various pieces of the associated towers. After some discussion, it became clear that this argument could not be used in this setting directly, and that there were some serious obstacles to employing even the general strategy. Namely, controlling the height and size of towers in the double-inducing construction was non-trivial.

4.2. *Marcus.* A. Quas, an expert in ergodic theory, described Marcus's theorem on mixing of horocycle flows, which relies crucially on proving equidistribution of geodesic segments when pushed forward under the horocycle flow. In our setting, geodesic segments in $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ translate to radial segments in the transversal, and the equidistribution desired becomes the following: for any suitably nice function v on Ω_1, we want, for any $z_0 \in \Omega$,

$$\int_0^b v(T^N z_{-t})dt \to b \int_{\Omega_1} v dm,$$

where $z_{-t} := g_{-t}z_0 := e^{-\frac{t}{2}}z_0$. Here, $g_{-t} = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}$. acts via the action on the lattice

$$\Lambda_{z_0} = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \mathbb{Z}^2.$$

The main thread of our discussions was developing a strategy to prove this equidistribution. Since if we were to replace the map T by the flow h_s, we know that this is true by Marcus, it became crucial to understand the time change created by inducing. Below, we discuss how this time change can be interpreted in many different ways, and in particular as a lattice point problem.

4.3. *Time Changes and Lattice Points.* Let $z_0 = (x, y) \in \Omega_1$, our standard BCZ transversal, and let $t \in [0, b]$ be a small parameter, and consider z_{-t} for $t \leq -2 \ln \frac{1}{x}$, which guarantees $z_{-t} \in \Omega_1$. Let $N > 0$, and let $s = s_{N,t}$ be such that

$$T^N(z_{-t}) = h_s g_{-t} z = g_{-t} h_{e^t s} z.$$

Then, since for all $v \in \Omega_1$, $g_t v \in \Omega_1$ for all $t > 0$, we have $h_{e^t s} z = g_t(g_{-t} h_{e^t s} z) \in \Omega_1$, so there is a $M_t > 0$ so that

$$h_{e^t s} z = T^{M_t}(z).$$

That is, we apply T^N to the entire radial segment $\{e^{\frac{t}{2}} z_0 : t \in [0, b]\}$, and want to express the point $T^N(e^{\frac{t}{2}} x, e^{\frac{t}{2}} y)$ as a point along the radial segment determined by $T^{M_t}(x, y)$, for some M_t. We want to understand the distribution of the quantity M_t along the segment $[0, b]$. The possible values of M_t range (roughly) between N and $e^b N$. Can we show that for an $M \in [N, e^b N]$ (a range of size $\approx bN$), that

$$|\{t \in [0, b] : M_t = M\}| \sim \frac{b}{bN} = \frac{1}{N}?$$

4.4. *Roof Functions.* The value $s = s_{N,t}$ is a partial sum of the roof function $R(x, y) = \frac{1}{xy}$ along the orbit of BCZ. Precisely, we have

$$s_{N,t} = \sum_{i=0}^{N} R(T^i(z_{-t})).$$

We also have

$$e^t s_{N,t} = \sum_{i=0}^{M_t - 1} R(T^i(z_0)).$$

This gives a heuristic for the expected size of M_t, using the ergodic theorem, since we expect the first sum to be $\approx N \int_\Omega R$ and the second to be $\approx M_t \int_\Omega R$, so we expect M_t to have size $e^t N_t$.

4.5. *Lattice Points.* We can interpret M_t and N in terms of (primitive) lattice points. $s_{N,t}$ is the slope of the N^{th} vector (where we order vectors by slope) in the lattice Λ_{z_0} in the vertical strip

$$V_{e^{-\frac{t}{2}}} = \{y \geq 0, 0 < x \leq e^{-\frac{t}{2}}\},$$

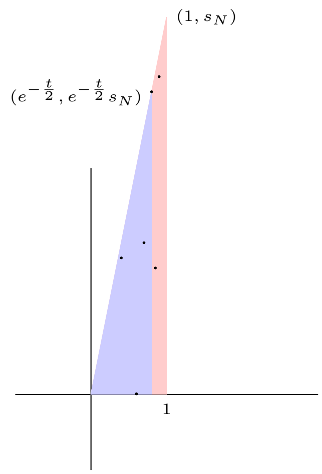
since this will be the N^{th} slope for the lattice $\Lambda_{z_{-t}}$ in the standard vertical strip $V_1 = \{y \geq 0, 0 < x \leq 1\}$. Then M_t is the total number of points in Λ_{z_0} in the triangle which is the standard trip V_1 bounded above by the line of slope $s_{N,t}$. That is, defining the triangles

$$T_t = \Delta((0,0), (1,0), (1, s_{N,t})) \text{ and } \tilde{T}_t = \Delta((0,0), (e^{-\frac{t}{2}}, 0), (e^{-\frac{t}{2}}, e^{-\frac{t}{2}} s_{N,t})),$$

we have (we are always counting only primitive points)

$$N = |\Lambda_{z_0} \cap \tilde{T}_t| \text{ and } M_t = |\Lambda_{z_0} \cap T_t|.$$

FIGURE 2. The triangle $\tilde{T}_t$ is in blue, and T_t is the union of the blue triangle and the red trapezoid. Marked points are primitive lattice points, so in this picture $N = 3$ and $M_t = 5$.



4.6. *M_t and Mixing.* Recall that to prove mixing, we want to prove equidistribution of geodesic segments, that is, we want to show that for any nice function v on Ω_1,

$$\int_0^b v(T^N z_{-t})dt \to b \int_{\Omega_1} vdm.$$

Using our definition of M_t, we have

$$\int_0^b v(T^N z_{-t})dt = \int_0^b v(T^{M_t} z_0)dt = \sum_{r=0}^{bN} v(T^{N+r} z_0)|\{t \in [0,b] : M_t = t + r\}|.$$

If we could show that

$$m_{t+r} = |\{t \in [0,b] : M_t = t + r\}| \sim 1/N,$$

then our integral would be an integral over the horocycle flow, and we could use unique ergodicity to get the equidistribution of this integral.

A natural interpretation of $|\{t \in [0,b] : M_t = t + r\}|$ is the *gap* between the r^{th} and $(r+1)^{st}$ x-coordinate (written in increasing order of x-coordinate) of primitive points in our lattice Λ_{z_0} in the trapezoidal strip with vertices at $(e^{-\frac{t}{2}}, 0), (1,0), (e^{-\frac{t}{2}}, e^{-\frac{t}{2}} s_{N,t}), (1, s_{N,t})$, that is, the symmetric difference between the triangles T_t and $\tilde{T}_t$.

4.7. *Independence.* It is not clear that the values $|\{t \in [0,b] : M_t = t+r\}|$ equidistribute. Another idea is to show that $T^k(z_0)$ (which yields, in particular, the horizontal component of the k^{th} vector that we see in the vertical strip V_1) becomes *independent* of the length $m_k = |\{t \in [0,b] : M_t = k\}|$. A precise version version of this is the following: fix $b > 0$, let $N \gg 0$, and consider the collection of points

$$\{(T^k(z_0), Nm_k : N \le k \le e^b N\} \subset \Omega \times \mathbb{R}^+.$$

Then we would like the uniform measure on this set to converge to some measure on $\Omega \times \mathbb{R}^+$ which looks like a product measure, of say, dm on Ω and an $\exp(-1)$ distribution on $\mathbb{R}^+$. We also note that

$$m_k = |\{t : R^{(N)}_{e^{-\frac{t}{2}}}(z_0) = k\}|,$$

where

$$R^{(N)}_{e^{-\frac{t}{2}}}(z_0) = \sum_{i=0}^{N-1} N_t(T^i_{e^{-\frac{t}{2}}}(z_0))$$

is the number of times the orbit of the point z_0 hits Ω_1 when it has hit the smaller transversal $\Omega_{e^{-\frac{t}{2}}}$ N times.

References

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