REPORT ON BIRS WORKSHOP 09W5102 “DEDEKIND SUMS IN GEOMETRY, TOPOLOGY, AND ARITHMETIC”

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1. Objectives of the workshop

The goal of the workshop was to explore the appearance of Dedekind sums [7, 18] in many different areas of geometry and number theory, with the hopes of illuminating new connections. We hoped that the balance between the lectures and free time as well as the intimate setting of the Banff Research Station would stimulate many informal discussions and collaborations. We were delighted to see that these objectives were fulfilled.

We would like to thank the BIRS staff for their hospitality and efficiency.

2. Topics of the workshop

2.1. **Special values of partial zeta functions.** Let F be a totally real number field of degree n, and let $\mathfrak{f}$, $\mathfrak{b}$ be two relatively prime ideals in $\mathcal{O}_F$. The partial zeta function attached to the ray class $\mathfrak{b}$ mod $\mathfrak{f}$ is defined by

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) := \sum_{\mathfrak{a} \equiv \mathfrak{b} \bmod \mathfrak{f}} N(\mathfrak{a})^{-s}, \quad \Re(s) > 1,$$

where $\mathfrak{a}$ runs over all all integral ideals in $\mathcal{O}_F$ such that the fractional ideal $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal generated by a totally positive number in the coset $1 + \mathfrak{f}\mathfrak{b}^{-1}$. A theorem of Klingen–Siegel shows that the special values $\zeta(\mathfrak{b}, \mathfrak{f}, s)$ at nonpositive integral values of s are rational. These special values are of keen interest in arithmetic because of their connections with the Brumer–Stark conjectures and because of relationships to class number formulas. Work of C. Meyer, Sczech, Shintani, and Zagier shows that these special values can be computed explicitly in terms of higher-dimensional Dedekind sums.

2.2. **Lattice-point enumeration in rational polyhedra.** While the first appearance of Dedekind sums in integer-point counting formulas goes back to a 1951 paper of Mordell [16], this connection was not well understood until the 90’s, when Pommersheim [17] showed how to compute the Ehrhart polynomial[1] of a 3-dimensional polytope using Dedekind sums as the only nontrivial ingredients. Formulas in higher dimensions (involving “higher-dimensional” Dedekind sums [8, 23]) and for polytopes with rational vertices (involving yet further generalizations of

[1] Given a polytope $P \subset \mathbb{R}^d$ with integral vertices, the Ehrhart polynomial of P is the counting function $\#\left(tP \cap \mathbb{Z}^d\right)$, which is a polynomial if we restrict t to positive integers, due to Ehrhart’s seminal work in the 60’s [4, 10].

Dedekind sums [2]) followed in the last decade. Recent activities involve computational complexity questions (e.g., what is the computationally efficient analogue of Dedekind's reciprocity law in higher dimensions?) and enumeration in the sense of listing lattice points (involving polynomial analogues of Dedekind sums due to Carlitz and Berndt–Dieter) [3].

2.3. **Dedekind sums in topology.** In topology, Dedekind sums appear when one computes a global geometric quantity (e.g., signature of a manifold, dimension of the space of sections of a holomorphic vector bundle over an orbifold). In many cases such quantities can be computed from local data, and the local data often involves Dedekind sums. For example, W. Meyer and Sczech [14] computed the Atiyah–Singer α-invariant $\alpha(s, E(A))$ of the 2-torus bundle $E(A)$ over a circle attached to a hyperbolic matrix $A \in SL_2(\mathbb{Z})$ in terms of Dedekind sums. Dedekind sums have also appeared in the study of the generalized Casson invariant and $SU(2)$-quantum invariants of rational homology 3-spheres. In the guise of Dedekind symbols, they appeared in the work of Kirby and Melvin [12] on the μ-invariants and the signature cocycle, and very recently in the work of Long and Reid [13], who constructed a generalization of the Dedekind symbol to fuchsian groups with cusp set $\mathbb{Q}$.

2.4. **Interrelationships.** Some interconnections between the above areas have been understood. For example, counting lattice points in integral polyhedra is the same as computing the dimension of the space of global sections of a line bundle over a projective toric variety. When the toric variety is a toric orbifold, one can apply Kawasaki's version of the Hirzebruch–Riemann–Roch theorem to obtain an expression for this dimension in terms of Dedekind sums. For another example, the computations of Meyer and Sczech show that the α-invariants of torus bundles $E(A)$ are related to special values of certain L-functions, which are then related to special values of partial zeta functions. Many other connections and relationships wait to be explored.

3. Contents of the talks

Many talks at the workshop reported presented many new results related to the above list of topics. Other talks presented new connections between geometry and number theory. In the following we summarize the contents of each talk. Junior speakers are indicated by a bullet (•).

3.1. **Dedekind sums in geometric and algebraic combinatorics. Ricardo Diaz** (University of Northern Colorado) spoke on *A Solid Angle Algorithm for Spherical Polytopes*. He outlined an algorithmic procedure for computing the spherical measure of spherical polytopes, based upon downward induction on dimension, via the Divergence Theorem. Potential applications include the determination of canonical solid-angle weights that behave additively under decomposition of lattice polytopes.

Karl Dilcher (Dalhousie University) spoke on *Reciprocity relations for Bernoulli numbers* [1, 9]. He started with the observation that several classical identities for Bernoulli numbers can be written as reciprocity relations, and then proved a new type of three-part reciprocity relation for Bernoulli numbers. As a consequence he obtain a quadratic recurrence for these numbers. This recurrence requires, surprisingly, the knowledge of only one third of the previous numbers. The talk ended with some other new convolution identities for Bernoulli numbers.

Todor Milev (•) (Jacobs University Bremen, Germany) spoke on *Partial fraction decompositions and an algorithm for computing the vector partition function* [15]. Milev's talk was based on works of M. Brion, A. Szenes and M. Vergne and is geared toward explicit computer realizations. In particular, he presented two algorithms for computing the vector partition function with respect to a finite set of vectors I as a quasipolynomial over a finite set of pointed polyhedral cones. He used his techniques to relate a result of P. Tumarkin and A. Felikson (and present an independent proof in the particular case of finite-dimensional root systems) to give bounds for the periods of the Kostant partition functions of E_6, E_7, E_8, F_4, G_2 (the periods are divisors of respectively 6, 12, 60, 12, 6). We note that Milev was a graduate student.

Giancarlo Urzua (•) (UMass Amherst) spoke on *Dedekind sums in Abelian covers and applications* [21]. Dedekind sums appear naturally in the computation of Chern numbers of Abelian covers of algebraic varieties. Urzua showed how, and some applications of this to Dedekind sums and to simply connected smooth projective surfaces of general type. For surfaces, his method to obtain high Chern ratios involves a large scale behavior of Dedekind sums and continued fractions described recently by K. Girstmair. Thanks to this, everything is encoded in certain invariants of the branch divisor. He described them and a bit of their geography.

Kevin Woods (•) (Oberlin College) spoke on *Counting With Rational Generating Functions* [22]. A step-polynomial is created by taking sums and products of the floor functions of degree one polynomials (in one or more variables). Like Dedekind sums, step-polynomials are closely tied to lattice point enumeration and related generating function methods. As an example, consider the Ehrhart quasi-polynomial, $f(t)$, of a rational polytope, P (that is, $f(t)$ counts the number of integer points in tP). One often considers the Hilbert series which is the generating function obtained by summing $f(t)x^t$ over all nonnegative t. This generating function has the advantage that it can be manipulated through algebraic means. On the other hand, we will see that $f(t)$ can be written as a step-polynomial (and the algorithm to find it is efficient). This representation has the advantage that it is an explicit function that can immediately be evaluated for any t. Fortunately, we do not have to choose between nimble generating functions and concrete step-polynomials, as one can convert back and forth between them in polynomial time (in fixed dimension).

3.2. **Dedekind sums in topology. Ruth Lawerence** (Hebrew University) gave a survey lecture on quantum invariants in topology, and explained how Dedekind sums arise in the quantum invariants of homology 3-spheres. This was one of the lectures that was videorecorded.

3.3. **Dedekind sums in number theory. Abdelmejid Bayad** (University Evry-Paris) spoke on *Some facets of multiple Dedekind-Rademacher sums.* Bayad introduced two kind of multiple Dedekind-Rademacher sums, in terms of Bernoulli and Jacobi modular forms. He proved their reciprocity laws, established the Hecke action on these sums, and obtained new Knopp–Petersson identies. He showed how to deduce Dedekind's, Rademacher's, Sczech's reciprocity formulas from his main results. Some applications in number theory (special values of some L-functions, periods, etc.) were discussed.

Pierre Charollois (•) (Institut de Mathématiques de Jussieu) spoke on *Integral Dedekind sums for* $GL_n(\mathbb{Q})$. This was a report on joint work with S. Dasgupta, based on the construction by R. Sczech of a rational valued cocycle for $GL_n(\mathbb{Q})$. (Both Dasgupta and Sczech were at the workshop.) Their refinement provides an integral valued cocycle, which can be expressed by a simpler fomula in terms of Bernoulli numbers.

Samit Dasgupta (•) (UC Santa Cruz) spoke on *Dedekind Sums and Gross–Stark Units*. In 2006 he stated a conjectural formula for Gross–Stark units over number fields. In this talk he discussed the role played by Dedekind sums in this formula. He also looked towards the function field setting for inspiration, where the conjectural formula may be proven following work of Hayes and using the theory of Drinfeld modules.

Richard Hill (University College London) spoke on *Shintani cocycles* [11]. He described an $(n-1)$-cocycle on the group $GL_n(\mathbb{Q})$, taking values in a space of power series called the Shintani functions. The coefficients of these power series are quite general Dedekind sums, and special values of the power series may be used to express special values of L-functions. The cocycle relation encodes the reciprocity laws for the Dedekind sums.

Veli Kurt (Akdeniz University, Turkey) spoke on *Higher Dimensional Dedekind Sums* [6]. The aim of this work is to construct new Dedekind type sums. Kurt and coauthors construct generating functions of Barnes-type multiple Frobenius-Euler numbers and polynomials. By applying Mellin transformation to these functions. They define Barnes type multiple L-functions which interpolate Frobenius–Euler numbers at negative integers. By using generalization of the Frobenius–Euler functions, they define generalized Dedekind type sums and prove corresponding reciprocity laws.

Sinai Robins (Nanyang Technological University, Singapore) spoke on *Some Polyhedral Dedekind sums*, which occur when we use the Poisson summation formula to analyze various sums over the integer points in a polytope. Similar sums occur in the work of Sczech and Gunnells (both of whom were at the workshop), but from a different perspective. It is often useful to deal directly with the infinite lattice sums, (as Sczech and Gunnells also do) that have the finite Dedekind sum properties, and here we find a similar phenomenon.

Robert Sczech (Rutgers University Newark) spoke on *Dedekind sums and derivatives of partial zeta functions in real quadratic fields at* $s = 0$. The classical Dedekind sums arise in formulas for special values of partial zeta values in real quadratic fields at $s = 0$. Sczech introduced a new type of Dedekind sums and showed how they can be used to calculate derivatives of partial zeta functions at $s = 0$ which show up in Stark's conjecture.

Yilmaz Simsek (Akdeniz University, Turkey) spoke on *Dedekind-type Sums* [20]. By p-adic q-Volkenborn integral and generating functions of Bernoulli numbers, he gave q-analogues of a family of zeta functions. He gave relations between p-adic q-Volkenborn integral and p-adic q-Dedekind -type sums.

Glenn Stevens (Boston University) spoke on *Milnor Algebras, Modular Symbols, and Values of L-functions*. He constructed a family of *Eisenstein modular symbols* over $GL_n(\mathbb{Q})$ taking values in the the Milnor algebra of a certain ring of trigonometric functions. Higher Dedekind sums arise as certain coefficients of this modular symbol and homological relations correspond to standard Dedekind reciprocity laws. A curious feature of the Eisenstein symbol (for $n = 2$) is that it contributes to both $\pm$-eigenspaces for complex conjugation acting on the cohomology of modular curves. This contrasts with periods of Eisenstein series, which contribute only to the *odd*-eignenspace. He illustrated this phenomena with a theorem of Cecelia Busuioc [5], proving congruence formulas for *even* critical values of L-functions of cusp forms satisfying Eisenstein congruences, as conjectured by Romyar Sharifi [19]. This was the second lecture that was videorecorded.

4. Feedback and response from the conference

The workshop was well-attended by researchers at all points in their careers, and from countries all around the world, including Canada, USA, Israel, Japan, Germany, UK, France, Singapore, and Turkey. After the completion of the workshop, the organizers received many positive comments from both senior and junior participants. One junior participant wrote *"I think this was a specially interesting conference because it brought together people from quite different mathematical backgrounds, who are nevertheless working on quite similar questions."* Another senior participant wrote *"Meeting with so many experts in such a diverse set of fields related to the simple unifying theme of Dedekind Sums was a rare pleasure which will undoubtedly influence my future research on arithmetic aspects of the theory."* Another mentioned that *"the combination of the workshop's focused topic and the experts who attended the workshop was extremely well tailored."* One even explicitly hoped that something similar will happen in the future: *"I hope that you can organize in the future another workshop in the same area."*

Based on these comments, we believe that the conference was successful. We also believe that by bringing together people from different fields who were all thinking about Dedekind sums, we have faciliated new collaborations. Because of the success of the workshop, we hope to apply to Banff in the future for another week, perhaps with the intent of running in 2012, so that participants can give updates and progress reports, and so that new junior people can be introduced to Dedekind sums.

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