# BIRS Workshop # 07w5503
# Geometric Inequalities

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## 1 Overview of the Field

Analytical methods are playing an ever increasing role in geometry. Sharp inequalities for integral functionals and for eigenvalues of the Laplacian contain much information about the geometry of the underlying space. Conversely, geometric ideas are crucial for understanding optimization problems, specifically problems with symmetry. In recent years, a web of new inequalities and surprising relationships with known inequalities have been discovered. In spectral geometry, recent results have included progress on some long-standing conjectures. New applications of geometric inequalities in kinetic theory and statistical mechanics are emerging. Some of these developments have been described in the recent surveys of Gardner [47], Villani [86] and Ashbaugh [5, 6].

### 1.1 Optimal transportation

Nowhere are geometry and analysis more tightly linked than in the Monge-Kantorovich theory of optimal transportation. Originally formulated as the practical problem of transporting one given mass distribution to another in a cost-minimizing matter, it has become a major tool in geometric analysis, while simultaneously giving rise to new geometric problems. Many of the geometric applications are based on theorems of Brenier [21] and McCann [69], which imply that any pair of probability measures can be connected by a transportation map with desirable monotonicity properties. In the late-1990's, a one-line proof of the isoperimetric inequality based on the Brenier-McCann map became part of the folklore in the area. With this map many classical and new inequalities with can be proved with nothing more than the arithmetic-geometric mean inequality, integration by parts, and change of variables. Among the results that have been obtained in this way are generalizations of Young's inequality by Barthe [13], a family of sharp Gagliardo-Nirenberg-Sobolev inequalities by Cordero-Erausquin, Nazaret and Villani [41], sharp Brezis-Lieb trace inequalities by Maggi and Villani [66], and some variants of the Gaussian correlation conjecture by Caffarelli [31] and Cordero-Erausquin [39]. Besides providing new geometric insight, optimal transportation proofs have led to the discovery of previously unknown "dual" versions of many classical inequalities [13, 3].

Optimal transportation also lends geometric meaning to the Wasserstein distances on probability measures. The function spaces defined by these distance functions have proved to be useful for the study of entropy inequalities and dissipative equations. Displacement convexity, which was discovered by McCann [68] and extended to Riemannian settings by Otto and Villani [73] and Cordero-Erausquin, McCann and Schmuckenschläger [40], has turned out to be the right notion for defining Ricci curvature bounds on a class of metric-measure spaces [63, 79]. This resolved a long-standing problem in synthetic geometry.

### 1.2 Symmetrization

The symmetric decreasing rearrangement has long been known to improve the value of physically relevant functionals such as the Coulomb energy of a charge distribution, the kinetic energy of a single particle in quantum mechanics, and the fundamental tone of a membrane. Originally devised for analytical proofs of the isoperimetric inequality by Steiner [77], rearrangements appeared in the 1970's and 1980's as key ingredients of Talenti's identification of the sharp constants in the Sobolev inequalities [81] and Lieb's corresponding results on the Hardy-Littlewood-Sobolev inequalities [60] Since then, much of the research in rearrangements has focused on technical questions, such as the continuity results of Almgren and Lieb [4], the search for a "master inequality" by Baernstein [11], and the characterization of equality cases in various rearrangement inequalities by Brothers and Ziemer [26], Burchard [27], and Chlebík, Cianchi, and Fusco [36]. Rearrangements have found many applications to geometric inequalities for eigenvalues of elliptic and higher order operators, starting with the Faber-Krahn inequality for the fundamental tone of a membrane, and continuing to the recent proof of the Payne-Pólya-Weinberger conjecture by Ashbaugh and Benguria [7, 8].

### 1.3 Nonlinear heat flows

Very recently, nonlinear heat flows have appeared independently in results of Carlen, Lieb and Loss [32] and Bennett, Carbery, Christ and Tao [17] on Young-type inequalities for multiple integrals on $\mathbf{R}^n$ and $\mathbf{S}^n$, and in Perelman's spectacular proofs [74] of the Poincaré and geometrization conjectures. The basic idea is that for a functional whose extremals include simple objects such as Gaussians in $\mathbf{R}^n$ or constant functions on a manifold, one should be able to construct a nonlinear diffusion that drives the functional monotonically towards its extremum while preserving the relevant side conditions. In other words, the problem is to construct a flow for which the functional in question acts as a Lyapunov function. This method also has a long history — we are aware of correlation inequalities that were obtained in this way by Pitt [75] and Herbst and Pitt [53], and suspect that there are earlier instances. Heat flow ideas have also been used for the Gaussian isoperimetric inequality by Bobkov [20] and Bakry and Ledoux [12].

## 2 Applications and Open Problems

Connections of the eigenvalues of the Laplacian (and related operators) with the geometry of the underlying domain have attracted the interest of mathematicians for at least the last 50 years, and are likely to do so for many years to come. A particularly stubborn but very interesting open geometric eigenvalue problem which might be amenable to proof via rearrangements is the Pólya-Szegö conjecture for the first buckling eigenvalue of a clamped plate. This problem involves the biharmonic operator and the result would be the analog of the Faber-Krahn inequality in this setting, that is, that among all domains of a given area, the first eigenvalue is minimized at the disk. In physical terms this says that among homogeneous plates of the same material and of a fixed area, with all possible shapes, the circular one buckles first when subjected to compressive loading of its edges. As a mathematical problem, the conjecture makes sense in any dimension. It has remained open since it was stated by Pólya and Szegö around 1950, despite attempts by various mathematicians to prove it. Under the assumption that a first eigenfunction of the problem exists which does not change sign, Szego was able to prove the conjecture in the 1950s. However, it is known that this assumption is rather restrictive and not at all the general case. Without the sign assumption, weaker estimates of the right general type have been obtained by Bramble and Payne [22], for n dimensions, by Ashbaugh and Laugesen [10]. A related conjecture for the vibration of a clamped plate (due, in the classical, two-dimensional case, to Rayleigh) is open in all dimensions above three, the cases of dimensions two and three having been established affirmatively by Nadirashvili [70, 71] and Ashbaugh and Benguria [8], respectively.

New applications for geometric inequalities are appearing in statistical mechanics. Mass transportation ideas have been used to study dissipative equations from kinetic theory since the late 1990's, and the area is evolving rapidly. Displacement convexity is being used in the work of Otto [72], Agueh [1] and others to prove logarithmic Sobolev inequalities and rates of convergence to equilibrium for nonlinear diffusion equations. There is reason to expect that displacement convexity will prove useful to estimate rates of decay for correlations in other models from statistical mechanics, with the goal of establishing phase transitions. The hope is that, in contrast with current methods that take advantage of the specific structure of models, convexity methods will be robust under small changes to the model.

For questions of convergence, stability, or robustness, a lower bound on the difference between the two sides of a geometric inequality in terms of some geometric quantity can be very valuable. For a long time, the best available result was a quantitative isoperimetric inequality due to R. R. Hall [52], which bounds the difference between a body in $\mathbf{R}^n$ and a suitably translated ball of the same volume (a measure of "asymmetry") from above in terms of the difference between their perimeters (the "isoperimetric deficit"). Fusco, Maggi, and Pratelli [45, 46] have sharpened Hall's result and obtained corresponding quantitative versions of Sobolev inequalities. Hall's inequality was used by Sznitman [80] and Povel [76] to understand the long-time behavior of Brownian motion with obstacles in dimensions three and above. Related asymmetry results have been used for dynamical stability results in Vlasov-Poisson and Vlasov-Maxwell systems citeGR,BG. It is a challenge to find quantitative inequalities for integral functionals that involve convolutions.

A larger challenge is posed by inequalities for functions defined on manifolds taking values in $\mathbf{R}^n$. Recently, Goldshtein and Troyanov [49] have proved Sobolev inequalities for differential forms on manifolds. Nothing is known about the sharp constants, which presumably encode geometric information about the manifold. Rearrangements do not apply to vector-valued functions, and it is hard to obtain inequalities on manifolds other than $\mathbf{S}^n$ and hyperbolic spaces. Heat flow methods may be the most promising approach here.

The regularity of optimal transportation plans has been the subject of many studies since the early 1990's. A major open problems are the precise geometric obstacles for the regularity of an optimal transportation plan for two given measures on differnt Rimennian manifolds. In the standard setup on $\mathbf{R}^n$, where the cost is given by the square of the Euclidean distance, The key geometric assumption is that the target measure should be supported on a convex set. Here, the regularity of the Brenier-McCann map was resolved (in the context of boundary-value problem for the Monge-Ampére equation) by Caffarelli [30] Delanoe [43] and Urbas [82]. Concave cost functions can also give rise to discontinuities even for smooth measures [67]. Finally, negative curvature of the underlying manifold can cause discontinuities, but positive curvature by itself does is not sufficient to guarantee regularity [57]. Recent results of Ma, Trudinger and Wang [65], Loeper [62] and Kim and McCann [58, 59] have opened the door towards a comprehensive regularity theory of optimal transportation.

# 3 Presentation Highlights

## 3.1 Optimal transportation

Alberto Bressan opened the workshop with a lecture on optimal transportation metrics for nonlinear wave equations. He discussed some examples where the solution does not depend continuously on the data in any of the natural Sobolev norms. However, the flow can be rendered Lipschitz continuous with respect a new distance function, which is determined by a problem of optimal transportation. This gives rise to a Riemannian structure in the problem. [24, 25].

Stamatis Dostoglou explained how to approximate solutions of the Navier-Stokes equation, using ides from optimal transportation. This gives rise to a new concept of weak solutions [44].

Franck Barthe extended the optimal transportation method for log-Sobolev type inequalities and isoperimetric inequalities. He described sufficient conditions for a measure to satisfy such an inequality that assume strong integrability, rather than convexity. As a result, he recovered precise concentration inequalities for log-concave measures, extended Bobkov's isoperimetric inequalities in this case, as well as Wang's extension of the Bakry-Emery criterion [14].

Dario Cordero-Erausquin talked about interpolation and geometric inequalities, discussing joint work with Bo'az Klartag. His goal was to find various ways of interpolating between norms to obtain sharpened versions of the Brunn-Minkowski and Santal0́ inequality [38].

Young-Heon Kim presented new results on curvature and the continuity of optimal transportation maps. He reported on recent results with Robert McCann of a semi-Riemannian metric associated with the cost function when transporting between two manifolds [58, 59]. This gives a general geometric framework for the regularity theory of Ma, Trudinger and Wang for optimal transportation plans [65, 62].

Guillaume Carlier gave an elementary variational proof of the classical theorems of Minkowsi and Alexandrov that guarantee the existence (and uniqueness up to translations) of a closed convex hypersurface with a given Gaussian curvature or with a given surface function. He emphasized the analogy with the classical optimal transportation problem and considered some applications to shape optimization [33].

### 3.2 Geometric flows

Tony Carbery discussed recent work on the Brascamp-Lieb inequalities for multilinear integrals of products of functions in several dimensions. He described the main tool, a monotonicity formula for positive solutions to heat equations in linear and multilinear setting [17].

Stefan Valdimarsson described all optimizers for this Brascamp-Lieb inequality. His proof combines the heat flow method with a careful analysis of the multilinear structure of the functional [83].

Aaron Smith presented a family of special solutions of Ricci flow on two-dimensional asymmetric cigars. These asymmetric cigars converge under the flow towards the standard symmetric cigar soliton, in the sense that they are bi-Lipschitz equivalent and the Lipschitz constant approaches one as time becomes large. He also described the precise exponential rate of convergence [29].

### 3.3 Sharp Gagliardo-Nirenberg and Sobolev inequalities

Rafael Benguria discussed optimal Gagliardo-Nirenberg inequalities for fourth order elliptic equations in one dimension. He compared minimization problems of Gagliardo-Nirenberg type on finite intervals with periodic boundary conditions with the corresponding problems on the whole real line. The main result is that the infimum is not achieved on the whole real line, but that it agrees with the minimum that is achieved on any finite interval [15].

Martial Agueh presented a method giving the sharp constants and optimal functions of all the GagliardoNirenberg inequalities involving the $L^p$-norm of the gradient. Optimal functions are explicitly derived from a specific non-linear ordinary differential equation which appears to be linear for a subclass of the GagliardoNirenberg inequalities or when the space dimension reduces to 1. The analysis includes also the sharp $L^p$-Nash inequalities [2].

Nicola Fusco discussed a quantitative version of the Sobolev inequality. Here, the difference between the two sides of the inequality is expressed in terms of the distance from the family of optimizers. In the proof, the inequality is proved first for radially decreasing functions, and then extended successively to $n$-symmetric functions and general functions in $W^{1,p}$, using symmetrization arguments [42].

### 3.4 Hardy inequalities

Several speakers addressed improved Hardy-type inequalities. Since these inequalities have no optimizers, it is a natural question whether the inequality can be improved by adding a suitable positive term to the left hand side.

Adele Ferone described such improved Hardy inequalities both in the classical and the limiting case. The relevant positive remainder terms depend on the distance from the family of "virtual" extremals [37].

Amir Moradifam gave criteria for a radially symmetric potential such that the remainder term in Hardy's inequality for a domain can be bounded from below by a potential term. His approach clarifies the issue behind the lack of an optimal improvement, while yielding other interesting "dual" inequalities. These results have immediate applications to the corresponding Schrödinger equations [48].

In the special case of a three-dimensional half-space, Rupert Frank showed that the sharp constant in the Hardy-Sobolev-Maz'ya inequality is given by the Sobolev constant. This is shown by a duality argument that relates the problem to a Hardy-Littlewood-Sobolev type inequality whose sharp constant is determined as well. [16]

Francesco Chiacchio reported on improved Hardy inequalities for the Gaussian measure, and explained their connection with Gross' log-Sobolev inequality. He also presented a family of factorized measures that enjoy isoperimetric inequalities, and used them to get sharp estimates for elliptic problems with degeneracy at infinity [23, 34].

### 3.5 Isoperimetric inequalities for eigenvalues

Marcello Lucia presented an inequality for the isoperimetric profile of a compact connected Riemannian manifold. This inequality can be used to bound the principal eigenvalue of the Laplacian on the manifold from below. As an application of the inequality, he obtains new uniqueness results for two-dimensional semilinear equations. [64]

Lotfi Hermi discussed how to use trace identities of the type derived by Harrell and Stubbe to produce universal bounds on the eigenvalues of the Dirichlet Laplacian. He then showed how to use these identities to produce new Weyl-type bounds for averages of eigenvalues, as well as an alternate proof of the Berezin-Li-You inequality as viewed by Laptev and Weidl [54].

### 3.6 Calculus of Variations and elliptic equations

Benjamin Stephens reported on the thread-wire problem, which is a minimal surface problem with a fixed boundary (given by a "wire") and a free boundary (given by a "thread"). He used isoperimetric arguments to show that if the length of the thread is close to the length of the wire, then a minimizing surface will remain close to the wire. An isoperimetric argument plays a key role in the proof [78].

Juncheng Wei considered a nonlinear elliptic eigenvalue problem with a supercritical negative exponent on a domain in $\mathbf{R}^n$. He showed that this problem as a unique minimal solution, provided that $\lambda > 0$ is small enough, and the branch of solutions must undergo infinitely many bifurcations or turning points [51].

Tobias Weth presented a result on radial symmetry of positive solutions to a class of semilinear polyharmonic Dirichlet problems in the unit ball. The result was obtained via a new variant of the moving plane method. In some special cases the result implies uniqueness of positive solutions [18].

Andrea Cianchi discussed isocapacitary inequalities that relate the relative capacity of a subset of an open domain $\Omega \subset \mathbf{R}^n$ to its Lebesgue measure. As a consequence, he obtained a priori estimates for nonlinear elliptic Neumann problems [35].

Antoine Henrot derived isoperimetric inequalities for the product of some moments of inertia on convex sets. As an application, we demonstrates an isoperimetric inequality for the product of the $n$ first nonzero eigenvalues of the Stekloff problem in $\mathbf{R}^n$ [53].

## 4 Outcome of the Meeting

A meeting between experts in optimal transportation, rearrangements, geometric flows and spectral geometry was long overdue. This workshop came at a time that saw an explosion of interest in geometric flows in response to Perelman's reslts on Ricci flow. At the same time, optimal transportation methods were just becoming accessible to non-experts through Villani's books [84, 85] available at www.umpa.ens-lyon.fr/ cvillani). The workshop brought together mathematicians working on geometric inequalities with colleagues interested in current and potential applications. The audience included specialists in optimal transportation, rearrangements, nonlinear heat flows, and spectral geometry, as well as researchers with interested in applications of optimal transportation to dissipative PDE and fluid mechanics, and a few participants with broader interests in geometric inequalities, geometric flows and calculus of variations.

Several subgroups had worked on closely related problems with different methods, sometimes with equivalent results, which suggested deeper deeper connections waiting to be explored. Arguably, rearrangements are just particular transportation plans, but the relationship with optimal transportation has yet to be made explicit and put to use. For instance, to our knowledge, optimal transportation methods have not been used in spectral geometry. Connections with the theory of Ricci curvature and Ricci flow, especially in non-smooth settings, have begun to emerge in preprints by Lott and Villani, Sturm, and Topping and McCann, while potential applications for geometric inequalities in statistical mechanics are emerging. By making these connections, we believe that the meeting has accelerated the rate of progress and has opened directions for future research.

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