Densest Packings of Spheres

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1 Introduction

A family of (not necessarily infinitely many) non-overlapping congruent balls in d-dimensional space of constant curvature is called a packing of congruent balls in the given d-space that is either in the Euclidean d-space $\mathbb{E}^d$ or in the spherical d-space $\mathbb{S}^d$ or in the hyperbolic d-space $\mathbb{H}^d$. The goal of this report is to give a state of the art description of d-dimensional sphere packings. On the one hand, the research on sphere packings seems to be one of the most active areas of (discrete) geometry on the other hand, it is one of the oldest areas of mathematics ever studied. The topics discussed in separate sections of this report are the following ones:

- Hadwiger numbers of convex bodies and kissing numbers of spheres;
- Touching numbers of convex bodies;
- Newton numbers of convex bodies;
- One-sided Hadwiger and kissing numbers;
- Contact graphs of finite packings and the combinatorial Kepler problem;
- Isoperimetric problems for Voronoi cells, the strong dodecahedral conjecture and the truncated octahedral conjecture;
- The strong Kepler conjecture;
- Bounds on the density of sphere packings in higher dimensions;
- Solidity and uniform stability.

Each section outlines the state of the art of relevant research along with some of the "most wanted" research problems. Generally speaking the material covered belongs to combinatorics, convexity and discrete geometry however, often the methods indicated cover a much broader spectrum of mathematics including computational geometry, hyperbolic geometry, the geometry of Banach spaces, coding theory, convex analysis, geometric measure theory, (geometric) rigidity, topology, linear programming and non-linear optimization.

2 Hadwiger numbers of convex bodies and kissing numbers of spheres

Let $\mathbf{K}$ be a convex body (i.e. a compact convex set with nonempty interior) in d-dimensional Euclidean space $\mathbb{E}^d, d \geq 2$. Then the Hadwiger number $H(\mathbf{K})$ of $\mathbf{K}$ is the largest number of non-overlapping translates of $\mathbf{K}$ that can all touch $\mathbf{K}$. An elegant observation of Hadwiger [49] is the following.

Theorem 2.1 *For every d-dimensional convex body* $\mathbf{K}$*,*

$$H(\mathbf{K}) \leq 3^d - 1,$$

where equality holds if and only if $\mathbf{K}$ *is an affine d-cube.*

On the other hand, in another elegent paper Swinnerton-Dyer [84] proved the following lower bound for Hadwiger numbers of convex bodies.

Theorem 2.2 *For every d-dimensional ($d \geq 2$)convex body* $\mathbf{K}$*,*

$$d^2 + d \leq H(\mathbf{K}).$$

Actually, finding a better lower bound for Hadwiger numbers of d-dimensional convex bodies is a highly challanging open problem for all $d \geq 4$. (It is not hard to see that the above theorem of Swinnerton-Dyer is sharp for dimensions 2 and 3.) The best lower bound known in dimensions $d \geq 4$ is due to Talata [85], who applying Dvoretzky's theorem on spherical sections of centrally symmetric convex bodies succeded to show the following inequality.

Theorem 2.3 *There exists an absolute constant $c > 0$ such that*

$$2^{cd} \leq H(\mathbf{K})$$

holds for every positive integer d and for every d-dimensional convex body $\mathbf{K}$*.*

Now, if we look at convex bodies different from a Euclidean ball in dimensions larger than 2, then our understanding of their Hadwiger numbers is very limited. Namely, we know the Hadwiger numbers of the following convex bodies different from a ball. The result for tetrahedra is due to Talata [86] and the rest was proved by Larman and Zong [60].

Theorem 2.4 *The Hadwiger numbers of tetrahedra, octahedra and rhombic dodecahedra are all equal to 18.*

In order to gain some more insight on Hadwiger numbers it is natural to pose the following question.

Problem 2.5 *For what integers k with $12 \leq k \leq 26$ does there exist a 3-dimensional convex body with Hadwiger number k? What is the Hadwiger number of a $d-$dimensional simplex (resp., crosspolytope) for $d \geq 4$?*

The second main problem in this section is fondly known as the kissing number problem. The kissing number τ_d is the maximum number of nonoverlapping d-dimensional balls of equal size that can touch a congruent one in $\mathbb{E}^d$. In three dimension this question was the subject of a famous discussion between Isaac Newton and David Gregory in 1964. So, it is not surprising that the literature on the kissing number problem is "huge". Perhaps the best source of information on this problem is the book [35] of Conway and Sloane. In what follows we give a short description of the present status of this problem.

$\tau_2 = 6$ is trivial. However, determining the value of τ_3 is not a trivial issue. Actually the first complete and correct proof of $\tau_3 = 12$ was given by Schütte and van der Waerden [82] in 1953. The subsequent (two pages) often cited proof of Leech [61], which is impressively short, contrary to the common belief does contain some gaps. It can be completed though, see for example, [66]. Further more recent proofs can be found in [29], [1] and in [72]. None of these are short proofs either and one may wonder whether there exists a proof of $\tau_3 = 12$ in THE BOOK at all. (For more information on this see the very visual paper [32].) Thus, we have the following theorem.

Theorem 2.6 $\tau_2 = 6$ *and* $\tau_3 = 12$*.*

The race for finding out the kissing numbers of Euclidean balls of dimension larger than 3 was always and is even today one of the most visible research projects of mathematics. Following the chronological ordering, here are the major inputs. Coxeter [36] conjectured and Böröczky [27] proved the following theorem, where $F_d(\alpha) = \frac{2^d U}{d! \omega_d}$ is the Schläfli function with U standing for the spherical volume of a regular spherical $(d-1)$-dimensional simplex of dihedral angle 2α and with ω_d denoting the surface volume of the d-dimensional unit ball.

Theorem 2.7 $\tau_d \leq \frac{2F_{d-1}(\beta)}{F_d(\beta)}$, *where* $\beta = \frac{1}{2}$*arcsec* d.

It was another breakthrough when Delsarte's linear programming method (for details see for example [77]) was applied to the kissing number problem and so, when Kabatiansky and Levenshtein [59] succeded to improve the upper bound of the previous theorem for large d as follows. The lower bound mentioned below was found by Wyner [87] several years earlier.

Theorem 2.8 $2^{0.2075d(1+o(1))} \leq \tau_d \leq 2^{0.401d(1+o(1))}$.

As the gap between the lower and upper bounds is exponential it was a great surprise when Levenshtein [61] and Odlyzko and Sloane [75] independently found the following exact values for τ_d.

Theorem 2.9 $\tau_8 = 240$ *and* $\tau_{24} = 196560$.

In addition, Bannai and Sloane [3] were able to prove the following.

Theorem 2.10 *There is a unique way (up to isometry) of arranging* 240 *(resp.,* 196560*) nonoverlapping unit spheres in* 8*-dimensional (resp.,* 24*-dimensional) Euclidean space such that they touch another unit sphere.*

The latest surprise came when Musin [70], [71] extending Delsarte's method found the kissing number of 4-dimensional Euclidean balls. Thus, we have

Theorem 2.11 $\tau_4 = 24$.

In connection with Musin's result we believe in the following conjecture.

Conjecture 2.12 *There is a unique way (up to isometry) of arranging* 24 *nonoverlapping unit spheres in* 4*-dimensional Euclidean space such that they touch another unit sphere.*

Using the spherical analogue of the technique devoloped in [11] K. Bezdek [22] gave a proof of the following theorem that one can regard as the local version of the above conjecture.

Theorem 2.13 *Take a unit ball* $\mathbf{B}$ *of* $\mathbb{E}^4$ *touched by 24 other (nonoverlapping) unit balls* $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{24}$ *with centers* $C_1, C_2, \ldots, C_{24}$ *such that the centers* $C_1, C_2, \ldots, C_{24}$ *form the vertices of a regular 24-cell* $\{3,4,3\}$ *in* $\mathbb{E}^4$. *Then there exists an* $\epsilon > 0$ *with the following property: if the nonoverlapping unit balls* $\mathbf{B}'_1, \mathbf{B}'_2, \ldots, \mathbf{B}'_{24}$ *with centers* $C'_1, C'_2, \ldots, C'_{24}$ *are chosen such that* $\mathbf{B}'_1, \mathbf{B}'_2, \ldots, \mathbf{B}'_{24}$ *are all tangent to* $\mathbf{B}$ *in* $\mathbb{E}^4$ *and for each* $i, 1 \leq i \leq 24$ *the Euclidean distance between* C_i *and* C'_i *is at most* ϵ, *then* $C'_1, C'_2, \ldots, C'_{24}$ *form the vertices of a regular 24-cell* $\{3,4,3\}$ *in* $\mathbb{E}^4$.

There is a great list of record kissing numbers in dimensions from 32 to 128 in [74] and also, we refer the interested reader to the paper [39] of Edel, Rains and Sloane for some amazingly elementary but efficient constructions.

3 Touching numbers of convex bodies

The touching number $t(\mathbf{K})$ of a convex body $\mathbf{K}$ in d-dimensional Euclidean space $\mathbb{E}^d$ is the largest possible number of mutually touching translates of $\mathbf{K}$ lying in $\mathbb{E}^d$. The elegant paper [37] of Danzer and Grünbaum gives a proof of the following fundamental inequality. In fact, this inequality was phrased by Petty [76] as well as by P. S. Soltan [83] in another equivalent form saying that the cardinality of an equilateral set in any d-dimensional normed space is at most 2^d.

Theorem 3.1 *For an arbitrary convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$

$$t(\mathbf{K}) \leq 2^d$$

with equality if and only if $\mathbf{K}$ *is an affine d-cube.*

In connection with the above inequality K. Bezdek and Pach [15] conjecture the following even stronger result.

Conjecture 3.2 *For any convex body* $\mathbf{K}$ *in* $\mathbb{E}^d, d \geq 3$ *the maximum number of pairwise tangent positively homothetic copies of* $\mathbf{K}$ *is not more than* 2^d.

This problem is still quite open. It seems that the only published upper bound is $3^d - 1$ in [15].

It is natural to ask for a non-trivial lower bound for $t(\mathbf{K})$. Brass [30] as an application of Dvoretzky's well-known theorem gave a partial answer for the existence of such a lower bound.

Theorem 3.3 *For each* k *there exists a* $d(k)$ *such that for any convex body* $\mathbf{K}$ *of* $\mathbb{E}^d$ *with* $d \geq d(k)$

$$k \leq t(\mathbf{K}).$$

It is remarkable that the natural sounding conjecture of Petty [76] stated next is still open for all $d \geq 4$.

Conjecture 3.4 *For each convex body* $\mathbf{K}$ *of* $\mathbb{E}^d, d \geq 4$

$$d + 1 \leq t(\mathbf{K}).$$

A generalization of the concept of touching numbers was introduced by K. Bezdek, M. Naszódi and B. Visy [19] as follows. The mth touching number (or the mth Petty number) $t(m, \mathbf{K})$ of a convex body $\mathbf{K}$ of $\mathbb{E}^d$ is the largest cardinality of (possibly overlapping) translates of $\mathbf{K}$ in $\mathbb{E}^d$ such that among any m translates always there are two touching ones. Note that $t(2, \mathbf{K}) = t(\mathbf{K})$. The following theorem proved by K. Bezdek, M. Naszódi and B. Visy [19] states some upper bounds for $t(m, \mathbf{K})$.

Theorem 3.5 *Let* $t(\mathbf{K})$ *be an arbitrary convex body in* $\mathbb{E}^d$. *Then*

$$t(m, \mathbf{K}) \leq \min\left\{(m-1)4^d, \binom{2^d + m - 1}{2^d}\right\}$$

holds for all $m \geq 2, d \geq 2$. *Also, we have the inequalities*

$$t(3, \mathbf{K}) \leq 2 \cdot 3^d, \;\; t(m, \mathbf{K}) \leq (m-1)[(m-1)3^d - (m-2)]$$

for all $m \geq 4, d \geq 2$. *Moreover, if* $\mathbf{B^d}$ *(resp.,* $\mathbf{C^d}$*) denotes a d-dimensional ball (resp., d-dimensional affine cube) of* $\mathbb{E}^d$, *then*

$$t(2, \mathbf{B^d}) = d + 1, \;\; t(m, \mathbf{B^d}) \leq (m-1)3^d, \;\; t(m, \mathbf{C^d}) = (m-1)2^d$$

hold for all $m \geq 2, d \geq 2$.

We cannot resist on raising the following question (for more details see [19]).

Problem 3.6 *Prove or disprove that if* $\mathbf{K}$ *is an arbitrary convex body in* $\mathbb{E}^d$ *with* $d \geq 2$ *and* $m > 2$, *then*

$$(m-1)(d+1) \leq t(m, \mathbf{K}) \leq (m-1)2^d.$$

4 Newton numbers of convex bodies

According to L. Fejes Tóth [44] the Newton number $N(\mathbf{K})$ of a convex body $\mathbf{K}$ in $\mathbb{E}^d$ is defined as the largest number of congruent copies of $\mathbf{K}$ that can touch $\mathbf{K}$ without having interior points in common. (Note that unlike in the case of Hadwiger numbers here it is not necessary at all to use translated copies of the given convex body. In fact, often it is better to use rotated or reflected ones.) For the special case when $\mathbf{K}$ is a ball we refer the reader to Section 2 of this paper. Here we focus on the case when $\mathbf{K}$ is different from a ball. Somewhat surprisingly, in this case only planar results are known. Namely, Linhart [65] and Böröczky [26] determined the Newton numbers of regular convex polygons.

Theorem 4.1 *If $N(n)$ denotes the Newton number of a regular convex n-gon in $\mathbb{E}^2$, then*

$$N(3) = 12, N(4) = 8 \text{ and } N(n) = 6 \text{ for all } n \geq 5.$$

L. Fejes Tóth [42] proved the following - in some cases quite sharp - upper bound for the Newton numbers of convex domains (i.e. compact convex sets with nonempty interior) in $\mathbb{E}^2$.

Theorem 4.2 *A convex domain with diameter D and minimum width W cannot be touched by more than*

$$\left[(4 + 2\pi)\frac{D}{W} + 2 + \frac{W}{D}\right]$$

non-overlapping congruent copies of it.

This result was improved by Schopp [81] as follows.

Theorem 4.3 *The Newton number of any convex domain of constant width in $\mathbb{E}^2$ is at most 7 and the Newton number of a Reuleaux triangle is exactly 7.*

We close this section with a rather natural question, which to the best of our knowledge has not been yet studied.

Problem 4.4 *Prove or disprove that the Newton number of a d-dimensional ($d \geq 3$) Euclidean cube is $3^d - 1$.*

5 One-sided Hadwiger and kissing numbers

K. Bezdek and P. Brass [20] assigned to each convex body $\mathbf{K}$ in $\mathbb{E}^d$ a specific positive integer called the one-sided Hadwiger number $h(\mathbf{K})$ as follows: $h(\mathbf{K})$ is the largest number of non-overlapping translates of $\mathbf{K}$ that touch $\mathbf{K}$ and that all lie in a closed supporting half-space of $\mathbf{K}$. In [20], using the Brunn-Minkowski inequality, K. Bezdek and P. Brass proved the following sharp upper bound for the one-sided Hadwiger numbers of convex bodies.

Theorem 5.1 *If $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$, then*

$$h(\mathbf{K}) \leq 2 \cdot 3^{d-1} - 1.$$

Moreover, equality is attained if and only if $\mathbf{K}$ is a d-dimensional affine cube.

The notion of one-sided Hadwiger numbers was introduced to study the (discrete) geometry of the so-called k^+-neighbour packings, which are packings of translates of a given convex body in $\mathbb{E}^d$ with the property that each packing element is touched by at least k others from the packing, where k is a given positive integer. As this area of discrete geometry has a rather large literature we refer the interested reader to [20] for a brief survey on the relevant results. Here, we emphasize the following corollary of the previous theorem proved also in [20].

Theorem 5.2 *If $\mathbf{K}$ is an arbitrary convex body in $\mathbb{E}^d$, then any k^+-neighbour packing by translates of $\mathbf{K}$ with $k \geq 2 \cdot 3^{d-1}$ must have a positive density in $\mathbb{E}^d$. Moreover, there is a $(2 \cdot 3^{d-1} - 1)^+$-neighbour packing by translates of a d-dimensional affine cube with density 0 in $\mathbb{E}^d$.*

It is obvious that the one-sided Hadwiger number of any circular disk in $\mathbb{E}^2$ is 4. However, the 3-dimensional analogue statement is harder to get. As it turns out the one-sided Hadwiger number of the 3-dimensional Euclidean ball is 9. One of the shortest proofs of this fact was found by A. Bezdek and K. Bezdek [10]. Since here we are studying Euclidean balls their one-sided Hadwiger numbers we simply call one-sided kissing numbers.

Theorem 5.3 *The one-sided kissing number of the 3-dimensional Euclidean ball is 9.*

As we have mentioned before Musin [71] has just announced a proof of the long-standing conjecture that the kissing number of the 4-dimensional Euclidean ball is 24. Based on this result K. Bezdek [22] gave a proof of the following.

Theorem 5.4 *The one-sided kissing number of the* 4*-dimensional Euclidean ball is either* 18 *or* 19*.*

The proof of the above theorem supports the following conjecture.

Conjecture 5.5 *The one-sided kissing number of the* 4*-dimensional Euclidean ball is* 18*.*

6 Contact graphs of finite packings and the combinatorial Kepler problem

Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d$. Then the contact graph of an arbitrary finite packing by non-overlapping translates of $\mathbf{K}$ in $\mathbb{E}^d$ is the (simple) graph whose vertices correspond to the packing elements and whose two vertices are connected by an edge if and only if the corresponding two packing elements touch each other. One of the most basic questions on contact graphs is to find out the maximum number of edges that a contact graph of n translates of the given convex body $\mathbf{K}$ can have in $\mathbb{E}^d$. Harborth [55] proved the following remarkable result on the contact graphs of congruent circular disk packings in $\mathbb{E}^2$.

Theorem 6.1 *The maximum number of touching pairs in a packing of n congruent circular disks in $\mathbb{E}^2$ is precisely*

$$\lfloor 3n - \sqrt{12n-3} \rfloor.$$

In a very recent paper [31] Brass extended the above result to the "unit circular disk packings" of normed planes as follows.

Theorem 6.2 *The maximum number of touching pairs in a packing of n translates of a convex domain $\mathbf{K}$ in $\mathbb{E}^2$ is $\lfloor 3n - \sqrt{12n-3} \rfloor$, if $\mathbf{K}$ is not a parallelogram, and $\lfloor 4n - \sqrt{28n-12} \rfloor$, if $\mathbf{K}$ is a parallelogram.*

The analogue question in the hyperbolic plane has been studied by Bowen in [23]. We prefer to quote his result in the following geometric way.

Theorem 6.3 *Consider circle packings in the hyperbolic plane, by finitely many congruent circles, which maximize the number of touching pairs for the given number of congruent circles. Then such a packing must have all of its centers located on the vertices of a triangulation of the hyperbolic plane by congruent equilateral triangles, provided the diameter D of the circles is such that an equilateral triangle in the hyperbolic plane of side length D has each of its angles equal to $\frac{2\pi}{N}$ for some $N > 6$.*

It is not hard to see that one can extend the above result to $\mathbb{S}^2$ exactly in the way as the above phrasing suggests. However, we get a more general approach if we do the following: Take n non-overlapping unit diameter balls in a convex position in $\mathbb{E}^3$, that is assume that there exists a 3-dimensional convex polyhedron whose vertices are center points moreover, each center point belongs to the boundary of that convex polyhedron, where $n \geq 4$ is a given integer. Obviously, the shortest distance among the center points is at least one. Then count the unit distances showing up between pairs of center points but, count only those pairs that generate a unit line segment on the boundary of the given 3-dimensional convex polyhedron. Finally, maximize this number for the given n and label this maximum by $c(n)$. The following theorem was found by D. Bezdek [12] who also pointed out its interesting relation to protein folding as well as to Dürer's unsolved geometric problem on edge-unfolding of convex polyhedra. He calls the convex polyhedra showing up in the theorem below "higher order deltahedra" mainly because they form an extension of "deltahedra" classified earlier by Freudenthal and van der Waerden in [47].

Theorem 6.4 *$c(n) \leq 3n-6$, where equality is attained for infinitely many n namely, for those for which there exists a* 3*-dimensional convex polyhedron whose each face is an edge-to-edge union of some regular triangles of side length one such that the total number of generating regular triangles on the boundary of the convex polyhedron is precisely $2n-4$ with a total number of $3n-6$ sides of length one and with a total number of n vertices.*

Now, we are ready to phrase the **Combinatorial Kepler Problem**. As its name suggests this problem is strongly related to the Kepler Conjecture on the densest unit sphere packings in $\mathbb{E}^3$ (for more details see Section 7 of this paper).

Problem 6.5 *For a given n find the largest number $K(n)$ of touching pairs in a packing of n congruent balls in $\mathbb{E}^3$.*

This problem is quite open. The first part of the following theorem was proved by D. Bezdek [12] the second part by K. Bezdek [22].

Theorem 6.6

(i) $K(4) = 6, K(5) = 9, K(6) = 12$ *and* $K(7) = 15$.

(ii) $K(n) < 6n - 0.59n^{\frac{2}{3}}$ *for all* $n \geq 4$.

We close this section with two upper bounds for the number of touching pairs in an arbitrary finite packing of translates of a convex body, proved by K. Bezdek in [18] . In order to state these theorems in a short way we need a bit of notation. Let $\mathbf{K}$ be an arbitrary convex body in $\mathbb{E}^d, d \geq 3$. Then let $\delta(\mathbf{K})$ denote the density of a densest packing of translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$. Moreover, let $\mathrm{Iq}(\mathbf{K}) = \frac{(\mathrm{Svol}_{d-1}(\mathrm{bd}\mathbf{K}))^d}{(\mathrm{Vol}_d(\mathbf{K}))^{d-1}}$ be the isoperimetric quotient of the convex body $\mathbf{K}$, where $\mathrm{Svol}_{d-1}(\mathrm{bd}\mathbf{K})$ denotes the $(d-1)$-dimensional surface volume of the boundary $\mathrm{bd}\mathbf{K}$ of $\mathbf{K}$ and $\mathrm{Vol}_d(\mathbf{K})$ denotes the d-dimensional volume of $\mathbf{K}$. Moreover, let $\mathbf{B}$ denote the closed d-dimensional ball of radius 1 centered at the origin in $\mathbb{E}^d$. Finally, let $\mathbf{K}_0 = \frac{1}{2}(\mathbf{K} + (-\mathbf{K}))$ be the normalized (centrally symmetric) difference body assigned to $\mathbf{K}$ with $H(\mathbf{K}_0)$ (resp., $h(\mathbf{K}_0)$) standing for the Hadwiger number (resp., one-sided Hadwiger number) of $\mathbf{K}_0$.

Theorem 6.7 *The number of touching pairs in an arbitrary packing of $n > 1$ translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$ is at most*

$$\frac{H(\mathbf{K}_0)}{2} \cdot n - \frac{1}{2^d \cdot (\delta(\mathbf{K}_0)^{\frac{(d-1)}{d}}} \cdot \Big(\frac{Iq(\mathbf{B})}{Iq(\mathbf{K}_0)}\Big)^{\frac{1}{d}} \cdot n^{\frac{(d-1)}{d}} - (H(\mathbf{K}_0) - h(\mathbf{K}_0) - 1).$$

Theorem 6.8 *The number of touching pairs in an arbitrary packing of $n > 1$ translates of the convex body $\mathbf{K}$ in $\mathbb{E}^d, d \geq 3$ is at most*

$$\frac{3^d - 1}{2} \cdot n - \frac{\omega_d^{\frac{1}{d}}}{2^{d+1}} \cdot n^{\frac{(d-1)}{d}},$$

where $\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ *is the volume of a d-dimensional ball of radius 1 in* $\mathbb{E}^d$.

7 Isoperimetric problems for Voronoi cells

Recall that a family of non-overlapping 3-dimensional balls of radii 1 in Euclidean 3-space, $\mathbb{E}^3$ is called a unit ball packing in $\mathbb{E}^3$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbb{E}^3$. The conjecture that the density of any unit ball packing in $\mathbb{E}^3$ is at most $\frac{\pi}{\sqrt{18}} = 0.74078\ldots$ is often attributed to Kepler that he stated in 1611. The problem of proving the Kepler conjecture appears as part of Hilbert's 18th problem [56]. Using an ingenious argument which works in any dimension, Rogers [79] obtained the upper bound $0.77963\ldots$ for the density of unit ball packings in $\mathbb{E}^3$. This bound has been improved by Lindsey [64], and Muder [68], [69] to $0.773055\ldots$. Hsiang [57], [58] proposed an elaborate line of attack (along the ideas of L. Fejes Tóth suggested 40 years earlier), but his claim that he settled Kepler's conjecture seems exaggerated. However, so far no one has found any serious gap in the approach of Hales [50], [51], [52], [53], although no one has been able to fully verify it either. This is not too surprising, given that the detailed argument is described in several papers and relies on long computer aided calculations of more than 5000 subproblems. Hales shows the following remarkable theorem.

Theorem 7.1 *The densest packing of unit balls in $\mathbb{E}^3$ has density $\frac{\pi}{\sqrt{18}}$, which is attained by the "cannonball packing".*

For several of the above mentioned papers Voronoi cells of unit ball packings play a central role. Recall that the Voronoi cell of a unit ball in a packing of unit balls in $\mathbb{E}^3$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As it is well-known, the Voronoi cells of a unit ball packing in $\mathbb{E}^3$ form a tiling of $\mathbb{E}^3$. One of the most attractive problems on Voronoi cells is the Dodecahedral Conjecture first phrased by L. Fejes Tóth in [40]. According to this the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^3$ is at least as large as the volume of a regular dodecahedron with inradius 1. Very recently Hales and McLaughlin [54] announced a solution to this problem:

Theorem 7.2 *The volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^3$ is at least as large as the volume of a regular dodecahedron with inradius* 1*.*

Now, we can make a step further and take a look of the following stronger version of the Dodecahedral Conjecture called the **Strong Dodecahedral Conjecture**. It was first articulated in [16].

Conjecture 7.3 *The surface area of any Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is at least as large as* $16.6508\ldots$ *the surface area of a regular dodecahedron of inradius* 1*.*

It is easy to see that if true, then the above conjecture implies the Dodecahedral Conjecture. The strongest inequality known towards the Strong Dodecahedral Conjecture is due to K. Bezdek and E. Daróczy-Kiss published in [21]. In order to phrase it properly we introduce a bit of terminology. A face cone of a Voronoi cell in a packing with unit balls in $\mathbb{E}^3$ is the convex hull of the face chosen and the center of the unit ball sitting in the given Voronoi cell. The surface area density of a unit ball in a face cone is simply the spherical area of the region of the unit sphere (centered at the apex of the face cone) that belongs to the face cone divided by the Euclidean area of the face. It should be clear from these definitions that if we have an upper bound for the surface area density in face cones of Voronoi cells, then the reciprocal of this upper bound times 4π (the surface area of a unit ball) is a lower bound for the surface area of Voronoi cells. Now, we are ready to state the main theorem of [21].

Theorem 7.4 *The surface area density of a unit ball in any face cone of a Voronoi cell in an arbitrary packing of unit balls of $\mathbb{E}^3$ is at most*

$$\frac{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)}{5\tan\left(\frac{\pi}{5}\right)} = 0.77836\ldots,$$

and so the surface area of any Voronoi cell in a packing with unit balls in E^3 is at least

$$\frac{20\pi\tan\left(\frac{\pi}{5}\right)}{-9\pi + 30\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{5}\right)\right)} = 16.1445\ldots.$$

Moreover, the above upper bound $0.77836\ldots$ for the surface area density is best possible in the following sense. The surface area density in the face cone of any n-sided face with $n = 4, 5$ of a Voronoi cell in an arbitrary packing of unit balls of $\mathbb{E}^3$ is at most

$$\frac{3(2-n)\pi + 6n\cdot\arccos\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{n}\right)\right)}{n\tan\left(\frac{\pi}{n}\right)}$$

and equality is achieved when the face is a regular n-gon inscribed in a circle of radius $\frac{1}{\sqrt{3}\cdot\cos\left(\frac{\pi}{n}\right)}$ and positioned such that it is tangent to the corresponding unit ball of the packing at its center.

The Kelvin problem asks for the surface minimizing partition of $\mathbb{E}^3$ into cells of equal volume. According to Lhuilier's memoir [63] of 1781, the problem has been described as one of the most difficult in geometry. The solution proposed by Kelvin is a natural generalization of the hexagonal honeycomb in $\mathbb{E}^2$. Take the Voronoi cells of the dual lattice giving the densest sphere packing. This gives truncated octahedra, the Voronoi cells of the body centered cubic lattice. A small deformation of the faces produces a minimal surface, which is Kelvin's proposed solution. Just recently Phelan and Weaire [78] produced a remarkable counter-example to the Kelvin conjecture. Their work indicates also that Kelvin's original question is even harder than it was expected. In fact, the following simplier and quite fundamental question seems to be still open. One can regard this as the isoperimetric inequality for parallelohedra and one can call the conjecture below the **Truncated Octahedral Conjecture**. (Recall that a parallelohedron is a 3-dimensional convex polyhedron that tiles $\mathbb{E}^3$ by translation.)

Conjecture 7.5 *The surface area of any parallelohedron of volume* 1 *in* $\mathbb{E}^3$ *is at least as large as the surface area of the truncated octahedral Voronoi cell of the body-centered cubic lattice of volume* 1 *in* $\mathbb{E}^3$.

8 The strong Kepler conjecture

In this section we propose a way to extend Kepler's conjecture to finite packings of congruent balls in 3-space of constant curvature that is in Euclidean 3-space $\mathbb{E}^3$, in spherical 3-space $\mathbb{S}^3$ and in hyperbolic 3-space $\mathbb{H}^3$. The idea goes back to the theorems of L. Fejes Tóth [41] in $\mathbb{E}^2$, J. Molnár [67] in $\mathbb{S}^2$ and K. Bezdek [13], [14] in $\mathbb{H}^2$ which in short, can be phrased as follows:

Theorem 8.1 *If at least two congruent circular disks are packed in a circular disk in the plane of constant curvature, then the packing density is always less than* $\frac{\pi}{\sqrt{12}}$.

The hyperbolic case of this theorem proved by K. Bezdek in [13] (see also [14]) seemed quite unexpected because there are (infinite) packings of congruent circular disks in $\mathbb{H}^2$ in which the density of a circular disk in its respective Voronoi cell is significantly larger than $\frac{\pi}{\sqrt{12}}$. Also, we note that the constant $\frac{\pi}{\sqrt{12}}$ is best possible in the above theorem. Last we have to mention that since the standard methods do not give a good definition of density in $\mathbb{H}^2$ (in fact all of them fail to work as it was observed by Böröczky [25]) and since even today we know only a rather "fancy" way of defining density in hyperbolic space (see the work of Bowen and Radin [24]) it seems important to study finite packings in bounded containers of the hyperbolic space where there is no complication with the proper definition of density. All this supports the idea of the following conjecture that we call the **Strong Kepler Conjecture**:

Conjecture 8.2 *The density of at least two non-overlapping congruent balls in a ball of the 3-space of constant curvature (having radius strictly less than* $\frac{\pi}{2}$ *in the case of* $\mathbb{S}^3$*) is always less than* $\frac{\pi}{\sqrt{18}} = 0.74048\ldots$.

The following theorem proved by K. Bezdek [22] supports the above conjecture.

Theorem 8.3 *The density of at least two non-overlapping congruent balls in a ball of the 3-space of constant curvature (having radius strictly less than* $\frac{\pi}{2}$ *in the case of* $\mathbb{S}^3$*) is always less than Rogers' upper bound for the density of packings of congruent balls in* $\mathbb{E}^3$ *that is less than* $0.77963\ldots$.

9 Bounds on the density of sphere packings in higher dimensions

Recall that a family of non-overlapping $d-$dimensional balls of radii 1 in the $d-$dimensional Euclidean space $\mathbb{E}^d$ is called a unit ball packing of $\mathbb{E}^d$. The density of the packing is the proportion of space covered by these unit balls. The sphere packing problem asks for the densest packing of unit balls in $\mathbb{E}^d$. Indubitably, of all problems concerning packing it was the sphere packing problem which attracted the most attention in the past decade. It has its roots in geometry, number theory and information theory and it is part of Hilbert's 18th problem. The reader is referred to [35] (especially the third edition, which has about 800 references covering 1988-1998) for further information, definitions and references. In what follows we report on a few selected developments some of which are fantastic recent news.

The Voronoi cell of a unit ball in a packing of unit balls in $\mathbb{E}^d$ is the set of points that are not farther away from the center of the given ball than from any other ball's center. As it is well-known the Voronoi cells of a unit ball packing in $\mathbb{E}^d$ form a tiling of $\mathbb{E}^d$. One of the most attractive results on the sphere packing problem was proved by C. A. Rogers [79] in 1958. It was rediscovered by Baranovskii [4] and extended to spherical and hyperbolic spaces by Böröczky [27]. It can be phrased as follows. Take a regular $d-$dimensional simplex of edge length 2 in $\mathbb{E}^d$ and then draw a $d-$dimensional unit ball around each vertex of the simplex. Let σ_d denote the ratio of the volume of the portion of the simplex covered by balls to the volume of the simplex. Then the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^d$ is at least $\frac{\omega_d}{\sigma_d}$, where ω_d denotes the volume of a $d-$dimensional unit ball. This has the following immediate corollary.

Theorem 9.1 *The (upper) density of any unit ball packing in $\mathbb{E}^d$ is at most σ_d.*

Daniel's asymptotic formula [80] yields that

$$\sigma_d = \frac{d}{e} 2^{-(0.5+o(1))d} \text{ (as } d \to \infty).$$

Then 20 years later, in 1978 Kabatjanskii and Levenshtein [59] improved this bound in the exponential order of magnitude as follows. They proved the following theorem.

Theorem 9.2 *The (upper) density of any unit ball packing in $\mathbb{E}^d$ is at most*

$$2^{-(0.599+o(1))d} \text{ (as } d \to \infty).$$

In fact, Rogers' bound is better than the Kabatjanskii-Levenshtein bound for $4 \leq d \leq 42$ and above that the Kabatjanskii-Levenshtein bound takes over ([35], p. 20).

There has been some very important recent progress concerning the existence of economical packings. On the one hand, improving earlier results, Ball [2] proved through a very elegant completely new variational argument the following statement. (See also [48] for a similar result of W. Schmidt on centrally symmetric convex bodies.)

Theorem 9.3 *For each d, there is a lattice packing of unit balls in $\mathbb{E}^d$ with density at least*

$$\frac{d-1}{2^{d-1}} \zeta(d),$$

where $\zeta(d) = \sum_{k=1}^{\infty} \frac{1}{k^d}$ is the Riemann zeta function.

On the other hand, for some small values of d, there are explicit (lattice) packings which give densities (considerably) higher than the bound just stated. The reader is referred to [35] and [73] for a comprehensive view of results of this type.

All these explicit constructions raise the well-known challenging question whether one can find a smaller upper bound than Rogers' bound for the density of unit ball packings, especially in low dimensions. The next theorem due to K. Bezdek [17] does exactly this by improving Rogers' upper bound for the density of unit ball packings in Euclidean $d-$space for all $d \geq 8$. Since this result extends also some of the results of Section 7 to higher dimensions we phrase it in details. For this we need a bit of notation. As usual, let $\mathrm{lin}(\dots)$, $\mathrm{aff}(\dots)$, $\mathrm{conv}(\dots)$, $\mathrm{Vol}_d(\dots)$, ω_d, $\mathrm{SVol}_{d-1}(\dots)$, $\mathrm{dist}(\dots)$, $\|\dots\|$ and $\mathbf{o}$ refer to the linear hull, the affine hull, the convex hull in $\mathbb{E}^d$, the $d-$dimensional Euclidean volume measure, the $d-$dimensional volume of a $d-$dimensional unit ball, the $(d-1)-$dimensional spherical volume measure, the distance function in $\mathbb{E}^d$, the standard Euclidean norm and to the origin in $\mathbb{E}^d$.

Let $\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\}$ be a $d-$dimensional simplex having the property that the linear hull $\mathrm{lin}\{\mathbf{w}_j - \mathbf{w}_i | i < j \leq d\}$ is orthogonal to the vector $\mathbf{w}_i$ in $\mathbb{E}^d, d \geq 8$ for all $1 \leq i \leq d-1$ that is let

$$\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \dots, \mathbf{w}_d\}$$

be a $d-$dimensional orthoscheme in $\mathbb{E}^d$ moreover, let

$$\|\mathbf{w}_i\| = \sqrt{\frac{2i}{i+1}} \text{ for all } 1 \leq i \leq d.$$

It is clear that in the right triangle $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ with right angle at the vertex $\mathbf{w}_{d-1}$ we have the inequality $\|\mathbf{w}_d - \mathbf{w}_{d-1}\| = \sqrt{\frac{2}{d(d+1)}} < \sqrt{\frac{2}{(d-1)d}} = \|\mathbf{w}_{d-1} - \mathbf{w}_{d-2}\|$ and therefore $\angle \mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_d < \frac{\pi}{4}$. Now, in the plane $\mathrm{aff}\{\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d\}$ of the triangle $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ let

$$\lhd\ \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$$

denote the circular sector of central angle $\angle \mathbf{w}_d\mathbf{w}_{d-2}\mathbf{w}_{d+1} = \frac{\pi}{4} - \angle \mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_d$ and of center $\mathbf{w}_{d-2}$ sitting over the circular arc with endpoints $\mathbf{w}_d, \mathbf{w}_{d+1}$ and radius $\|\mathbf{w}_d - \mathbf{w}_{d-2}\| = \|\mathbf{w}_{d+1} - \mathbf{w}_{d-2}\|$ such that $\lhd\ \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$ and $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ are adjacent along the line segment $\mathbf{w}_{d-2}\mathbf{w}_d$ and are separated by the line of $\mathbf{w}_{d-2}\mathbf{w}_d$. Then let

$$D(\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d, \mathbf{w}_{d+1}) = \triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d \cup \lhd\ \mathbf{w}_{d-2}\mathbf{w}_d\mathbf{w}_{d+1}$$

be the convex domain generated by the triangle $\triangle\ \mathbf{w}_{d-2}\mathbf{w}_{d-1}\mathbf{w}_d$ with constant angle

$$\angle \mathbf{w}_{d-1}\mathbf{w}_{d-2}\mathbf{w}_{d+1} = \frac{\pi}{4}.$$

Now, let

$$W = \mathrm{conv}\left(\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_{d-3}\} \cup D(\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d, \mathbf{w}_{d+1})\right)$$

be the $d-$dimensional wedge (or cone) with $(d-1)-$dimensional base

$$Q_W = \mathrm{conv}\left(\{\mathbf{w}_1, \ldots, \mathbf{w}_{d-3}\} \cup D(\mathbf{w}_{d-2}, \mathbf{w}_{d-1}, \mathbf{w}_d, \mathbf{w}_{d+1})\right) \quad \text{and} \quad \text{apex } \mathbf{o}.$$

Finally, if $B = \{\mathbf{x} \in \mathbb{E}^d |\ \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq 1\}$ denotes the $d-$dimensional unit ball centered at the origin $\mathbf{o}$ of and $S = \{\mathbf{x} \in \mathbb{E}^d |\ \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| = 1\}$ denotes the $(d-1)-$dimensional unit sphere centered at $\mathbf{o}$, then let

$$\widehat{\sigma}_d = \frac{\mathrm{SVol}_{d-1}(W \cap S)}{\mathrm{Vol}_{d-1}(Q_W)} = \frac{\mathrm{Vol}_d(W \cap B)}{\mathrm{Vol}_d(W)}$$

be the the surface density (resp., volume density) of the unit sphere S (resp., of the unit ball B) in the wedge W. For the sake of completeness we remark that as the regular $d-$dimensional simplex of edge length 2 can be dissected into $(d+1)!$ pieces each being congruent to $\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\}$ therefore

$$\sigma_d = \frac{\mathrm{Vol}_d(\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\} \cap B)}{\mathrm{Vol}_d(\mathrm{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\})}.$$

Now, we are ready to state the main result of [17]. Recall that the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbb{E}^d$ is defined as the ratio of the surface area of the unit sphere to the surface area of its Voronoi cell.

Theorem 9.4 *The surface area of any Voronoi cell in a packing of unit balls in the $d-$dimensional Euclidean space $\mathbb{E}^d, d \geq 8$ is at least $\frac{d \cdot \omega_d}{\widehat{\sigma}_d}$, that is the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbb{E}^d, d \geq 8$ is at most $\widehat{\sigma}_d$. Thus, the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^d, d \geq 8$ is at least $\frac{\omega_d}{\widehat{\sigma}_d}$ and so, the (upper) density of any unit ball packing in $\mathbb{E}^d, d \geq 8$ is at most $\widehat{\sigma}_d < \sigma_d$.*

In fact, K. Bezdek [22] extended the above theorem to spherical space ($\mathbb{S}^d$) as well as to hyperbolic space ($\mathbb{H}^d$) in the following local sense. Consider packings of congruent balls of small radii only. Then for sufficiently small radii r of the given space $\mathbb{S}^d$ (resp., $\mathbb{H}^d$) one can define the quantity $\widehat{\sigma}_{\mathbb{S}^d}(r) = \frac{\mathrm{Vol}_d(W \cap B)}{\mathrm{Vol}_d(W)}$ (resp., $\widehat{\sigma}_{\mathbb{H}^d}(r) = \frac{\mathrm{Vol}_d(W \cap B)}{\mathrm{Vol}_d(W)}$) in a way very similar to the Euclidean case. (Here we simply ommit the obvious details.) With this notation the following theorem holds.

Theorem 9.5 *Consider an arbitrary packing of spheres of radius r in $\mathbb{S}^d$ (resp., $\mathbb{H}^d$) with $d \geq 8$. Then there exists an $r(d) > 0$ such that the (volume) density of each ball (of the given packing) in its respective Voronoi cell is at most $\widehat{\sigma}_{\mathbb{S}^d}(r)$ (resp., $\widehat{\sigma}_{\mathbb{H}^d}(r)$) provided that $r \leq r(d)$.*

Further improvements on the upper bound $\widehat{\sigma}_d$ of K. Bezdek for the dimensions from 4 to 36 have been obtained very recently by Cohn and Elkies [33]. They developed an analogue for sphere packing of the linear programming bounds for error correcting codes, and used it to prove new upper bounds for the density of sphere packings, which are better than K. Bezdek's upper bounds $\widehat{\sigma}_d$ for the dimensions 4 through 36. Their method together with the best known sphere packings yield the following remarkable theorem in dimensions 8 and 24.

Theorem 9.6 *The density of the densest unit ball packing in $\mathbb{E}^8$ (resp., $\mathbb{E}^{24}$) is at least $0.2536\dots$ (resp., $0.00192\dots$) and is at most $0.2537...$ (resp., $0.00196\dots$).*

Cohn and Elkies [33] conjecture that their approach can be used to solve the sphere packing problem in $\mathbb{E}^8$ (resp., $\mathbb{E}^{24}$).

Conjecture 9.7 *The E_8 root lattice (resp., the Leech lattice) that produces the corresponding lower bound in the previous theorem in fact, represents the largest possible density for unit sphere packings in $\mathbb{E}^8$ (resp., $\mathbb{E}^{24}$).*

If linear programming bounds can indeed be used to prove optimality of these lattices, it would not come as a complete surprise, because for example, the kissing number problem in these dimensions was solved similarly (for more details see Section 2).

Last but not least we mention the following striking result of Cohn and Kumar [34] according to which the Leech lattice is the densest lattice packing in $\mathbb{E}^{24}$. (The densest lattices have been known up to dimension 8.)

Theorem 9.8 *The Leech lattice is the unique densest lattice in $\mathbb{E}^{24}$, up to scaling and isometries of $\mathbb{E}^{24}$.*

We close this section with a short summary on the recent progress of L. Fejes Tóth's [45] "sausage conjecture" that is one of the main problems of the theory of finite sphere packings. According to this conjecture if in $\mathbb{E}^d, d \geq 5$ we take $n \geq 1$ non-overlapping unit balls, then the volume of their convex hull is at least as large as the volume of the convex hull of the "sausage arrangement" of n non-overlapping unit balls under which we mean an arrangement whose centers lie on a line of $\mathbb{E}^d$ such that the unit balls of any two consecutive centers touch each other. By optimizing the methods developed by Betke, Henk and Wills [7], [8] finally, Betke and Henk [6] succeded to prove the sausage conjecture of L. Fejes Tóth in any dimension of at least 42. Thus, we have the following natural looking but, far not trivial theorem.

Theorem 9.9 *The sausage conjecture holds in $\mathbb{E}^d$ for all $d \geq 42$.*

It remains a highly interesting challange to prove or disprove the sausage conjecture of L. Fejes Tóth for the dimensions between 5 and 41.

Conjecture 9.10 *Let $5 \leq d \leq 41$ be given. Then the volume of the convex hull of $n \geq 1$ non-overlapping unit balls in $\mathbb{E}^d$ is at least as large as the volume of the convex hull of the "sausage arrangement" of n non-overlapping unit balls which is an arrangement whose centers lie on a line of $\mathbb{E}^d$ such that the unit balls of any two consecutive centers touch each other.*

10 Solidity and uniform stability

The notion of solidity, introduced by L. Fejes Tóth [43] to overcome difficulties of the proper definition of density in the hyperbolic plane, has been proved very useful and stimulating. Roughly speaking a family of convex sets generating a packing is said to be solid if no proper rearrangement of any finite subset of the packing elements can provide a packing. More concretely, a circle packing in the plane of constant curvature is called solid if no finite subset of the circles can be rearranged such that the rearranged circles together with the rest of the circles form a packing not congruent to the original. An (easy) example for solid circle packings is the family of incircles of a regular tiling $\{p,3\}$ for any $p \geq 3$. In fact, a closer look of this example led L. Fejes Tóth [46] to the following simple sounding but difficult problem: he conjectured that the incircles of a regular tiling $\{p,3\}$ form a strongly solid packing for any $p \geq 5$, i.e. by removing any circle from the packing the remaining circles still form a solid packing. This conjecture has been verified for $p = 5$ by Böröczky [28] and Danzer [38] and for $p \geq 8$ by A. Bezdek [9]. Thus, we have the following theorem.

Theorem 10.1 *The incircles of a regular tiling $\{p, 3\}$ form a strongly solid packing for $p = 5$ and for any $p \geq 8$.*

The outstanding open question left is the following.

Conjecture 10.2 *The incircles of a regular tiling $\{p, 3\}$ form a strongly solid packing for $p = 6$ as well as for $p = 7$.*

In connection with solidity and finite stability (of circle packings) the notion of uniform stability (of sphere packings) has been introduced by A. Bezdek, K. Bezdek and R. Connelly [11]. According to this a sphere packing (in the space of constant curvature) is said to be uniformly stable if there exists an $\epsilon > 0$ such that no finite subset of the balls of the packing can be rearranged such that each ball is moved by a distance less than ϵ and the rearranged balls together with the rest of the balls form a packing not congruent to the original one. Now, suppose that $\mathcal{P}$ is a packing of (not necessarily) congruent balls in $\mathbb{E}^d$. Let G be the contact graph of $\mathcal{P}$, where the centers of the balls serve as the vertices of G and an edge is placed between two vertices when the corresponding two balls are tangent. The following basic principle can be used to show that many packings are uniformly stable.

Theorem 10.3 *Suppose that $\mathbb{E}^d$ can be tiled face-to-face by congruent copies of finitely many convex polytopes $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ such that the vertices and edges of that tiling form the vertex and edge system of the contact graph G of the packing $\mathcal{P}$ of some balls in $\mathbb{E}^d$. If each $\mathbf{P}_i$ is strictly locally volume expanding with respect to G, then the packing $\mathcal{P}$ is uniformly stable.*

By taking a closer look of the Delaunay tilings of some lattice sphere packings one can derive the following corollary (for more details see [11]).

Theorem 10.4 *The densest lattice sphere packings $A_2, A_3, D_4, D_5, E_6, E_7, E_8$ up to dimension 8 are all uniformly stable.*

Last we mention another corollary (for details see [11]), which was observed also by Bárány and Dolbilin [5] and which supports the above mentioned conjecture of L. Fejes Tóth.

Theorem 10.5 *Consider the triangular packing of circular disks of equal radii in $\mathbb{E}^2$ where each disk is tangent to exactly six others. Remove one disk to obtain the packing $\mathcal{P}'$. Then the packing $\mathcal{P}'$ is uniformly stable.*

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