JOINT DYNAMICS

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This workshop focused on three themes: recent advances in ergodic theory, rigidity, and tiling systems. Therefore, we divide our report accordingly.

ERGODIC THEORY

Ergodic theory, as it arises in dynamics, concerns itself principally with the existence and structure of invariant probability measures for the dynamics. This study breaks naturally into two parts, the nature and origin of invariant Borel measures for some group or semigroup of continuous actions on some space and the structure and invariants of such an action, up to measurable conjugacy, for some fixed invariant measure. Both these areas were represented in the presentations on the Ergodic Theory day at the workshop.

From the Ergodic Theorem, the study of the origin and nature of invariant measures, at least for amenable group actions, is really the study of the statistical properties of orbits. The shift action on tiling systems provide excellent examples of this. As the space of tilings under the usual local finiteness assumptions, form a compact metric space, one is guaranteed the existence of invariant probability measures. Penrose tilings tilings, for example, are uniquely ergodic, there is only one shift invariant and ergodic probability measure. If we let B_r be the ball of radius r in $\mathbb{R}^2$ and $\sigma_{\vec{v}}$ be the shift action of tilings, then this unique ergodicity is equivalent to saying that for all continuous functions f the ergodic averages

$$A(x, B_r) = \frac{1}{V(B_r)} \int_{\vec{v} \in B_r} f(\sigma_{\vec{v}}(x))\, d\vec{v}$$

converge uniformly as $r \to \infty$. The limit of course is the integral of f. When such an average at a point x converges to the integral with respect to some measure μ for all continuous f, the point is said to be "generic" for μ. In fact most systems that can be described a substitution systems are uniquely ergodic. At the other extreme, there are $\mathbb{R}^n$ tiling systems whose invariant measures model all $\mathbb{R}^n$ dynamical systems of entropy below the topological entropy of the tiling system.

Two of the presentations at the workshop concerned the exotic statistical behavior of orbits and the corresponding exotic behavior of invariant measures for natural systems. Anthony Quas, with his coauthors Emmanuel Lesigne and Mate Wierdl, has been investigation the behavior of orbits of cartesian powers of the Morse system. This system is a uniquely ergodic substitution and is perhaps the simplest system more general than a compact group rotation as it is a 2-point extension of the dyadic odometer. They have shown that up to the third power all points are generic for a measure, the natural extension of the corresponding measure for the odometer factor. For the 4th power though points arise which are not generic for any measure.

Chris Hoffman, with his coauthors N. Berger and V. Sidoravicius discussed certain generalizations of Markov processes. We remind the reader that a finite state Markov process is given by a transition matrix of conditional probabilities and that usually, by the Perron-Frobenius theorem, these transition probabilities can arise from only one possible invariant measure. What has been shown here is that, generalizing the Markov case, if the dependence of the transition probabilities on the distant past decays rapidly enough, then again one has this uniqueness. On the other hand, if it does not, then a large number of rather exotic measures can arise, all with this common transition probability.

The remaining three presentations concerned the structure of systems with a fixed measure. This other half of ergodic theory then concerns investigations of ergodic theorems, recurrence properties, mixing properties, entropy and the nature of invariant sub-algebras and joinings.

Bryna Kra discussed a tremendously exciting new development in the study of Furstenberg's theory of multiple recurrence, due to her and B. Host. In his classical work from the 70's, leading to the ergodic proof of Szemeredi's theorem, Furstenberg showed that for T an ergodic map, for all $k \in \mathbb{N}$ and sets $A_1, \dots, A_k$ of positive measure that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \prod_{n=1}^{k-1} \chi_{A_i}(T^{ji}(x)) > 0.$$

This is saying that intersections of the form

$$T^j(A_1) \cap T^{2j}(A_2) \cap \cdots \cap T^{kj}(A_j)$$

have positive measure for many values of j.

What Kra and Host have show is that this $\limsup$ is actually a limit in L^2 and moreover this limit function is measurable with respect to a sub-algebra where the dynamics is that of a rotation on a k-step nil-manifold. This is probably just the opening chapter in a long story to be developed on their techniques.

A significant and fundamental direction of ergodic theory in the past fifteen years has been an attempt to lift the deeply understood theory of actions of a single transformations to actions of larger groups. Joint dynamics would concern this for higher rank abelian groups and semigroups. The work of Ornstein and Weiss in the late 80's has shown that the natural limit for natural generalizations are the amenable groups. Dan Rudolph described the current state of a broad approach to such an effort through what is called the "orbit transference method". The core observation here is that all ergodic and free measure preserving actions of discrete amenable groups are orbit equivalent. Based on this Rudolph and B. Weiss developed an approach to lifting results for $\mathbb{Z}$ actions to the general discrete amenable action. Results concerning mixing properties are natural candidates for this method. For example using it one can show that K-systems (systems of completely positive entropy) are mixing of all orders and have countable Haar spectrum. The work discussed concerned a class of theorems showing that mixing properties of a base system lift to isometric extensions of that system, if the extension is weakly mixing. Interestingly, for the mixing property itself the method has not yet proven successful.

The talk of Jean-Paul Thouvenot, "A new information theoretical viewpoint in ergodic theory" presented a structure of ideas, that if successful, would finally offer tools to understand one of the deepest open areas of ergodic theory: To what extent is the weak Pinsker property universal? This property says that for every $\varepsilon > 0$ the system is measurably conjugate to the direct product of a Bernoulli system and a system of entropy at most ε. The structure outlined by Thouvenot would indicate the property is not universal.

The talks themselves and the problem session after offered a broad range of open questions from study of explicit systems arising from tilings and trees to the general theory of measure preserving dynamics. It would not be appropriate to go into further detail here. In addition private discussions among the participants were active and vital. Some very hopeful developments arose from these discussions in a number of areas.

RIGIDITY PHENOMENA

Rigidity is a highly successful theme in several mathematical disciplines, which roughly means that a weak assumption about a mathematical objects (e.g. a map between two spaces) implies a much stronger conclusion (e.g. that this map needs to be of a particularly nice and regular type which is easy to classify).

In the context of this workshop, a particular class of this general theme was considered, namely regarding natural actions of commutative semigroups (such as $\mathbb{Z}_+^d$, $\mathbb{Z}^d$ or $\mathbb{R}^d$ with $d > 1$) which are neither cyclic nor a finite extension of a cyclic group or semigroup.

Specifically considered were two types of such higher rank actions:

1. $\mathbb{Z}^d$ or $\mathbb{Z}_+^d$-actions by endomorphism of a compact abelian group X. The simplest example is when X is a torus. By suitably extending X to a larger compact abelian group $\tilde{X}$ one can reduce most aspects of the study of $\mathbb{Z}_+^d$-action by endomorphisms of X to a $\mathbb{Z}^d$-action by automorphisms of $\tilde{X}$.

2. Actions of multidimensional $\mathbb{R}$-diagonalizable subgroups of an Algebraic group G with $\mathbb{R}$-rank at least 2 on quotients G/Γ with $\Gamma < G$ discrete.

A basic problem for these actions is to classify the invariant probability measures (if one works in the measurable category) or the closed invariant sets (in the topological category). Results of these type can be used to prove other rigidity statements, for example partial results towards classification of invariant measures have been used to show that isomorphisms between some such systems need to be algebraic, as well as information about joinings. The converse is also true: in Furstenberg's original paper introducing the subjects, joinings were used to classify closed invariant sets.

The prototype example of such action is the action generated by the maps $x \mapsto nx \bmod 1$ and $x \mapsto mx \bmod 1$ with n, m relatively prime (or more generally multiplicatively independent, i.e. $\log n / \log m \notin \mathbb{Q}$) on the one torus $\mathbb{R}/\mathbb{Z}$. For this example, the closed invariant sets have been completely classified by Furstenberg in 1967, but understanding the invariant probability measures is one of the big open problems in the subject. If μ is such an invariant measure, and if one assumes one of the maps, say $x \mapsto nx \bmod 1$, has positive entropy then it is a theorem of Johnson-Rudolph that μ is Lebesgue.

New results discussed. The last year has been an exciting time with much progress, which was discussed at the workshop. A very partial (and somewhat arbitrary) list of these results is the following:

1. progress regarding classification of measure theoretic isomorphisms of such higher dimensional abelian actions was reported, both in the context of symmetric spaces and for much more general actions on compact abelian group by commuting automorphisms (even when each automorphism has infinite entropy).

2. There are fundamental difficulties in extending the Johnson-Rudolph theorem to more general situations. Katok and Spatzier have been able to do so some years ago, but in most situations they obtained a substantially weaker result than Rudolph's theorem, with substantially reduced applications.

   recently, there has been substantial progress in this direction. Applications include a proof of arithmetic quantum unique ergodicity, and a partial results towards Littlewood's conjecture regarding Diophantine approximations.

3. new and exciting results regarding mixing properties of actions on zero dimensional compact abelian groups by commuting automorphisms were presented, with applications to rigidity.

Partial list of problems from problem session. A fruitful problem session was held. Among the problems presented were the following:

1. $\times n, \times m$-action on $\mathbb{R}/\mathbb{Z}$, with n, m multiplicatively independent.

(a) Furstenberg's original conjecture was that any nonatomic $\times n, \times m$ invariant measure is Lebesgue. Mixing properties of the action seem to play a role. The question was raised whether assuming additional mixing properties, for example mixing of all orders, implies anything about the measure without an entropy assumption.

(b) Host has an alternative approach to Rudolph's theorem which gives a stronger results, namely that if μ is a $\times m$-ergodic measure on $\mathbb{R}/\mathbb{Z}$ with positive entropy then for μ almost every x, $\left\{n^j x \bmod 1\right\}_j$ is a quick distributed (with respect to the uniform measure) on $\mathbb{R}/\mathbb{Z}$. His proof works only for m, n relatively prime; more recently it has been shown that a similar result works unless n divides a power of m. It is interesting to check whether a Host type theorem holds also for n,m multiplicatively independent but n does divide a power of m (e.g. $m = 30; n = 2$).

(c) An alternative formulation of Furstenberg's conjecture can be given in terms of since joinings: any measurable self joining of the measure preserving $\mathbb{Z}^2$ action generated by $\times n, \times m$ on $\mathbb{R} divided \mathbb{Z}$ equipped with Lebesgue measure is either the trivial joinings or a joinings which is given by a graph of a measure theoretic endomorphism commuting with $\times n, \times m$ (which in this case can only be the map $\times k$ for some integer k).

(d) In the topological category, the following seemingly simple question is surprisingly challenging: what are the possible limit sets of $\left\{n^i m^j \mathbf{x} \bmod \mathbb{Z}^2\right\}_{i,j\to\infty}$ for $\mathbf{x} \in \mathbb{R}^2/\mathbb{Z}^2$? For $n = 10, m = 20$ it is known this limit sets can be L-shaped. It is possible for n, m relatively prime?

2. As mentioned earlier, many problems about actions of semigroups of endomorphisms can be translated two questions about actions of groups of automorphisms. An interesting possible direction of research is trying to study intrinsic properties of the non invertible case of actions by endomorphisms.

3. For $\mathbb{Z}^d$ actions on zero dimensional compact abelian groups (such as Ledrappier's example), it is well-known that there can be many irregular invariant measures. Some classification or natural conjecture about these measures is required. In particular, for example, it is known such Haar measure is the only measure which is mixing (indeed only measure which is mixing in a certain critical direction). It at the only measure was full support? Is it the only measure with a single element of the action acting ergodically?

TILING DYNAMICAL SYSTEMS, SUBSTITUTIONS, AND RELATED PROBLEMS

Let M be a Riemannian homogeneous space (such as $\mathbb{Z}^d$, $\mathbb{R}^d$, or the hyperbolic plane), and pick a point to be the origin. Let G be a closed subgroup of the group of isometries of M. Let X be a collection of tilings of M, with the topology that two tilings are ϵ-close if they agree on a ball of size $1/\epsilon$ around the origin, up to the action of an ϵ-small element of G. We assume that X is closed under the action of G, and that X is compact. This implies that X has finitely many distinct tiles and finitely many "patches" of any given size, up to the action of G. A tiling dynamical system is the action of G on X, or the action of a closed subgroup $\Gamma \subset G$, when X is Γ-invariant. In fact, it is not necessary to restrict oneself to tilings: this construction easily extends to spaces of Delone sets (i.e. uniformly discrete relatively dense sets), packings, coverings, etc.

Tiling dynamical systems can be viewed as generalized symbolic dynamical systems in which the tiles play the role of the symbols. Both multidimensionality and the connectedness (in most cases) of the acting group play important roles in the theory. In particular, the theory of multidimensional shifts of finite type can be regarded as a part of this more general theory.

Considering the transitive dynamical system generated by a single tiling, it is possible to express classical symmetry theory in dynamical terms. For instance, when the group acting on the tiling space is $\mathbb{R}^d$, freeness of the action is equivalent to the tiling being aperiodic. Most of the examples that have been studied fall into one of three classes: finite type tilings, substitution tilings or quasiperiodic tilings. The case which has been studied the most is $M = G = \Gamma = \mathbb{R}^d$ (such tilings are often called *translationally finite*). However, one of the active areas of current research deals with the case of $M = \mathbb{R}^d$ and G the full Euclidean group, with the action of $\Gamma = \mathbb{R}^d \subset G$. Such tilings are exemplified by the pinwheel tiling of the plane in which tiles occur in infinitely many orientations. Another area of recent activity concerns the case of $M = \mathbb{H}^2$ and $G = PSL(2, \mathbb{R})$.

Of the three classes indicated above, **finite type tilings** are the broadest, and there are consequently fewer general results than for the others. Finite type tiling spaces are defined by "local rules," by analogy with shifts of finite type. The theory splits into a zero entropy theory, exemplified by the "marked" Penrose tilings, and a positive entropy theory, exemplified by the Rudolph tilings. The positive entropy theory has mostly been studied in the context of multidimensional shifts of finite type. However, the universal modeling property exhibited by Rudolph tilings suggests many rich possibilities for positive entropy results. The zero entropy theory has been studied more extensively. Many examples tend to be strictly ergodic, a phenomenon that is strictly multidimensional (in the finite type case). This is perhaps explained by the fact that most of the examples in this class arise from the other two general classes discussed below.

Substitution tilings generalize one dimensional substitution dynamical systems. Examples in this class are strictly ergodic and entropy zero. They can have pure point spectrum, mixed spectrum, or be weakly mixing. In the translationally finite case, they are never strongly mixing (metrically at least). Penrose tilings, with their substitution rule, are a pure point spectrum example in this class. This class of examples has interesting connections with number theory (the theory of beta expansions in particular) and with the theory of Markov partitions. A remarkable fact is that every substitution tiling dynamical system has a finite type almost 1:1 cover. This provides a huge number of examples of finite type tilings.

Here, we take **quasiperiodic tilings** to mean a projection of a "higher-dimensional" periodic structure to $\mathbb{R}^d$. One way to think of these examples is as generalizations of Sturmian dynamical systems. They are all translationally finite. Examples in this class are strictly ergodic and have pure point spectrum. Since pure point spectrum can be interpreted in terms of diffraction, these tilings have been a popular source of models for quasicrystals. The question of when examples in this class are finite type has been answered in a variety of special cases, and seems to depend in a sensitive way on number theoretic properties of the parameters. Some tilings of this class are also known to arise from substitutions. For example, the Penrose tilings belong to this class too!

Next we mention several specific areas of recent activity, with interesting open problems. There is no way that this account can be comprehensive, so we mostly limit ourselves to the directions discussed at the workshop. Obviously, we will have to describe them in very general terms omitting all the technicalities.

Complexity. The complexity $C(n)$ is the function which counts the number of distinct n^d patches in a tiling space. Entropy is zero if and only if complexity is sub-exponential in n^d. Quasiperiodic and substitution tilings all have sub-polynomial complexity. However, while any strictly ergodic finite type tiling has entropy zero, it seems possible *a priori* for it to have complexity higher than any polynomial. Do such examples really exist?

Computation of algebraic topological invariants (e.g., cohomology groups) for tiling spaces. Technically these are homeomorphism invariants, but any homeomorphism of tiling spaces is actually an orbit equivalence, so these groups carry a great deal of dynamical information. Anderson and Putnam (1998) associated a stationary inverse limit space with a translationally finite substitution tiling, which allowed them to compute the cohomology and K-theory of the space of tilings. This construction turned out to be very useful and was extended to much more general settings by Gähler, Bellissard-Benedetti-Gambaudo, Kellendonk, Ormes-Radin-Sadun, and others. In particular, tiling spaces are inverse limits even when there is no substitution involved. Moreover, in the translationally finite case, the tiling spaces are Cantor set fiber bundles (Sadun-Williams 2003), but the general case is not known.

Conjugacies. For symbolic subshifts, the classical Curtis-Lyndon-Hedlund theorem states that a topological conjugacy can be represented by a sliding block code. For tiling dynamical systems there is a natural analog, usually called "mutual local derivability." Mutually locally derivable tilings generate topologically conjugate systems. However, the converse is false, in general, as was shown by Radin-Sadun (1998) and Petersen (1999). Recently, Holton-Radin-Sadun proved that the converse does hold for pinwheel-like spaces. Along the way they proved the invertibility of the substitution in this setting, complementing the results of Solomyak (1998) for translationally finite substitution tilings. In another direction, Clark and Sadun explored when "deformed" tilings give rise to topologically conjugate systems and found conjugacy invariants using cohomology groups mentioned above.

Optimally dense packings. L. Bowen, Radin, and their co-authors used the ergodic theory approach to investigate optimally dense packings in the Euclidean and, most notably, the hyperbolic, spaces. They obtained uniqueness and symmetry results, organizing the solutions to the densest packings problems through invariant measures on the topological space of packings.

Model sets and pure point diffraction. In the theory of aperiodic systems, which is motivated by the physics of quasicrystals, one of the central questions is to classify (describe) systems with pure point spectrum. Physicists are concerned with the *diffraction spectrum*, but as demonstrated by Dworkin (1993) and Lee-Moody-Solomyak (2002), under general assumptions pure point diffraction is equivalent to pure discrete spectrum in the sense of ergodic theory. A general "cut and projection formalism" (extending the quasiperiodic tilings mentioned above) was developed by Moody, Baake, Schlottmann, and their collaborators, building on earlier work of Meyer. Its main object is a "model set" obtained by projecting a (subset of a) lattice in $\mathcal{G} \times \mathbb{R}^d$ to $\mathbb{R}^d$, where $\mathcal{G}$ is a locally compact abelian group. All such systems have pure discrete spectrum. On the other hand, all the main examples of substitution systems with pure discrete spectrum seem to be representable in the model set framework, often with $\mathcal{G}$ being non-Archimedean! How general this phenomenon is remains an open problem. Baake and Moody recently developed an approach to systems with pure point diffraction via the theory of almost periodic measures, which leads to the cut and project formalism and may help resolve this question.

Spectrum and geometric models for substitution tiling systems. There are still many open problems, even for one-dimensional symbolic substitution systems. In particular, we do not know when the spectrum is (a) pure discrete, (b) pure singular (but there is always a singular component). A class of nontrivial higher-dimensional systems with a Lebesgue component in its spectrum was recently found by Priebe Frank (2003). Geometric models for substitutions with pure discrete spectrum are closely related to model sets discussed in the previous item. Much less studied are geometric models for weakly mixing systems. Recently, Fitzkee-Hockett-Robinson (2000) showed that certain examples may be realized as almost 1:1 factors of pseudo-Anosov diffeomorphisms. Very little is known when the substitution tiling is not translationally finite; in particular, it would be interesting to decide whether the pinwheel tiling dynamical system is strongly mixing (or even topologically mixing).