Stochastic Partial Differential Equations

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The conference was attended by 41 participants, including Ph.D. students, postdoctoral fellows, young researchers and international leaders in stochastic PDE's and related fields. An attempt was made to introduce new ideas from deterministic PDE (Souganidis' talk on homogenization in PDE) and statistical physics (Thomas' lecture on a stochastic wave equation approach to non-equilibrium heat flow). It was evident that the level of the talks throughout the meeting, both with respect to the science and level of exposition was unusually high. What follows is an attempt to focus on some of the key topics discussed at the meeting both in the lectures and in the informal meeting rooms.

1 Stochastic Differential Equations and Partial Differential Equations

Nick Krylov discussed uniqueness problems for the finite dimensional stochastic differential equation (SDE)

$$dX_t = a(X_t)dW_t, \qquad X_0 = x \in \mathbb{R}^d; \tag{1}$$

here W is a d-dimensional Brownian motion, and $a(x)$, $x \in \mathbb{R}^d$ is bounded measureable family of $d \times d$ matrices. There are two important types of uniqueness: strong, which means that the solution X is a function of the driving Brownnian motion W, and weak, which means uniqueness of the probability law of the solution X. For Lipschitz a Itô [11] proved strong uniqueness, and for continuous uniformly elliptic a Stroock and Varadhan [21] proved weak uniqueness.

Suppose that a is uniformly elliptic. A counterexample of Barlow [1] shows that strong uniqueness may fail even if $d = 1$, while Nadirashvili [19] proved that if $d \geq 3$ then weak uniqueness need not hold.

Krylov then described some recent and very interesting results on the set of initial x for which weak uniqueness holds. His first main result was that if weak uniqueness holds for all $x \in D - \{x_0\}$, where x_0 is an interior point of the domain D, then weak uniqueness holds for all $x \in D$. By iteratively applying this result the exceptional set can be enlarged to sets with a cluster point, infinitely many cluster points, etc. By combining this theorem with an older (and largely overlooked result) of Lorenzi [14] which allows discontinuity on hyperplanes, one can derive a number of uniqueness results including those of Bass and Pardoux in [2].

Rich Bass spoke of ongoing work of Athreya and Perkins, which develops an idea of Walsh and uses the Fourier transform to study parabolic SPDE's on a compact set via a related class of infinite dimensional Ornstein-Uhlenbeck SDE's. Essentially one has

$$dX_t = a(X_t)dW_t - b(X_t)dt, \tag{2}$$

where W is an infinite dimensional Brownian motion, and $a(\cdot)$ and $b(\cdot)$ are Hölder continuous, and a is uniformly elliptic. For this problem the interesting question is weak uniqueness.

The analysis of (2) assumes only Hölder continuity of the coefficients and uses a perturbation technique to obtain Schauder type estimates on the derivatives of the resolvent in terms of a norm associated with the semigroup from which you are perturbing. A second part of the analysis uses infinite dimensional localization methods to reduce to the perturbative setting. The norm used here turns out to coincide with that used by Bass and Perkins in [3] to handle a class of singular SDE's arising in the theory of interactive superprocesses in both the finite and countable dimension cases. Hence the norm, used already in work of Canarsa and DaPrato [6] in a different context, would appear to apply more widely than originally thought, and is being used by Dawson and Perkins to analyze another singular SDE arising in the renormalization analysis of Dawson, Greven et al. It was also interesting to compare these techniques with those of DaPrato and his co-authors, thanks to the presence at the workshop of Lorenzo Zambotti.

P. Souganidis discussed a different kind of interaction between randomness and differential equations in a survey of new methods in the theory of homogenization. He considered the stochastic homogenization of the Hamilton-Jacobi equation (1) $u^\epsilon_t + H(x/\epsilon, Du^\epsilon, \omega) = 0$ in $\mathbf{R}^N \times (0,\infty)$, $u^\epsilon = u_0$ on $\mathbf{R}^N \times \{0\}$, and assumed that the Hamiltonian is superlinear and convex with respect to the spatial variable. By a generalization of the Lax-Oleĭnik formula, he showed that the solution u of (1) can be expressed in terms of a "fundamental" solution. Then the behavior of the latter as $\epsilon \to 0$ can be analyzed by Γ-convergence type arguments and the ergodic theorem. He proved convergence toward the unique viscosity solution $u \in \mathrm{BUC}(\mathbf{R}^N \times [0,\infty))$ of $u_t + \overline{H}(Du) = 0$ in $\mathbf{R}^N \times (0,\infty)$, $u = u_0$ on $\mathbf{R}^N \times \{0\}$, where $\overline{H}$ is a convex function.

2 Superprocesses

Superprocesses arise as solutions to a particularly delicate parabolic SPDE driven by white noise. The basic example is the equation for super-Brownian motion:

$$\frac{\partial u}{\partial t} = \Delta u + u^{1/2}\dot{W}. \tag{3}$$

Here $\dot{W}$ is space time white noise, and $u = u(x,t)(\omega)$, $x \in D \subset \mathbb{R}^d$, $t \in [0,\infty)$. This equation is delicate because if $d \geq 2$ solutions must be interpreted in a generalized sense, and the equation is nonlinear. Weak uniqueness of non-negative solutions has been known for some time, but strong uniqueness (that is, whether u is measurable with respect to the white noise $\dot{W}$) remains open.

Nevertheless, the special nature of the equation allows for a remarkable array of exact calculations. Leonid Mytnik reported on his work with J.-F. Le Gall and Ed Perkins ([18]). This includes the resolution of a well-known problem on the existence of a continuous density for stable branching in 1 dimension but not in higher dimensions. One of the areas of application of superprocesses is in the study of a class of nonlinear boundary value problems of the form

$$\Delta u = u^{1+\beta} \text{ on } D.$$

The existence of a continuous exit measure density allows for a general probabilistic representation of u in terms of this exit density from D, the boundary behaviour of u at its well-behaved points and the set of points at which it explodes badly. This is one refinement of a far-reaching program of Dynkin, Kuznetsov, Le Gall and Veron.

One exciting recent development of superprocesses is its emergence as universal limit for stochastic spatial systems near criticality above the critical dimension. Remco van der Hofstad reported on his result with Gordon Slade [10] on critical oriented percolation in $\mathbb{Z}_+ \times \mathbb{Z}^d$. They have proved that if $d \geq 4$ then the finite dimensional distributions converge to those of the canonical measure of super-Brownian motion. His presentation included a nice explanation of the lace expansion, a combinatorial method introduced by Brydges and Spencer [5]. Super-Brownian motion has emerged as a universal scaling limit for a variety of interacting particle systems, percolation models and combinatorial structures over the past eight years. This field started with a 1987 conjecture of Rick Durrett that the one dimensional long range contact process when rescaled converges to super-Brownian motion with a killing term proportional to the density. This was confirmed in 1995 by Mueller and Tribe [16] and the higher dimensional analogues were proved by Durrett and Perkins [8]. In 1993 David Aldous conjectured that rescaled lattice trees above 8 dimensions should converge to Integrated Super-excursion (ISE), that is the integral of a super-Brownian cluster conditioned to have mass one. This was confirmed by Derbez and Slade [9] using the lace expansion. A number of results, some using the lace expansion and others using martingale methods have been proved since. A good survey of these results appears in [20]. One interesting development mentioned by van der Hofstad was his ongoing work with Akira Sakai extending the above results of Durrett and Perkins to short range critical contact processes above 4 dimensions.

A key unsolved problem for oriented percolation are the precise asymptotics for the survival probability of the critical cluster as time gets large. One expects it to be that of critical branching random walk with parameters given by their general limit theorem. This is the analogue of the classical Bramson-Griffeath result voter model, but the absence of a useful dual makes the question much harder. A second hard unresolved problem, discussed at the meeting is the extension of van der Hofstad's convergence theorem to the level of processes. Tightness for these self-repellent spatial processes appears to be a hard problem. If we are to find a way of combining martingale methods with the lace expansion, the Markovian setting of oriented percolation would seem to be the best setting for doing so. Mark Holmes, a Ph.D. student at UBC is currently working on this with van der Hofstad, Gord Slade and Ed Perkins.

3 Parabolic SPDE

An important outstanding problem in superprocesses is the question of pathwise uniqueness of the SPDE (3), giving the density of super-Brownian motion in one dimension. Lorenzo Zambotti presented a fascinating calculation in the context of solutions to the reflecting parabolic stochastic PDE of Nualart-Pardoux on $[0,1]$:

$$\frac{\partial u}{\partial t} = \Delta u + \dot{W} + \eta,\ u \geq 0,\ u(t,0) = u(t,1) = 0,$$
$$\eta(\cdot, x) \text{ increasing with support contained in } \{t : u(t,x) = 0\}.$$

Solutions to this SPDE are Hölder of index $1/4$ in time and so for a fixed x the process $u(\cdot, x)$ is not a semimartingale. In spite of this, he was able to use chaos expansions to derive an Itô type formula for smooth functions of $u(t,x)$. These kind of calculations have been tried, mainly without success, to extend standard pathwise uniqueness arguments for SDE's to SPDE's. Many people in the room found this to be familiar territory, except that Zambotti was able to see it through in this context. Mytnik, Barlow and Perkins have a program for pathwise uniqueness involving signed solutions to the SPDE for the density of super-Brownian motion and a competing species model in which particles of two types annihilate when they collide. Showing equivalence between these models is one step in this program and requires a calculation like that of Zambotti's but for a convex function. It is safe to say this will lead to a future work among these participants.

Zambotti also described a number of explicit calculations he could do with solutions of this equation using reversibility with respect to the law of the 3-dimensional Bessel bridge. Especially interesting was his joint work with Mueller and Dalang on the maximum number of zeros of the solutions to Bessel type SPDE's. The answer is shown to be 3 or 4 for an appropriate range of dimensions.

This can of course be thought of as potential theoretic results for a class of infinite dimensional solutions to parabolic SPDE's. It is related to the results presented by Eulalia Nualart for hyperbolic SPDE's: see Section 4 below.

Although the connection with parabolic SPDE is not immediately apparent, Jeremy Quastel's lecture on the superdiffusivity of the asymmetric exclusion process generated considerable interest here. He considered an exclusion process on $\mathbb{Z}^d$ with initial law independent coin tossing with mean 1/2, say. Particles move according to a kernel with non-zero mean provided the selected site is vacant. It is known that the asymptotic mean square displacement of a tagged particle at time t is diffusive (ie. grows linearly in t) in dimensions 3 or more but the behaviour in 1 or 2 dimensions was open. A quick analysis of the generator shows that this may be viewed as a discretization of the stochastic Burger's equation

$$\frac{\partial u}{\partial t} = \nabla u^2 + \Delta u + \nabla \dot{W}.$$

Quastel outlined the proof for $d = 1$ of a result with H.T. Yau, Salmhofer and Landim which showed that the system is super-diffusive in one or two dimensions as the mean square displacement grows at least as fast as $t^{5/4}$ if $d = 1$ and $t(\log t)^{2/3}$ if $d = 2$. The conjectured rates are $t^{4/3}$ and $t(\log t)^{1/2}$, respectively. The result reduces to studying the asymptotics of the resolvent acting on a particular second order polynomial. They consider the generator and hence resolvent as an asymmetric perturbation of the symmetric exclusion process. The latter maps degree n polynomials to degree n polynomials but the symmetric operator increases the degree. The standard resolvent equation is therefore not closed, a familiar dilemna for a variety of SPDE's. Nonetheless they are able to truncate these equations and show the resulting iterated solutions provide upper (even iterations) and lower (odd iterations) bounds for the quantity of interest. A particular special inequality on degree two polynomials then leads to the above bound. For $d = 2$ H.T. Yau was able to analyze higher order iterations to get the conjectured result. Although one feels such special inequalties may be rare, one certainly wants to return to other settings where moment equations are not closed in case a similar trick can give some information here.

Not unrelated to the above was Boris Rozovsky's lecture on stochastic Navier-Stokes equations in two spatial dimensions and his proposed method of calculating moments of the solutions. The onset of turbulence in fluid mechanics is often associated with some random forcing terms in classical Navier-Stokes. Again one finds the moment equations associated with these stochastic equations are not closed and hence computations are problematic. He showed that 2-d stochastic NS has a unique strong solution which could be represented as the conditional expectation (given the driving white noise) of the flow of solutions of a related SDE. The latter involves non-Lipschitz coefficients and hence may be non-unique (this idea also appears in work of Le Jan and Raimond [13]). The chaos expansion of the strong solution is then used to effectively estimate the moments.

4 Hyperbolic SPDE's

Eulalia Nualart gave a systematic account of the problem of characterizing polar sets, mainly in the context of systems of hyperbolic SPDE's, using the Malliavin calculus to get estimates on the actual densities of the conditional increments of the solutions. In particular, she showed that a compact set K in Euclidean space is polar for the solution to a non-linear hyperbolic system of d SPDE's in two coordinates if and only if $\mathrm{Cap}_{d-4}(K) = 0$. Here $\mathrm{Cap}_{d-4}(\cdot)$ is the Newtonian capacity. This extends results of Khoshnevisan and Shi [12] for the Brownian sheet, and appears to be the first potential-theoretic result for non-linear SPDE's. The parabolic case is somewhat more difficult due to the different structure of the underlying filtrations. The partial result obtained to date (joint with R. Dalang) for a d-dimensional system of non-linear heat equations in one spatial dimension extends the results from the linear case obtained by Mueller and Tribe [17]. For the moment, the results here are less complete but lead to the interesting conjecture that for d-dimensional parabolic SPDE's driven by space-time white noise, the critical dimension for polarity for the range is $d - 6$. The sufficiency of this condition (dimension less that $d - 6$ implies polarity) has been proved.

Olivier Lévèque presented some results concerning non-linear hyperbolic SPDE's in spatial dimensions greater than one. One central difficulty is that the Green's function is generally not a function but a measure or a (Schwarz) distribution. For instance, for the wave equation, the Green's function is an unbounded function in dimension 2, a positive measure in dimension 3, and a distribution but not a measure in dimensions 4 or more. Therefore, J.B. Walsh's notion of (worthy) martingale measure (see [22]) must be extended in order to give an integral formulation of the SPDE. In the linear case, for the equation on the whole space, O. Lévèque presented a necessary and sufficient condition for the existence of a function-valued solution, and a related but slightly stronger condition for the existence of a random field solution (the main difference between the two notions of solution is that the latter is defined for every point in space-time and is continuous in mean-square). Under the stronger condition, he established the existence of a random field solution to non-linear forms of the equation.

Olivier Lévèque also considered the case of hyperbolic equations in a ball with isotropic driving noise concentrated on a sphere inside the ball. In this case, he presented a sufficient condition for existence of the solution in the linear case, and showed that the condition is necessary if the sphere is the boundary of the ball.

Larry Thomas showed how a stochastic wave equation could be used to model non-equilibrium dynamics of heat flow. This involved a stochastic wave equation driven by Gaussian noise that is modeled by Brownian motions. He presented a global existence theorem for the solution of the equation.

5 The Brownian sheet

R. Dalang and T. Mountford presented recent results concerning the Hausdorff dimension of the boundary of individual excursions of the Brownian sheet. A central step is the determination of gambler's ruin probabilities for additive Brownian motion, which was the subject of R. Dalang's lecture. Given that the value at the origin of an additive Brownian motion is $x \in (0,1)$, what is the probability $\mathcal{E}(x)$ that there exists a path in the plane, starting at the origin, along which the additive Brownian motion hits 1 before 0? The solution to this problem relies on an algorithm developed by Dalang and Walsh [7], which makes it possible to express the problem as an absorption probability for a discrete-time continuous state space Markov chain of order 2. An exact explicit formula has been obtained, which implies in particular that as $x \downarrow 0$, $\mathcal{E}(x) \simeq x^{\lambda_1}$, where

$$\lambda_1 = \frac{1}{2}\left(5 - \sqrt{13 + 4\sqrt{5}}\right).$$

In his lecture, T. Mountford explained how to use this result to show that the Hausdorff dimension of the boundary of individual excursions of the Brownian sheet is $\frac{3}{2} - \frac{\lambda_1}{2}$. Indeed, the first issue is to characterize those points which are on the boundary of an individual excursion of height at least one. Such points must satisfy two conditions: they are on the level set, and from these points, there is a curve along which the Brownian sheet hits 1 before 0. These two events are nearly independent, and because the Brownian sheet can be locally approximated by additive Brownian motion, the gambler's ruin probability mentioned above comes into play. However, additional ingredients are needed, already for an upper bound, because there is an error term in the approximation: a modification of the Dalang-Walsh algorithm is needed which is stable under continuous perturbations of the sample paths. The lower bound is more delicate, since this involves looking at the probability of escaping simulatenously from two distinct points and getting bounds on the probability of this event. These bounds have been obtained and establish the result mentioned above.

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