The weak Lie 2-algebra of multiplicative forms on a quasi-Poisson groupoid

Honglei Lang (China Agricultural University) joint with Zhuo Chen and Zhangju Liu

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Motivation

For a Poisson manifold $(N, P \in \mathfrak{X}^2(N))$, we have a Lie algebra homo.

$$(\Omega^1(N), [\cdot, \cdot]_P) \xrightarrow{P^{\sharp}} (\mathfrak{X}^1(N), [\cdot, \cdot]_S),$$

where
$$[df_1, df_2]_P = d\{f_1, f_2\}$$
 for $f_1, f_2 \in C^{\infty}(N)$, or,

$$[\alpha,\beta]_P = L_{P^{\sharp}\alpha}\beta - L_{P^{\sharp}\beta}\alpha - dP(\alpha,\beta), \qquad \forall \alpha,\beta \in \Omega^1(N).$$

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Consider the Koszul bracket on $\Omega^{\bullet}(N)$:

$$[\alpha,\beta]_P = (-1)^{k-1} (\mathcal{L}_P(\alpha \wedge \beta) - \mathcal{L}_P(\alpha) \wedge \beta) - \alpha \wedge \mathcal{L}_P \beta, \qquad \alpha \in \Omega^k(N), \beta \in \Omega^l(N),$$

where $\mathcal{L}_P = \iota_P \circ d - d \circ \iota_P : \Omega^n(N) \to \Omega^{n-1}(N)$.

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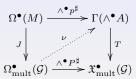
where $\mathcal{L}_P = \iota_P \circ d - d \circ \iota_P : \Omega^n(N) \to \Omega^{n-1}(N)$. Then we have a graded Lie algebra homo.

$$(\Omega^{\bullet}(N), [\cdot, \cdot]_{P}) \xrightarrow{\wedge^{\bullet} P^{\sharp}} (\mathfrak{X}^{\bullet}(N), [\cdot, \cdot]_{S}).$$

It is an isomorphism when P is symplectic.

Goal

For a quasi-Poisson Lie groupoid $(\mathcal{G} \rightrightarrows M, P, \Phi)$, construct two Lie 2-algebras and a weak morphism:





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Geometric structures on Lie groupoids



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Lie groupoids

A groupoid is a small category such that every arrow has an inverse. Namely, it has the data source, target and multiplication

$$s, t: \mathcal{G} \rightrightarrows M, \qquad m: \mathcal{G}^{(2)} \to M$$

satisfying the associativity, units, inverse properties. If \mathcal{G} and M are smooth manifolds and the structure maps are smooth, it is called a Lie groupoid.



A. Weinstein, Groupoids. Unifying internal and external symmetry, A tour through some examples, *Notices of The AMS* 43 (1996), no. 7, 744-752.

Lie algebroids

A Lie algebroid is a vector bundle $A \to M$ with a Lie bracket on the space of sections and a bundle map $\rho: A \to TM$ such that

$$[x, fy]_A = f[x, y]_A + \rho(x)(f)y, \quad \forall x, y \in \Gamma(A), f \in C^{\infty}(M).$$

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Given a Lie groupoid $\mathcal G$, the v.b. $A:=\ker s_*|_M\subset T\mathcal G|_M\to M$ is a Lie algebroid with

$$\overrightarrow{[x,y]_A} := [\overrightarrow{x}, \overrightarrow{y}]_S, \qquad \rho = t_*.$$

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Lie's third theorem for Lie algebroids:



M. Crainic and R. L. Fernandes, Integrability of Lie brackets, Ann. of Math. (2) 157 (2003), 61-98.

Examples

- Lie group/Lie algebra;
- Lie group bundle/Lie algebra bundle;
- Pair groupoid $M \times M \rightrightarrows M / TM$;
- Action groupoid $G \times M \Rightarrow M/Action$ algebroid $\mathfrak{g} \times M \to M$;
- Fundamental groupoid $\Pi(M) \rightrightarrows M$ consisting of homotopy classes of paths relative to fixed end points, concatenation of paths/TM;
- Gauge groupoid $Q \times Q/G \rightrightarrows M$ for a principal G-bundle $Q \to M/\text{Atiyah}$ algebroid TQ/G;
- Symplectic groupoid of a Poisson manifold $(M, P) / T_P^* M$.

A Lie 2-algebra (Baez-Crans) is a 2-term L_{∞} -algebra (Schlessinger-Stasheff). It has the data

- $d: \mathfrak{g}_{-1} \to \mathfrak{g}_0;$
- 2-bracket $[\cdot,\cdot]_2:\mathfrak{g}_0\wedge\mathfrak{g}_0\to\mathfrak{g}_0$ and $\triangleright:=[\cdot,\cdot]_2:\mathfrak{g}_0\wedge\mathfrak{g}_{-1}\to\mathfrak{g}_{-1};$
- 3-bracket $[\cdot, \cdot, \cdot]_3 : \wedge^3 \mathfrak{g}_0 \to \mathfrak{g}_{-1}$

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- 3-bracket $[\cdot,\cdot,\cdot]_3:\wedge^3\mathfrak{g}_0\to\mathfrak{g}_{-1}$
- s.t., for all $w, x, y, z \in \mathfrak{g}_0$ and $u, v \in \mathfrak{g}_{-1}$,
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- $(1) \ \ [du,v]_2=-[dv,u]_2, \qquad d[x,u]_2=[x,du]_2;$
- $(2) \quad [[x,y]_2,z]_2 + [[y,z]_2,x]_2 + [[z,x]_2,y]_2 = d[x,y,z]_3;$
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- $(3) \ \ [[x,y]_2,u]_2+[[y,u]_2,x]_2+[[u,x]_2,y]_2=[x,y,du]_3;$
- (4) " d_{CE} " $[\cdot, \cdot, \cdot]_3 = 0$, i.e.

$$\begin{split} &-[w,[x,y,z]_3]_2-[y,[x,z,w]_3]_2+[z,[x,y,w]_3]_2+[x,[y,z,w]_3]_2\\ =&\quad [[x,y]_2,z,w]_3-[[x,z]_2,y,w]_3+[[x,w]_2,y,z]_3+[[y,z]_2,x,w]_3\\ &-[[y,w]_2,x,z]_3+[[z,w]_2,x,y]_3. \end{split}$$

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- (2) $[[x,y]_2,z]_2 + [[y,z]_2,x]_2 + [[z,x]_2,y]_2 = d[x,y,z]_3;$
- $(3) \quad [[x,y]_2,u]_2 + [[y,u]_2,x]_2 + [[u,x]_2,y]_2 = [x,y,du]_3;$
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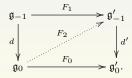
Example

- strict: $\mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathfrak{g})$;
- skeletal: $\mathbb{R} \xrightarrow{0} \mathfrak{g}$, where $[x, y, z]_3 = K(x, [y, z])$.

Definition

Let $\mathfrak g$ and $\mathfrak g'$ be Lie 2-algebras. A Lie 2-algebra homomorphism consists of

- a chain map $F_0: \mathfrak{g}_0 \to \mathfrak{g}'_0, F_1: \mathfrak{g}_{-1} \to \mathfrak{g}'_{-1},$
- a skew-symmetric bilinear map $F_2: \wedge^2 \mathfrak{g}_0 \to \mathfrak{g}'_{-1}$, such that, for $x, y, z \in \mathfrak{g}_0$ and $u \in \mathfrak{g}_{-1}$.
 - (1) $F_0[x, y]_2 [F_0(x), F_0(y)]_2' = d'F_2(x, y),$
 - (2) $F_1[x, u]_2 [F_0(x), F_1(u)]_2' = F_2(x, d(u)),$
 - $(3) \ F_1[x,y,z]_3 [F_0(x),F_0(y),F_0(z)]_3' = [F_0(x),F_2(y,z)]_2' F_2([x,y]_2,z) + c.p..$



A k-vector field $\Pi \in \mathfrak{X}^k(G)$ on a Lie group G is multiplicative if

$$\Pi_{gr} = L_{g*}\Pi_r + R_{r*}\Pi_g.$$

Or, the graph of the group mult is a coisotropic submanifold of $G\times G\times \overline{G}$ w.r.t. $\Pi\times \Pi\times (-1)^{k-1}\Pi$.

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Definition (Drinfeld)

A Poisson Lie group is a pair (G, P), where $P \in \mathfrak{X}^2_{\text{mult}}(G)$ and [P, P] = 0.

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Example

- $(\mathfrak{g}^*, P_{KKS})$ is a Poisson Lie group.
- SL(n) = SU(n)SB(n), $SL(n)^* = B_+ \times_H B_-$, etc.

Definition (Iglesias Ponte-Laurent Gengoux-Xu)

A k-vector field $\Pi \in \mathfrak{X}^k(\mathcal{G})$ is multiplicative, if the graph of groupoid multiplication is a coisotropic submanifold in $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ w.r.t. $\Pi \times \Pi \times (-1)^{k-1}\Pi$.

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Equivalently (Xu),

- ① $\Pi_{gr} = L_{b_g*}\Pi_r + R_{b'_r*}\Pi_g L_{b_g*}R_{b'_r*}\Pi_x$, where b_g and b'_r are bisections passing g and r. (iff $[\Pi, \overrightarrow{u}]_S$ is right-invariant, denoted by $\overline{\delta_{\Pi}(u)}$, thus $\delta_{\Pi}: \Gamma(A) \to \Gamma(\wedge^k A)$).
- ② for any $\xi \in \Omega^1(M)$, $\iota_{t^*(\xi)}\Pi$ is right-invariant $(\overline{\delta_{\Pi}(f)} = [\Pi, t^*f])$.
- 3 M is a coisotropic submanifold of \mathcal{G} ($\prod_{M}(\xi_1, \dots, \xi_k) = 0, \forall \xi_i \in A^*$).

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- 3 M is a coisotropic submanifold of $\mathcal{G}(\Pi|_{M}(\xi_{1},\cdots,\xi_{k})=0, \forall \xi_{i}\in A^{*}).$

Example

For
$$\tau \in \Gamma(\wedge^k A)$$
, $\overrightarrow{\tau} - \overleftarrow{\tau} \in \mathfrak{X}^k_{\text{mult}}(\mathcal{G})$.

A graded Lie 2-algebra structure

Theorem (Bonechi-Ciccoli-Laurent Gengoux-Xu)

There is a natural strict graded Lie 2-algebra on the complex

$$\Gamma(\wedge^{\bullet} A) \to \mathfrak{X}^{\bullet}_{\text{mult}}(\mathcal{G}), \qquad \tau \mapsto \overrightarrow{\tau} - \overleftarrow{\tau}.$$

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$$[\Pi, \tau] = \delta_{\Pi}(\tau) \in \Gamma(\wedge^{k+l-1}A), \qquad \Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \tau \in \Gamma(\wedge^l A).$$

For $\bullet = 1$ case, see Berwick Evans-Lerman and Ortiz-Waldron's works.

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For $\bullet=1$ case, see Berwick Evans-Lerman and Ortiz-Waldron's works. Facts:

- The Lie 2-algebras for Morita equiv. groupoids are homotopy equivalent. So they are viewed as multivector fields on the corres. differentiable stack.
- Maurer-Cartan elements of this Lie 2-algebra are quasi-Poisson strs on the groupoid. Shifted Poisson stacks correspond to Morita equivalence classes of quasi-Poisson groupoids.

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Question: Do we have Lie 2-algebra structures on multiplicative forms?

Multiplicative forms on Poisson groupoids

Definition (Weinstein, Bursztyn-Cabrera)

A k-form $\Theta \in \Omega^k(\mathcal{G})$ is multiplicative if it satisfies

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta,$$

where $m, \operatorname{pr}_1, \operatorname{pr}_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ are the multiplication and the two projections.

Or, the graph of groupoid multiplication is isotropic.

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Example

- For a Lie group G, $\Omega_{\text{mult}}^k(G) = 0$ for $k \geq 2$.
- $\Omega^1_{\text{mult}}(V) = V^*$ for a vector space V.

$$\begin{split} [\alpha,\beta]_P &= L_{P^{\sharp}\alpha}\beta - L_{P^{\sharp}\beta}\alpha - dP(\alpha,\beta), \qquad \alpha,\beta \in \Omega^1(\mathcal{G}), \\ [\alpha,\beta]_P &= (-1)^{k-1}(\mathcal{L}_P(\alpha \wedge \beta) - \mathcal{L}_P(\alpha) \wedge \beta) - \alpha \wedge \mathcal{L}_P\beta, \quad \alpha \in \Omega^k(\mathcal{G}), \beta \in \Omega^l(\mathcal{G}). \end{split}$$

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• Question: Are multiplicative 1-forms closed w.r.t $[\cdot, \cdot]_P$?

$$\begin{split} [\alpha,\beta]_P &=& L_{P^{\sharp}\alpha}\beta - L_{P^{\sharp}\beta}\alpha - dP(\alpha,\beta), \qquad \alpha,\beta \in \Omega^1(\mathcal{G}), \\ [\alpha,\beta]_P &=& (-1)^{k-1}(\mathcal{L}_P(\alpha \wedge \beta) - \mathcal{L}_P(\alpha) \wedge \beta) - \alpha \wedge \mathcal{L}_P\beta, \quad \alpha \in \Omega^k(\mathcal{G}), \beta \in \Omega^l(\mathcal{G}). \end{split}$$

• Question: Are multiplicative 1-forms closed w.r.t $[\cdot, \cdot]_P$? Yes, by developing Cartan calculus.

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 Yes, by developing Cartan calculus.
- Question: Are all multiplicative forms closed w.r.t the Koszul bracket?
 Not obvious. They are not closed under the wedge product.

Lie 2-algebras on multiplicative forms of Poisson groupoids

Theorem (Ortiz-Waldron, Chen-L-Liu)

For a Poisson groupoid (G, P), we have a natural strict graded Lie 2-algebra

$$\Omega^{\bullet}(M) \xrightarrow{e} (\Omega^{\bullet}_{\text{mult}}(\mathcal{G}), [\cdot, \cdot]_{P}), \qquad \gamma \mapsto s^{*}\gamma - t^{*}\gamma,$$

where the action is determined by

$$s^*(\Theta \triangleright \gamma) = [\Theta, s^*\gamma]_P, \qquad \forall \Theta \in \Omega^{\bullet}_{\mathrm{mult}}(\mathcal{G}), \gamma \in \Omega^{\bullet}(M).$$

Moreover, (P^{\sharp}, p^{\sharp}) is a graded Lie 2-algebra homomorphism:

$$\begin{array}{ccc} \Omega^{\bullet}(M) & \stackrel{p^{\sharp}}{\longrightarrow} \Gamma(\wedge^{\bullet}A) & . \\ J & & \downarrow^{T} \\ \Omega^{\bullet}_{\mathrm{mult}}(\mathcal{G}) & \stackrel{P^{\sharp}}{\longrightarrow} \mathfrak{X}^{\bullet}_{\mathrm{mult}}(\mathcal{G}) \end{array}$$

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$$\downarrow J \qquad \qquad \downarrow T$$

$$\Omega^{\bullet}_{\text{mult}}(\mathcal{G}) \xrightarrow{P^{\sharp}} \mathfrak{X}^{\bullet}_{\text{mult}}(\mathcal{G})$$

Question: What about quasi-Poisson groupoids? Can we get weak Lie 2-algebras?

Quasi-Poisson groupoids

A quasi-Poisson groupoid is a groupoid $\mathcal G$ with $P\in\mathfrak X^2_{\mathrm{mult}}(\mathcal G)$ and $\Phi\in\Gamma(\wedge^3A)$ s.t.

$$\frac{1}{2}[P,P] = \overrightarrow{\Phi} - \overleftarrow{\Phi}, \qquad [P,\overrightarrow{\Phi}] = 0.$$

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Example

 $(G, P = 0, \Phi)$ is a quasi-Poisson Lie group, where $\Phi = K(\cdot, [\cdot, \cdot]) \in \wedge^3 \mathfrak{g}^* \cong \wedge^3 \mathfrak{g}$.

Example

Let G be a semi-simple Lie group. Then $(G \triangleright G \rightrightarrows G, P, \Phi)$ is a quasi-Poisson groupoid, where

$$\begin{split} P_{g,s} &= & \frac{1}{2} \sum_{i} \overleftarrow{e_{i}^{2}} \wedge \overrightarrow{e_{i}^{2}} - \overleftarrow{e_{i}^{2}} \wedge \overleftarrow{e_{i}^{1}} - \overrightarrow{\left(\operatorname{Ad}_{g^{-1}} e_{i} \right)^{2}} \wedge \overrightarrow{e_{i}^{2}}, \\ \Phi &= & \frac{1}{4} K(\cdot, [\cdot, \cdot]) \in \wedge^{3} \mathfrak{g}^{*} \cong \wedge^{3} \mathfrak{g}, \end{split}$$

where $\{e_i\}$ is a basis of \mathfrak{g} .

Linear quasi-Poisson groupoids

The dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} with the Kirillov-Kostant-Souriau Poisson structure is a Poisson Lie group.

Example

The dual of a Lie 2-algebra $\mathfrak{g}_{-1} \xrightarrow{d} \mathfrak{g}_{0}$ is a quasi-Poisson groupoid (quasi-Poisson 2-group). It is an action groupoid: $g \triangleright m = d^{T}g + m$:

$$\mathcal{G}: \mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^* \rightrightarrows \mathfrak{g}_{-1}^*, \quad (h, g \triangleright m)(g, m) = (h + g, m),$$

$$[\cdot,\cdot]_2 \to P \in \mathfrak{X}^2_{\text{mult}}(\mathcal{G}), \qquad [\cdot,\cdot,\cdot]_3 \to \Phi \in \Gamma(\wedge^3 A).$$

Weak Lie 2-algebras of multiplicative forms on quasi-Poisson groupoids

Theorem (Chen-L-Liu)

Let (\mathcal{G}, P, Φ) be a quasi-Poisson groupoid. Then the triple

$$\Omega^1(M) \xrightarrow{J} \Omega^1_{\text{mult}}(\mathcal{G}), \qquad J(\gamma) := s^* \gamma - t^* \gamma,$$

is a weak Lie 2-algebra, where the bracket on $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ is $[\cdot,\cdot]_P$, the action and 3-bracket

$$\triangleright: \Omega^1_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^1(M) \to \Omega^1(M), \qquad \text{and} \qquad [\cdot, \cdot, \cdot]_3: \wedge^3 \Omega^1_{\mathrm{mult}}(\mathcal{G}) \to \Omega^1(M)$$

are determined by

$$\begin{split} s^*(\Theta \triangleright \gamma) &= [\Theta, s^* \gamma]_P, \\ s^*[\Theta_1, \Theta_2, \Theta_3]_3 &= L_{\overleftarrow{\Phi}(\Theta_1, \Theta_2, \cdot)} \Theta_3 + c.p. - 2d \overleftarrow{\Phi}(\Theta_1, \Theta_2, \Theta_3). \end{split}$$

Important formulas

$$\begin{split} [\Theta_1, [\Theta_2, \Theta_3]_P]_P + c.p. &= &-\frac{1}{2} L_{[P,P](\Theta_1, \Theta_2)} \Theta_3 + c.p. + d\big([P,P](\Theta_1, \Theta_2, \Theta_3)\big); \\ P^{\sharp}[\Theta_1, \Theta_2]_P - [P^{\sharp}\Theta_1, P^{\sharp}\Theta_2]_S &= &\frac{1}{2} [P,P](\Theta_1, \Theta_2), \qquad \Theta_i \in \Omega^1(\mathcal{G}). \end{split}$$

Important formulas

$$[\Theta_{1}, [\Theta_{2}, \Theta_{3}]_{P}]_{P} + c.p. = -\frac{1}{2}L_{[P,P](\Theta_{1},\Theta_{2})}\Theta_{3} + c.p. + d([P,P](\Theta_{1},\Theta_{2},\Theta_{3}));$$

$$P^{\sharp}[\Theta_{1}, \Theta_{2}]_{P} - [P^{\sharp}\Theta_{1}, P^{\sharp}\Theta_{2}]_{S} = \frac{1}{2}[P,P](\Theta_{1},\Theta_{2}), \qquad \Theta_{i} \in \Omega^{1}(\mathcal{G}).$$

Key step:

$$\begin{split} s^*\left(\Theta_1 \rhd [\Theta_2,\Theta_3,\Theta_4]_3 + c.p. - \left([[\Theta_1,\Theta_2]_P,\Theta_3,\Theta_4]_3 + c.p.\right)\right) \\ = & \quad \iota_{[P,\overleftarrow{\Phi}](\Theta_1,\Theta_2,\Theta_3,\cdot)} d\Theta_4 + c.p. + d[P,\overleftarrow{\Phi}](\Theta_1,\Theta_2,\Theta_3,\Theta_4). \end{split}$$

Main Theorem

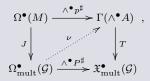
Theorem (Chen-L-Liu)

Let (G, P, Φ) be a quasi-Poisson groupoid. Then

(a) The triple $\Omega^{\bullet}(M) \xrightarrow{J} \Omega^{\bullet}_{\mathrm{mult}}(\mathcal{G})$ is a graded Lie 2-algebra, where the bracket on $\Omega^{\bullet}_{\mathrm{mult}}(\mathcal{G})$ is $[\cdot,\cdot]_{P}$, the action $\triangleright: \Omega^{p}_{\mathrm{mult}}(\mathcal{G}) \times \Omega^{q}(M) \to \Omega^{p+q-1}(M)$ and the 3-bracket $[\cdot,\cdot,\cdot]_{3}: \Omega^{p}_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^{q}_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^{s}_{\mathrm{mult}}(\mathcal{G}) \to \Omega^{p+q+s-2}(M)$ are

$$\begin{array}{lcl} s^*(\Theta \rhd \gamma) & = & [\Theta, s^*\gamma]_P, \\ s^*[\Theta_1, \Theta_2, \Theta_3]_3 & = & d\iota_{(\iota_{(\iota_{\overleftarrow{\Phi}}\Theta_1)}\Theta_2)}\Theta_3 + \left(\iota_{(\iota_{(\iota_{\overleftarrow{\Phi}}\Theta_1)}\Theta_2)}d\Theta_3 + c.p.\right). \end{array}$$

(b) There is a weak morphism of graded Lie 2-algebras



where $\nu: \Omega^p_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^q_{\mathrm{mult}}(\mathcal{G}) \to \Gamma(\wedge^{p+q-1}A)$ is defined by

$$\nu(\Theta_1, \Theta_2) = (\mathrm{id} \otimes \wedge^{p+q-2} p^{\sharp}) \big(\iota_{\Phi}(\theta_1 \wedge \theta_2) \big).$$

For a tensor field $T \in \mathcal{T}^{k,l}(\mathcal{G})$ on \mathcal{G} and $\Theta \in \Omega^p(\mathcal{G})$, define $\iota_T \Theta \in \mathcal{T}^{k-1,l+p-1}(\mathcal{G})$:

$$\iota_T \Theta = \iota_{X_1 \wedge \dots \wedge X_k \otimes \beta} \Theta := \sum_i (-1)^{k-i} X_1 \wedge \dots \widehat{X_i} \wedge X_k \otimes (\beta \wedge \iota_{X_i} \Theta). \tag{1}$$

Lemma

- (a) For all $T \in \mathcal{T}^{k,l}_{\mathrm{mult}}(\mathcal{G})$ and $\Theta \in \Omega^p_{\mathrm{mult}}(\mathcal{G})$, we have $\iota_T \Theta \in \mathcal{T}^{k-1,l+p-1}_{\mathrm{mult}}(\mathcal{G})$;
- (b) For $u \in \Gamma(\wedge^k A)$, $\gamma \in \Omega^l(M)$ and $\Theta \in \Omega^p_{\text{mult}}(\mathcal{G})$, we have

$$\iota_{\overleftarrow{u}\otimes s^*\gamma}\Theta = \overleftarrow{v}\otimes s^*\mu,$$

for some $v \in \Gamma(\wedge^{k-1}A)$ and $\mu \in \Omega^{l+p-1}(M)$.

For a tensor field $T \in \mathcal{T}^{k,l}(\mathcal{G})$ on \mathcal{G} and $\Theta \in \Omega^p(\mathcal{G})$, define $\iota_T \Theta \in \mathcal{T}^{k-1,l+p-1}(\mathcal{G})$:

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Corollary

Given a quasi-Poisson groupoid (G, P, Φ) , we have a graded Lie algebra homo:

$$(\Omega_{\mathrm{mult}}^{\bullet}(\mathcal{G})/\sim, [\cdot, \cdot]_{P}) \xrightarrow{P^{\sharp}} (\mathfrak{X}_{\mathrm{mult}}^{\bullet}(\mathcal{G})/\sim, [\cdot, \cdot]_{S}).$$

$$P^{\sharp}[\Theta_1, \Theta_2]_P - [P^{\sharp}\Theta_1, P^{\sharp}\Theta_2] = \overrightarrow{\Phi(\theta_1, \theta_2)} - \overleftarrow{\Phi(\theta_1, \theta_2)},$$

where $\theta_i = \operatorname{pr}_{A^*} \Theta_i|_M \in \Gamma(A^*)$. Information on stack.

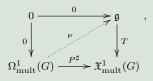
Example: Lie group case

There are only nontrivial multiplicative 1-forms on Lie groups.

Example

If (G, P, Φ) with $\Phi \in \wedge^3 \mathfrak{g}$ is a quasi-Poisson Lie group, then

- $(\Omega^1_{\mathrm{mult}}(G), [\cdot, \cdot]_P)$ is a Lie algebra, which is isomorphic to $((\mathfrak{g}^*)^G, [\cdot, \cdot]_*)$;
- $\bullet \ (P^{\sharp},0,\nu)$ is a weak homomorphism between two strict Lie 2-algebras:



where $\nu : \wedge^2 \Omega^1_{\text{mult}}(G) \to \mathfrak{g}$ is given by

$$\nu(\Theta_1, \Theta_2) = -\Phi(\theta_1, \theta_2), \qquad \theta_i = \operatorname{pr}_{\mathfrak{q}^*} \Theta_i.$$

A weak morphism between two strict Lie 2-algebras.

Example: Linear quasi-Poisson groupoid case

The dual of a Lie 2-algebra $\mathfrak{g}_{-1} \xrightarrow{d} \mathfrak{g}_{0}$ gives rise to a quasi-Poisson groupoid:

$$\mathcal{G}:\mathfrak{g}_0^*\rhd\mathfrak{g}_{-1}^*\rightrightarrows\mathfrak{g}_{-1}^*,\qquad (h,g\rhd m)(g,m)=(h+g,m).$$

Example

Fix a decomposition $\mathfrak{g}_0 = \operatorname{Im} d \oplus (\operatorname{coker} d)$. We have an isomorphism

$$\Omega^1_{\mathrm{mult}}(\mathfrak{g}_0^* \rhd \mathfrak{g}_{-1}^*) \cong C^\infty(\mathfrak{g}_{-1}^*, \mathrm{Im} d) \oplus C^\infty(\mathfrak{g}_{-1}^*, \mathrm{coker} d)^{\mathfrak{g}_0^*} \oplus C^\infty_{\mathrm{mult}}(\mathfrak{g}_0^* \rhd \mathfrak{g}_{-1}^*, \ker d).$$

$$\mathfrak{X}^1_{\mathrm{mult}}(\mathfrak{g}_0^* \rhd \mathfrak{g}_{-1}^*) \cong C^\infty(\mathfrak{g}_{-1}^*, \mathrm{Im} d^T) \oplus C^\infty(\mathfrak{g}_{-1}^*, \mathrm{coker} d^T)^{\mathfrak{g}_0^*} \oplus C^\infty_{\mathrm{mult}}(\mathfrak{g}_0^* \rhd \mathfrak{g}_{-1}^*, \ker d^T).$$

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Multiplicative w.r.t both the groupoid and abelian group structures.

$$\Omega^1_{\mathrm{bmult}}(\mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^*) = \mathfrak{g}_0;$$

$$\mathfrak{X}^1_{\mathrm{bmult}}(\mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^*) \quad = \quad \mathrm{End}_0(\mathfrak{g}^*) := \{(A,B) \in \mathrm{End}(\mathfrak{g}_0^*) \oplus \mathrm{End}(\mathfrak{g}_{-1}^*) | d^T \circ A = B \circ d^T \}.$$

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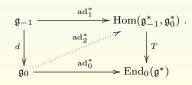
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The Lie 2-algebra morphism is the coadjoint action $ad^* : \mathfrak{g} \to \operatorname{End}(\mathfrak{g}^*)$:



Quasi-Lie bialgebroids

Theorem (Drinfeld)

There is a one-one correspondence between connected and simply connected Poisson Lie groups and Lie bialgebras:

$$(G,\pi_G) \xrightarrow{1-1} (\mathfrak{g} = T_eG, d_* = d_e\pi), \qquad \big((d_e\pi)_x = (L_{\tilde{x}}\pi)(e)\big).$$

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(Quasi-)Poisson Lie groupoids \longrightarrow (quasi-)Lie bialgebroids A quasi-Lie bialgebroid is a triple (A, d_*, Φ) consisting of a Lie algebroid A, a section $\Phi \in \Gamma(\wedge^3 A)$, and a deg 1 derivation $d_* : \Gamma(A) \to \Gamma(\wedge^2 A)$ satisfying

$$d_*[x,y] = [d_*x,y] + [x,d_*y], d_*^2 = -[\Phi,\cdot], d_*\Phi = 0.$$

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Lie 2-algebras of IM 1-forms on quasi-Lie bialgebroids

An IM 1-form on a Lie algebroid A is a pair (ν, θ) with $\nu : A \to T^*M$ and $\theta \in \Gamma(A^*)$ s.t.

$$(d_A\theta)(x,y) = \langle \nu(x), \rho(y) \rangle, \qquad \nu[x,y] = L_{\rho x}\nu(y) - \iota_{\rho y}d\nu(x), \qquad \forall x,y \in \Gamma(A).$$

Theorem (Chen-L.-Liu)

Let (A, d_*, Φ) be a quasi-Lie bialgebroid. There is a Lie 2-algebra structure:

$$\Omega^{1}(M) \xrightarrow{j} \mathrm{IM}^{1}(A), \qquad j(\gamma) = (-\iota_{\rho(\cdot,\cdot)} d\gamma, -\rho^{*}\gamma),$$

where the 2-bracket on $IM^1(A)$ is

$$[(\nu,\theta),(\nu',\theta')] = \left(\nu\rho_*^*\nu' - \nu'\rho_*^*\nu + \nu(L_{\theta'}(\cdot)) - L_{\rho_*\theta'}\nu(\cdot) - \nu'(L_{\theta}(\cdot)) + L_{\rho_*\theta}\nu'(\cdot),[\theta,\theta']_*\right),$$

the action of $IM^1(A)$ on $\Omega^1(M)$ is

$$(\nu, \theta) \triangleright \gamma = \nu(\rho_*^* \gamma) + L_{\rho_* \theta} \gamma,$$

and the 3-bracket $[\cdot,\cdot,\cdot]_3: \wedge^3 \mathrm{IM}^1(A) \to \Omega^1(M)$ is

$$[(\nu_1, \theta_1), (\nu_2, \theta_2), (\nu_3, \theta_3)]_3 = \nu_1(\Phi(\theta_2, \theta_3)) + c.p. + d\Phi(\theta_1, \theta_2, \theta_3).$$

Proposition

Let (A, d_*, Φ) be a quasi-Lie bialgebroid. There is a Lie 2-algebra homomorphism $(\psi_0, \rho_*^*, \psi_2)$:

$$\Omega^{1}(M) \xrightarrow{\rho_{*}^{*}} \Gamma(A) ,$$

$$\downarrow \downarrow \qquad \qquad \downarrow t$$

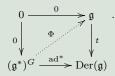
$$IM^{1}(A) \xrightarrow{\psi_{0}} Der^{1}(A)$$

where $\psi_0(\nu, \theta) = \rho_*^* \nu(\cdot) + L_{\theta}(\cdot)$ and $\psi_2 : \wedge^2 \mathrm{IM}^1(A) \to \Gamma(A)$ is given by

$$\psi_2((\nu, \theta), (\nu', \theta')) = \Phi(\theta, \theta').$$

Example

Let $(\mathfrak{g}, d_*, \Phi)$ be a quasi-Lie bialgebra. The Lie 2-algebra morphism becomes

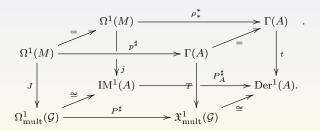


Summary

If a quasi-Poisson groupoid (\mathcal{G}, P, Φ) is s-connected and simply connected, we have the two universal lifting theorems:

$$\delta:\ \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G})\xrightarrow{1-1}\mathrm{Der}^1(A),\qquad \sigma:\ \Omega^1_{\mathrm{mult}}(\mathcal{G})\xrightarrow{1-1}\mathrm{IM}^1(A).$$

And then



Future work

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- (2) Can we construct such Lie 2-algebras on multiplicative forms of quasi-symplectic groupoids?
- (3) What are the algebraic structures on affine multivector fields and forms?

Thanks for your attention!