# The weak Lie 2-algebra of multiplicative forms on a quasi-Poisson groupoid

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# Motivation

For a Poisson manifold $(N, P \in \mathfrak{X}^2(N))$, we have a Lie algebra homo.

$$(\Omega^1(N), [\cdot,\cdot]_P) \xrightarrow{P^\sharp} (\mathfrak{X}^1(N), [\cdot,\cdot]_S),$$

where $[df_1, df_2]_P = d\{f_1, f_2\}$ for $f_1, f_2 \in C^\infty(N)$, or,

$$[\alpha, \beta]_P = L_{P^\sharp \alpha}\beta - L_{P^\sharp \beta}\alpha - dP(\alpha, \beta), \qquad \forall \alpha, \beta \in \Omega^1(N).$$

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Consider the Koszul bracket on $\Omega^\bullet(N)$:

$$[\alpha,\beta]_P = (-1)^{k-1}(\mathcal{L}_P(\alpha\wedge\beta) - \mathcal{L}_P(\alpha)\wedge\beta) - \alpha\wedge\mathcal{L}_P\beta, \qquad \alpha\in\Omega^k(N), \beta\in\Omega^l(N),$$

where $\mathcal{L}_P = \iota_P \circ d - d \circ \iota_P : \Omega^n(N) \to \Omega^{n-1}(N)$.

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where $\mathcal{L}_P = \iota_P \circ d - d \circ \iota_P : \Omega^n(N) \to \Omega^{n-1}(N)$. Then we have a graded Lie algebra homo.

$$(\Omega^\bullet(N), [\cdot,\cdot]_P) \xrightarrow{\wedge^\bullet P^\sharp} (\mathfrak{X}^\bullet(N), [\cdot,\cdot]_S).$$

It is an isomorphism when $P$ is symplectic.

# Goal

For a quasi-Poisson Lie groupoid $(\mathcal{G} \rightrightarrows M, P, \Phi)$, construct two Lie 2-algebras and a weak morphism:

$$\begin{tikzcd}
\Omega^\bullet(M) \arrow[r, "\wedge^\bullet p^\sharp"] \arrow[d, "J"'] & \Gamma(\wedge^\bullet A) \arrow[d, "T"] \\
\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \arrow[r, "\wedge^\bullet P^\sharp"] \arrow[ur, dotted, "\nu"] & \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})
\end{tikzcd}$$

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# Geometric structures on Lie groupoids

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# Lie groupoids

A groupoid is a small category such that every arrow has an inverse. Namely, it has the data source, target and multiplication

$$s, t : \mathcal{G} \rightrightarrows M, \qquad m : \mathcal{G}^{(2)} \to M$$

satisfying the associativity, units, inverse properties. If $\mathcal{G}$ and $M$ are smooth manifolds and the structure maps are smooth, it is called a Lie groupoid.

📄 A. Weinstein, Groupoids. Unifying internal and external symmetry, A tour through some examples, *Notices of The AMS* 43 (1996), no. 7, 744-752.

# Lie algebroids

A Lie algebroid is a vector bundle $A \to M$ with a Lie bracket on the space of sections and a bundle map $\rho : A \to TM$ such that

$$[x, fy]_A = f[x, y]_A + \rho(x)(f)y, \qquad \forall x, y \in \Gamma(A), f \in C^\infty(M).$$

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Given a Lie groupoid $\mathcal{G}$, the v.b. $A := \ker s_*|_M \subset T\mathcal{G}|_M \to M$ is a Lie algebroid with

$$\overrightarrow{[x, y]_A} := [\overrightarrow{x}, \overrightarrow{y}]_S, \qquad \rho = t_*.$$

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Lie's third theorem for Lie algebroids:

M. Crainic and R. L. Fernandes, Integrability of Lie brackets, *Ann. of Math. (2)* 157 (2003), 61-98.

# Examples

- Lie group/Lie algebra;
- Lie group bundle/Lie algebra bundle;
- Pair groupoid $M \times M \rightrightarrows M$/ $TM$;
- Action groupoid $G \times M \rightrightarrows M$/Action algebroid $\mathfrak{g} \times M \to M$;
- Fundamental groupoid $\Pi(M) \rightrightarrows M$ consisting of homotopy classes of paths relative to fixed end points, concatenation of paths/ $TM$;
- Gauge groupoid $Q \times Q/G \rightrightarrows M$ for a principal $G$-bundle $Q \to M$/Atiyah algebroid $TQ/G$;
- Symplectic groupoid of a Poisson manifold $(M, P)$/ $T^*_P M$.

# Lie 2-algebras

A Lie 2-algebra (Baez-Crans) is a 2-term $L_\infty$-algebra (Schlessinger-Stasheff). It has the data

- $d : \mathfrak{g}_{-1} \to \mathfrak{g}_0$;
- 2-bracket $[\cdot,\cdot]_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \to \mathfrak{g}_0$ and $\triangleright := [\cdot,\cdot]_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$;
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s.t., for all $w, x, y, z \in \mathfrak{g}_0$ and $u, v \in \mathfrak{g}_{-1}$,

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(4) "$d_{CE}$"$[\cdot,\cdot,\cdot]_3 = 0$, i.e.

$$\begin{aligned} & -[w, [x, y, z]_3]_2 - [y, [x, z, w]_3]_2 + [z, [x, y, w]_3]_2 + [x, [y, z, w]_3]_2 \\ = \quad & [[x, y]_2, z, w]_3 - [[x, z]_2, y, w]_3 + [[x, w]_2, y, z]_3 + [[y, z]_2, x, w]_3 \\ & -[[y, w]_2, x, z]_3 + [[z, w]_2, x, y]_3. \end{aligned}$$

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### Example

- strict: $\mathfrak{g} \xrightarrow{\text{ad}} \mathrm{Der}(\mathfrak{g})$;
- skeletal: $\mathbb{R} \xrightarrow{0} \mathfrak{g}$, where $[x, y, z]_3 = K(x, [y, z])$.

**Definition**

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie 2-algebras. A Lie 2-algebra homomorphism consists of

- a chain map $F_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0$, $F_1 : \mathfrak{g}_{-1} \to \mathfrak{g}'_{-1}$,
- a skew-symmetric bilinear map $F_2 : \wedge^2 \mathfrak{g}_0 \to \mathfrak{g}'_{-1}$,

such that, for $x, y, z \in \mathfrak{g}_0$ and $u \in \mathfrak{g}_{-1}$.

(1) $F_0[x, y]_2 - [F_0(x), F_0(y)]'_2 = d' F_2(x, y)$,

(2) $F_1[x, u]_2 - [F_0(x), F_1(u)]'_2 = F_2(x, d(u))$,

(3) $F_1[x, y, z]_3 - [F_0(x), F_0(y), F_0(z)]'_3 = [F_0(x), F_2(y, z)]'_2 - F_2([x, y]_2, z) + c.p..$

$$\begin{tikzcd}
\mathfrak{g}_{-1} \arrow[r, "F_1"] \arrow[d, "d"'] & \mathfrak{g}'_{-1} \arrow[d, "d'"] \\
\mathfrak{g}_0 \arrow[r, "F_0"] \arrow[ur, dotted, "F_2"] & \mathfrak{g}'_0 .
\end{tikzcd}$$

# Multiplicative multivector fields on Lie groups

A $k$-vector field $\Pi \in \mathfrak{X}^k(G)$ on a Lie group $G$ is multiplicative if

$$\Pi_{gr} = L_{g*}\Pi_r + R_{r*}\Pi_g .$$

Or, the graph of the group mult is a coisotropic submanifold of $G \times G \times \overline{G}$ w.r.t. $\Pi \times \Pi \times (-1)^{k-1}\Pi$.

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$\mathfrak{X}^k_{\mathrm{mult}}(V) = \mathrm{Hom}(\wedge^k V^*, V^*)$ for a vector space $V$ (abelian Lie group).

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$C^\infty(G)$ is a Poisson algebra and

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**Example**

- $(\mathfrak{g}^*, P_{KKS})$ is a Poisson Lie group.
- $SL(n) = SU(n)SB(n)$, $SL(n)^* = B_+ \times_H B_-$, etc.

# Multiplicative multivector fields on Lie groupoids

**Definition (Iglesias Ponte-Laurent Gengoux-Xu)**

*A k-vector field* $\Pi \in \mathfrak{X}^k(\mathcal{G})$ *is multiplicative, if the graph of groupoid multiplication is a coisotropic submanifold in* $\mathcal{G} \times \mathcal{G} \times \mathcal{G}$ *w.r.t.* $\Pi \times \Pi \times (-1)^{k-1}\Pi$.

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Equivalently (Xu),

1. $\Pi_{gr} = L_{b_g *}\Pi_r + R_{b'_r *}\Pi_g - L_{b_g *}R_{b'_r *}\Pi_x$, where $b_g$ and $b'_r$ are bisections passing $g$ and $r$. (iff $[\Pi, \overrightarrow{u}]_S$ is right-invariant, denoted by $\overrightarrow{\delta_\Pi(u)}$, thus $\delta_\Pi : \Gamma(A) \to \Gamma(\wedge^k A)$).
2. for any $\xi \in \Omega^1(M)$, $\iota_{t^*(\xi)}\Pi$ is right-invariant ($\overrightarrow{\delta_\Pi(f)} = [\Pi, t^* f]$).
3. $M$ is a coisotropic submanifold of $\mathcal{G}$ ($\Pi|_M(\xi_1, \cdots, \xi_k) = 0, \forall \xi_i \in A^*$).

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**Example**

For $\tau \in \Gamma(\wedge^k A)$, $\overrightarrow{\tau} - \overleftarrow{\tau} \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G})$.

# A graded Lie 2-algebra structure

Theorem (Bonechi-Ciccoli-Laurent Gengoux-Xu)

*There is a natural strict graded Lie 2-algebra on the complex*

$$\Gamma(\wedge^\bullet A) \to \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}), \qquad \tau \mapsto \overrightarrow{\tau} - \overleftarrow{\tau}.$$

$$[\Pi, \overrightarrow{\tau} - \overleftarrow{\tau}] = \overrightarrow{\delta_\Pi \tau} - \overleftarrow{\delta_\Pi \tau}.$$

$$[\Pi, \tau] = \delta_\Pi(\tau) \in \Gamma(\wedge^{k+l-1} A), \qquad \Pi \in \mathfrak{X}^k_{\mathrm{mult}}(\mathcal{G}), \tau \in \Gamma(\wedge^l A).$$

For $\bullet = 1$ case, see Berwick Evans-Lerman and Ortiz-Waldron's works.

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Facts:

- The Lie 2-algebras for Morita equiv. groupoids are homotopy equivalent. So they are viewed as multivector fields on the corres. differentiable stack.
- Maurer-Cartan elements of this Lie 2-algebra are quasi-Poisson strs on the groupoid. Shifted Poisson stacks correspond to Morita equivalence classes of quasi-Poisson groupoids.

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Question: Do we have Lie 2-algebra structures on multiplicative forms?

# Multiplicative forms on Poisson groupoids

**Definition (Weinstein, Bursztyn-Cabrera)**

*A $k$-form $\Theta \in \Omega^k(\mathcal{G})$ is multiplicative if it satisfies*

$$m^*\Theta = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta,$$

*where $m, \mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}^{(2)} \to \mathcal{G}$ are the multiplication and the two projections.*

Or, the graph of groupoid multiplication is isotropic.

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**Example**

- For a Lie group $G$, $\Omega^k_{\mathrm{mult}}(G) = 0$ for $k \geq 2$.
- $\Omega^1_{\mathrm{mult}}(V) = V^*$ for a vector space $V$.

Given a $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$, consider

$$\begin{aligned}
[\alpha,\beta]_P &= L_{P^\sharp\alpha}\beta - L_{P^\sharp\beta}\alpha - dP(\alpha,\beta), \qquad \alpha,\beta \in \Omega^1(\mathcal{G}),\\
[\alpha,\beta]_P &= (-1)^{k-1}(\mathcal{L}_P(\alpha\wedge\beta) - \mathcal{L}_P(\alpha)\wedge\beta) - \alpha\wedge\mathcal{L}_P\beta, \quad \alpha\in\Omega^k(\mathcal{G}), \beta\in\Omega^l(\mathcal{G}).
\end{aligned}$$

Given a $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$, consider

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- Question: Are multiplicative 1-forms closed w.r.t $[\cdot,\cdot]_P$?

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  Yes, by developing Cartan calculus.

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  Yes, by developing Cartan calculus.
- Question: Are all multiplicative forms closed w.r.t the Koszul bracket?

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- Question: Are multiplicative 1-forms closed w.r.t $[\cdot,\cdot]_P$?
  Yes, by developing Cartan calculus.
- Question: Are all multiplicative forms closed w.r.t the Koszul bracket?
  Not obvious. They are not closed under the wedge product.

# Lie 2-algebras on multiplicative forms of Poisson groupoids

**Theorem (Ortiz-Waldron, Chen-L-Liu)**

*For a Poisson groupoid $(\mathcal{G}, P)$, we have a natural strict graded Lie 2-algebra*

$$\Omega^\bullet(M) \xrightarrow{e} (\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), [\cdot,\cdot]_P), \qquad \gamma \mapsto s^*\gamma - t^*\gamma,$$

*where the action is determined by*

$$s^*(\Theta \triangleright \gamma) = [\Theta, s^*\gamma]_P, \qquad \forall \Theta \in \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}), \gamma \in \Omega^\bullet(M).$$

*Moreover, $(P^\sharp, p^\sharp)$ is a graded Lie 2-algebra homomorphism:*

$$\begin{tikzcd}
\Omega^\bullet(M) \arrow[r, "p^\sharp"] \arrow[d, "J"'] & \Gamma(\wedge^\bullet A) \arrow[d, "T"] \\
\Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \arrow[r, "P^\sharp"] & \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})
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Question: What about quasi-Poisson groupoids? Can we get weak Lie 2-algebras?

# Quasi-Poisson groupoids

A quasi-Poisson groupoid is a groupoid $\mathcal{G}$ with $P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G})$ and $\Phi \in \Gamma(\wedge^3 A)$ s.t.

$$\frac{1}{2}[P,P] = \overrightarrow{\Phi} - \overleftarrow{\Phi}, \qquad [P, \overrightarrow{\Phi}] = 0.$$

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**Example**

*$(G, P = 0, \Phi)$ is a quasi-Poisson Lie group, where $\Phi = K(\cdot,[\cdot,\cdot]) \in \wedge^3\mathfrak{g}^* \cong \wedge^3\mathfrak{g}$.*

**Example**

*Let $G$ be a semi-simple Lie group. Then $(G \triangleright G \rightrightarrows G, P, \Phi)$ is a quasi-Poisson groupoid, where*

$$\begin{aligned} P_{g,s} &= \frac{1}{2}\sum_i \overleftarrow{e_i^2} \wedge \overrightarrow{e_i^2} - \overleftarrow{e_i^2} \wedge \overleftarrow{e_i^1} - \overrightarrow{(\mathrm{Ad}_{g^{-1}} e_i)^2} \wedge \overrightarrow{e_i^2}, \\ \Phi &= \frac{1}{4}K(\cdot,[\cdot,\cdot]) \in \wedge^3\mathfrak{g}^* \cong \wedge^3\mathfrak{g}, \end{aligned}$$

*where $\{e_i\}$ is a basis of $\mathfrak{g}$.*

# Linear quasi-Poisson groupoids

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with the Kirillov-Kostant-Souriau Poisson structure is a Poisson Lie group.

**Example**

*The dual of a Lie 2-algebra* $\mathfrak{g}_{-1} \xrightarrow{d} \mathfrak{g}_0$ *is a quasi-Poisson groupoid (quasi-Poisson 2-group). It is an action groupoid:* $g \triangleright m = d^T g + m$*:*

$$\mathcal{G} : \mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^* \rightrightarrows \mathfrak{g}_{-1}^*, \quad (h, g \triangleright m)(g, m) = (h + g, m),$$

$$[\cdot,\cdot]_2 \to P \in \mathfrak{X}^2_{\mathrm{mult}}(\mathcal{G}), \qquad [\cdot,\cdot,\cdot]_3 \to \Phi \in \Gamma(\wedge^3 A).$$

# Weak Lie 2-algebras of multiplicative forms on quasi-Poisson groupoids

**Theorem (Chen-L-Liu)**

*Let $(\mathcal{G}, P, \Phi)$ be a quasi-Poisson groupoid. Then the triple*

$$\Omega^1(M) \xrightarrow{J} \Omega^1_{\mathrm{mult}}(\mathcal{G}), \qquad J(\gamma) := s^*\gamma - t^*\gamma,$$

*is a weak Lie 2-algebra, where the bracket on $\Omega^1_{\mathrm{mult}}(\mathcal{G})$ is $[\cdot,\cdot]_P$, the action and 3-bracket*

$$\triangleright : \Omega^1_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^1(M) \to \Omega^1(M), \qquad \text{and} \qquad [\cdot,\cdot,\cdot]_3 : \wedge^3\Omega^1_{\mathrm{mult}}(\mathcal{G}) \to \Omega^1(M)$$

*are determined by*

$$\begin{aligned} s^*(\Theta \triangleright \gamma) &= [\Theta, s^*\gamma]_P, \\ s^*[\Theta_1, \Theta_2, \Theta_3]_3 &= L_{\overleftarrow{\Phi}(\Theta_1,\Theta_2,\cdot)}\Theta_3 + c.p. - 2d\overleftarrow{\Phi}(\Theta_1, \Theta_2, \Theta_3). \end{aligned}$$

# Important formulas

$$
\begin{aligned}
[\Theta_1, [\Theta_2, \Theta_3]_P]_P + c.p. &= -\frac{1}{2} L_{[P,P](\Theta_1,\Theta_2)}\Theta_3 + c.p. + d\big([P,P](\Theta_1,\Theta_2,\Theta_3)\big); \\
P^\sharp[\Theta_1, \Theta_2]_P - [P^\sharp\Theta_1, P^\sharp\Theta_2]_S &= \frac{1}{2}[P,P](\Theta_1,\Theta_2), \qquad \Theta_i \in \Omega^1(\mathcal{G}).
\end{aligned}
$$

# Important formulas

$$\begin{aligned} [\Theta_1,[\Theta_2,\Theta_3]_P]_P + c.p. &= -\frac{1}{2}L_{[P,P](\Theta_1,\Theta_2)}\Theta_3 + c.p. + d\big([P,P](\Theta_1,\Theta_2,\Theta_3)\big);\\ P^\sharp[\Theta_1,\Theta_2]_P - [P^\sharp\Theta_1,P^\sharp\Theta_2]_S &= \frac{1}{2}[P,P](\Theta_1,\Theta_2), \qquad \Theta_i\in\Omega^1(\mathcal{G}). \end{aligned}$$

Key step:

$$\begin{aligned} & s^*\Big(\Theta_1 \triangleright [\Theta_2,\Theta_3,\Theta_4]_3 + c.p. - \big([[\Theta_1,\Theta_2]_P,\Theta_3,\Theta_4]_3 + c.p.\big)\Big)\\ = \quad & \iota_{[P,\overleftarrow{\Phi}](\Theta_1,\Theta_2,\Theta_3,\cdot)}d\Theta_4 + c.p. + d[P,\overleftarrow{\Phi}](\Theta_1,\Theta_2,\Theta_3,\Theta_4). \end{aligned}$$

# Main Theorem

**Theorem (Chen-L-Liu)**

*Let $(\mathcal{G}, P, \Phi)$ be a quasi-Poisson groupoid. Then*

(a) *The triple $\Omega^\bullet(M) \xrightarrow{J} \Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is a graded Lie 2-algebra, where the bracket on $\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})$ is $[\cdot,\cdot]_P$, the action $\triangleright : \Omega^p_{\mathrm{mult}}(\mathcal{G}) \times \Omega^q(M) \to \Omega^{p+q-1}(M)$ and the 3-bracket $[\cdot,\cdot,\cdot]_3 : \Omega^p_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^q_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^s_{\mathrm{mult}}(\mathcal{G}) \to \Omega^{p+q+s-2}(M)$ are*

$$\begin{aligned} s^*(\Theta \triangleright \gamma) &= [\Theta, s^*\gamma]_P, \\ s^*[\Theta_1, \Theta_2, \Theta_3]_3 &= d\iota_{(\iota_{(\iota_{\overleftarrow{\Phi}}\Theta_1)}\Theta_2)}\Theta_3 + \big(\iota_{(\iota_{(\iota_{\overleftarrow{\Phi}}\Theta_1)}\Theta_2)} d\Theta_3 + c.p.\big). \end{aligned}$$

(b) *There is a weak morphism of graded Lie 2-algebras*

$$\begin{tikzcd} \Omega^\bullet(M) \arrow[r, "\wedge^\bullet p^\sharp"] \arrow[d, "J"'] & \Gamma(\wedge^\bullet A) \arrow[d, "T"] \\ \Omega^\bullet_{\mathrm{mult}}(\mathcal{G}) \arrow[r, "\wedge^\bullet P^\sharp"] \arrow[ur, dotted, "\nu"] & \mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G}) \end{tikzcd},$$

*where $\nu : \Omega^p_{\mathrm{mult}}(\mathcal{G}) \wedge \Omega^q_{\mathrm{mult}}(\mathcal{G}) \to \Gamma(\wedge^{p+q-1} A)$ is defined by*

$$\nu(\Theta_1, \Theta_2) = (\mathrm{id} \otimes \wedge^{p+q-2} p^\sharp)\big(\iota_\Phi(\theta_1 \wedge \theta_2)\big).$$

For a tensor field $T \in \mathcal{T}^{k,l}(\mathcal{G})$ on $\mathcal{G}$ and $\Theta \in \Omega^p(\mathcal{G})$, define $\iota_T\Theta \in \mathcal{T}^{k-1,l+p-1}(\mathcal{G})$:

$$\iota_T\Theta = \iota_{X_1\wedge\cdots\wedge X_k\otimes\beta}\Theta := \sum_i (-1)^{k-i} X_1 \wedge \cdots \widehat{X_i} \wedge X_k \otimes (\beta \wedge \iota_{X_i}\Theta). \tag{1}$$

**Lemma**

(a) *For all* $T \in \mathcal{T}^{k,l}_{\mathrm{mult}}(\mathcal{G})$ *and* $\Theta \in \Omega^p_{\mathrm{mult}}(\mathcal{G})$, *we have* $\iota_T\Theta \in \mathcal{T}^{k-1,l+p-1}_{\mathrm{mult}}(\mathcal{G})$;

(b) *For* $u \in \Gamma(\wedge^k A), \gamma \in \Omega^l(M)$ *and* $\Theta \in \Omega^p_{\mathrm{mult}}(\mathcal{G})$, *we have*

$$\iota_{\overleftarrow{u}\otimes s^*\gamma}\Theta = \overleftarrow{v} \otimes s^*\mu,$$

*for some* $v \in \Gamma(\wedge^{k-1} A)$ *and* $\mu \in \Omega^{l+p-1}(M)$.

For a tensor field $T \in \mathcal{T}^{k,l}(\mathcal{G})$ on $\mathcal{G}$ and $\Theta \in \Omega^p(\mathcal{G})$, define $\iota_T\Theta \in \mathcal{T}^{k-1,l+p-1}(\mathcal{G})$:

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### Corollary

*Given a quasi-Poisson groupoid* $(\mathcal{G}, P, \Phi)$, *we have a graded Lie algebra homo:*

$$(\Omega^\bullet_{\mathrm{mult}}(\mathcal{G})/\sim, [\cdot,\cdot]_P) \xrightarrow{P^\sharp} (\mathfrak{X}^\bullet_{\mathrm{mult}}(\mathcal{G})/\sim, [\cdot,\cdot]_S).$$

$$P^\sharp[\Theta_1, \Theta_2]_P - [P^\sharp\Theta_1, P^\sharp\Theta_2] = \overrightarrow{\Phi(\theta_1, \theta_2)} - \overleftarrow{\Phi(\theta_1, \theta_2)},$$

where $\theta_i = \mathrm{pr}_{A^*}\Theta_i|_M \in \Gamma(A^*)$. Information on stack.

# Example: Lie group case

There are only nontrivial multiplicative 1-forms on Lie groups.

**Example**

If $(G, P, \Phi)$ with $\Phi \in \wedge^3\mathfrak{g}$ is a quasi-Poisson Lie group, then

- $(\Omega^1_{\mathrm{mult}}(G), [\cdot,\cdot]_P)$ is a Lie algebra, which is isomorphic to $((\mathfrak{g}^*)^G, [\cdot,\cdot]_*)$;
- $(P^\sharp, 0, \nu)$ is a weak homomorphism between two strict Lie 2-algebras:

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"'] & \mathfrak{g} \arrow[d, "T"] \\
\Omega^1_{\mathrm{mult}}(G) \arrow[r, "P^\sharp"] \arrow[ur, dotted, "\nu"] & \mathfrak{X}^1_{\mathrm{mult}}(G)
\end{tikzcd} ,$$

where $\nu : \wedge^2\Omega^1_{\mathrm{mult}}(G) \to \mathfrak{g}$ is given by

$$\nu(\Theta_1, \Theta_2) = -\Phi(\theta_1, \theta_2), \qquad \theta_i = \mathrm{pr}_{\mathfrak{g}^*}\Theta_i.$$

A weak morphism between two strict Lie 2-algebras.

# Example: Linear quasi-Poisson groupoid case

The dual of a Lie 2-algebra $\mathfrak{g}_{-1} \xrightarrow{d} \mathfrak{g}_0$ gives rise to a quasi-Poisson groupoid:

$$\mathcal{G} : \mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^* \rightrightarrows \mathfrak{g}_{-1}^*, \qquad (h, g \triangleright m)(g, m) = (h + g, m).$$

**Example**

Fix a decomposition $\mathfrak{g}_0 = \mathrm{Im} d \oplus (\mathrm{coker} d)$. We have an isomorphism

$$\Omega^1_{\mathrm{mult}}(\mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^*) \cong C^\infty(\mathfrak{g}_{-1}^*, \mathrm{Im} d) \oplus C^\infty(\mathfrak{g}_{-1}^*, \mathrm{coker} d)^{\mathfrak{g}_0^*} \oplus C^\infty_{\mathrm{mult}}(\mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^*, \ker d).$$

$$\mathfrak{X}^1_{\mathrm{mult}}(\mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^*) \cong C^\infty(\mathfrak{g}_{-1}^*, \mathrm{Im} d^T) \oplus C^\infty(\mathfrak{g}_{-1}^*, \mathrm{coker} d^T)^{\mathfrak{g}_0^*} \oplus C^\infty_{\mathrm{mult}}(\mathfrak{g}_0^* \triangleright \mathfrak{g}_{-1}^*, \ker d^T).$$

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Multiplicative w.r.t both the groupoid and abelian group structures.

$$\begin{aligned} \Omega^1_{\mathrm{bmult}}(\mathfrak{g}_0^* \rhd \mathfrak{g}_{-1}^*) &= \mathfrak{g}_0; \\ \mathfrak{X}^1_{\mathrm{bmult}}(\mathfrak{g}_0^* \rhd \mathfrak{g}_{-1}^*) &= \mathrm{End}_0(\mathfrak{g}^*) := \{(A, B) \in \mathrm{End}(\mathfrak{g}_0^*) \oplus \mathrm{End}(\mathfrak{g}_{-1}^*) | d^T \circ A = B \circ d^T\}. \end{aligned}$$

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The Lie 2-algebra morphism is the coadjoint action $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$:

$$\begin{tikzcd}
\mathfrak{g}_{-1} \arrow[r, "\mathrm{ad}_1^*"] \arrow[d, "d"'] & \mathrm{Hom}(\mathfrak{g}_{-1}^*, \mathfrak{g}_0^*) . \arrow[d, "T"] \\
\mathfrak{g}_0 \arrow[r, "\mathrm{ad}_0^*"] \arrow[ur, dotted, "\mathrm{ad}_2^*"] & \mathrm{End}_0(\mathfrak{g}^*)
\end{tikzcd}$$

# Quasi-Lie bialgebroids

**Theorem (Drinfeld)**

*There is a one-one correspondence between connected and simply connected Poisson Lie groups and Lie bialgebras:*

$$(G, \pi_G) \xrightarrow{1-1} (\mathfrak{g} = T_eG, d_* = d_e\pi), \qquad \big((d_e\pi)_x = (L_{\tilde{x}}\pi)(e)\big).$$

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(Quasi-)Poisson Lie groupoids ⟶ (quasi-)Lie bialgebroids

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(Quasi-)Poisson Lie groupoids $\longrightarrow$ (quasi-)Lie bialgebroids
A quasi-Lie bialgebroid is a triple $(A, d_*, \Phi)$ consisting of a Lie algebroid $A$, a section $\Phi \in \Gamma(\wedge^3 A)$, and a deg 1 derivation $d_* : \Gamma(A) \to \Gamma(\wedge^2 A)$ satisfying

$$d_*[x, y] = [d_* x, y] + [x, d_* y], \qquad d_*^2 = -[\Phi, \cdot], \qquad d_* \Phi = 0.$$

# Lie 2-algebras of IM 1-forms on quasi-Lie bialgebroids

An IM 1-form on a Lie algebroid $A$ is a pair $(\nu, \theta)$ with $\nu : A \to T^*M$ and $\theta \in \Gamma(A^*)$ s.t.

$$(d_A\theta)(x, y) = \langle \nu(x), \rho(y)\rangle, \qquad \nu[x, y] = L_{\rho x}\nu(y) - \iota_{\rho y} d\nu(x), \qquad \forall x, y \in \Gamma(A).$$

**Theorem (Chen-L.-Liu)**

*Let* $(A, d_*, \Phi)$ *be a quasi-Lie bialgebroid. There is a Lie* 2*-algebra structure:*

$$\Omega^1(M) \xrightarrow{j} \mathrm{IM}^1(A), \qquad j(\gamma) = (-\iota_{\rho(\ \cdot\ )} d\gamma, -\rho^*\gamma),$$

*where the 2-bracket on* $\mathrm{IM}^1(A)$ *is*

$$[(\nu, \theta), (\nu', \theta')] = \big(\nu\rho_*^*\nu' - \nu'\rho_*^*\nu + \nu(L_{\theta'}(\cdot)) - L_{\rho_*\theta'}\nu(\cdot) - \nu'(L_\theta(\cdot)) + L_{\rho_*\theta}\nu'(\cdot), [\theta, \theta']_*\big),$$

*the action of* $\mathrm{IM}^1(A)$ *on* $\Omega^1(M)$ *is*

$$(\nu, \theta) \triangleright \gamma = \nu(\rho_*^*\gamma) + L_{\rho_*\theta}\gamma,$$

*and the 3-bracket* $[\cdot, \cdot, \cdot]_3 : \wedge^3\mathrm{IM}^1(A) \to \Omega^1(M)$ *is*

$$[(\nu_1, \theta_1), (\nu_2, \theta_2), (\nu_3, \theta_3)]_3 = \nu_1(\Phi(\theta_2, \theta_3)) + c.p. + d\Phi(\theta_1, \theta_2, \theta_3).$$

**Proposition**

*Let* $(A, d_*, \Phi)$ *be a quasi-Lie bialgebroid. There is a Lie 2-algebra homomorphism* $(\psi_0, \rho_*^*, \psi_2)$*:*

$$\begin{tikzcd}
\Omega^1(M) \arrow[r, "\rho_*^*"] \arrow[d, "j"'] & \Gamma(A) \arrow[d, "t"] \\
\mathrm{IM}^1(A) \arrow[r, "\psi_0"] \arrow[ur, dotted, "\psi_2"] & \mathrm{Der}^1(A)
\end{tikzcd} ,$$

*where* $\psi_0(\nu, \theta) = \rho_*^* \nu(\cdot) + L_\theta(\cdot)$ *and* $\psi_2 : \wedge^2 \mathrm{IM}^1(A) \to \Gamma(A)$ *is given by*

$$\psi_2((\nu, \theta), (\nu', \theta')) = \Phi(\theta, \theta').$$

**Example**

Let $(\mathfrak{g}, d_*, \Phi)$ be a quasi-Lie bialgebra. The Lie 2-algebra morphism becomes

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"'] & \mathfrak{g} \arrow[d, "t"] \\
(\mathfrak{g}^*)^G \arrow[r, "\mathrm{ad}^*"] \arrow[ur, dotted, "\Phi"] & \mathrm{Der}(\mathfrak{g})
\end{tikzcd} .$$

# Summary

If a quasi-Poisson groupoid $(\mathcal{G}, P, \Phi)$ is $s$-connected and simply connected, we have the two universal lifting theorems:

$$\delta: \ \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}) \xrightarrow{1-1} \mathrm{Der}^1(A), \qquad \sigma: \ \Omega^1_{\mathrm{mult}}(\mathcal{G}) \xrightarrow{1-1} \mathrm{IM}^1(A).$$

And then

$$\begin{tikzcd}
 & \Omega^1(M) \arrow[rr, "\rho_*^*"] \arrow[dd, "j"] & & \Gamma(A) \quad . \arrow[dd, "t"] \\
\Omega^1(M) \arrow[ur, "="] \arrow[rr, "p^\sharp"] \arrow[dd, "J"'] & & \Gamma(A) \arrow[ur, "="'] \arrow[dd] & \\
 & \mathrm{IM}^1(A) \arrow[rr, "T" description, "P_A^\sharp"{pos=0.8}] & & \mathrm{Der}^1(A). \\
\Omega^1_{\mathrm{mult}}(\mathcal{G}) \arrow[ur, "\cong"] \arrow[rr, "P^\sharp"] & & \mathfrak{X}^1_{\mathrm{mult}}(\mathcal{G}) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

# Future work

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(3) What are the algebraic structures on affine multivector fields and forms?

Thanks for your attention!