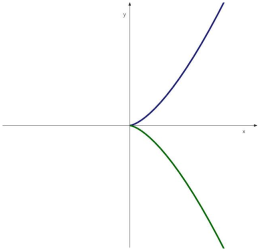
The ring of differential operators on a monomial curve is a Hopf algebroid



Ulrich Krähmer (joint work with Myriam Mahaman)

Part 1: Ingredients

$A = k[X]$

1. Fix a field $k \supseteq \mathbb{Q}$ and $p_1, \dots, p_d \in \mathbb{N}$ with $gcd(p_1, \dots, p_d) = 1$, i.e. such that there are $n_1, \dots, n_d \in \mathbb{Z}$ with $n_1 p_1 + \cdots + n_d p_d = 1$.
2. The monoid $\mathrm{span}_{\mathbb{N}}(p_1, \dots, p_d)$ is called a **numerical semigroup**.

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2. The monoid $\mathrm{span}_{\mathbb{N}}(p_1, \ldots, p_d)$ is called a **numerical semigroup**.
3. The k-algebra $A = k[X] = k[t^{p_1}, \ldots, t^{p_d}] \subseteq k[t]$ is the coordinate ring of a cuspidal curve. For example, for $p_1 = 2, p_2 = 3$, we have

$$X = \{(x, y) \in k^2 \mid x^3 = y^2\}$$

from the title page.

The main result

Theorem (K., Mahaman 2024)

The ring $\mathcal{D}(A)$ of differential operators on A is a (left) Hopf algebroid.

1. That is: If M, N are $\mathcal{D}(A)$-modules, then $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$ carry natural $\mathcal{D}(A)$-module structures so that $\mathcal{D}(A)$-**Mod** is closed monoidal.
2. The theorem could be derived from one by Smith and Stafford, or from one by Ben-Zvi and Nevins, but we use a result on the descent of Hopf algebroid structures applied to $A \to k[t, t^{-1}]$.

Augmented A-rings

1. An A-**ring** is a ring morphism $\eta\colon A \to H$.
2. An **augmentation** is an A-linear splitting

$$\varepsilon\colon H \to A, \quad \varepsilon \circ \eta = \mathrm{id}_A$$

   for which $\ker \varepsilon \subseteq H$ is a left ideal.
3. Simplest case: $A \subseteq H \subseteq \mathrm{End}_k(A)$, $\varepsilon(h) := h(1)$.

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4. View H as A-module via multiplication from the left.
5. Assume now that H carries an A-linear coassociative coproduct

$$\Delta\colon H \to H \otimes_A H, \quad h \mapsto h_{(1)} \otimes_A h_{(2)}$$

   which is counital with counit given by ε.

Bialgebroids ($\times_A$-bialgebras)

1. Note: $(g \otimes_A h)(x \otimes_A y) = (gx \otimes_A hy)$ makes no sense on $H \otimes_A H$,

   $$(g \otimes_A h)(ax \otimes_A y) = gax \otimes_A hy \neq gx \otimes_A hay = (g \otimes_A h)(x \otimes_A ay).$$

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2. Solution (Sweedler 1974): Assume the coproduct is an algebra map

   $$\Delta \colon H \to H \times_A H \subseteq H \otimes_A H,$$

   where $H \times_A H$ is the set of $\sum_i g_i \otimes_A h_i \in H \otimes_A H$ for which

   $$\sum_i g_i a \otimes_A h_i = \sum_i g_i \otimes_A h_i a, \quad a \in A.$$

Hopf algebroids

1. It is less obvious how to define Hopf algebroids. We will use:

Definition (Schauenburg 1999)

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2. This does not imply the existence of an antipode, but corresponds to a closed monoidal category structure on H-**Mod** compatible with the forgetful functor to A-**Mod**.
3. Variations with antipodes satisfying various axioms were formulated by Lu, Böhm–Szlachány, and by Böhm.

$\mathcal{D}(A)$

1. The inclusion $A \subseteq \mathrm{End}_k(A)$ identifies the elements $a \in A$ with the multiplication operators

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Definition

The A-ring $\mathcal{D}(A)$ of k-**linear differential operators** over A is the filtered k-subalgebra $\mathcal{D}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{D}(A)^n \subseteq \mathrm{End}_k(A)$, where

1. $\mathcal{D}(A)^0 = A$,
2. $\mathcal{D}(A)^n = \{D \in \mathrm{End}_k(A) \mid Da - aD \in \mathcal{D}(A)^{n-1} \, \forall a \in A\}$ for $n \geqslant 1$.

$\mathcal{D}(A)^1$

1. In particular: $D \in \mathcal{D}(A)^1$ iff for all $a \in A$ there exists $c \in A$ with

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2. Define $\partial(a) := c = (D - D(1))(a)$. This is a derivation:
$$\partial(ab) = D(ab) - D(1)ab = aD(b) + \partial(a)b - D(1)ab = a\partial(b) + \partial(a)b.$$

Lemma

There is an A-linear iso $\mathcal{D}(A)^1 \to \mathrm{Der}_k(A) \oplus A, D \mapsto (D - D(1), D(1))$.

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Lemma

There is an A-linear iso $\mathcal{D}(A)^1 \to \mathrm{Der}_k(A) \oplus A, D \mapsto (D - D(1), D(1))$.

3. In general, the A-linear embedding $\mathcal{D}(A)^{n-1} \subseteq \mathcal{D}(A)^n$ does not split!

The descent theorem

1. Let $G \subseteq \mathrm{End}_k(K)$ be a Hopf algebroid over $K \supseteq A$ and set

$$H := \{h \in G \mid h(a) \in A \ \forall a \in A\},$$
$$R := \{H \to A, h \mapsto \sum_i a_i h(b_i) \mid a_i, b_i \in A\}.$$

2. We call H R-**locally projective** over A if for all $h \in H$ there is $\pi = \sum_i r_i \otimes_A g_i \in R \otimes_A H \subseteq \mathrm{End}_A(H)$ with $\pi(h) = \sum_i r_i(h) g_i = h$.

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Theorem (K., Mahaman 2024)

If H is R-locally projective and $K \otimes_A H \to G$, $b \otimes_A h \mapsto bh$ is an isomorphism, then the Hopf algebroid structure of G restricts to H.

Part 2: Motivation

The Nakai conjecture I

Theorem (Grothendieck 1967, Sweedler 1974)

If X is smooth, we have $\mathcal{D}(k[X]) \cong U(k[X], \mathrm{Der}_k(k[X]))$.

1. Here the right hand side is the universal enveloping algebra of the Lie-Rinehart algebra $(k[X], \mathrm{Der}_k(k[X]))$.
2. This is the universal $k[X]$-ring which contains $\mathrm{Der}_k(k[X])$ as Lie algebra over k such that

$$\partial\delta - \delta\partial = [\partial, \delta], \quad \partial, \delta \in \mathrm{Der}_k(k[X]),$$

with $a\partial$ being the $k[X]$-module structure on $\mathrm{Der}_k(k[X])$, and with

$$\partial a - a\partial = \partial(a), \quad \partial \in \mathrm{Der}_k(k[X]), a \in k[X].$$

The Nakai conjecture II

1. Example: For $A = k[t]$, $\mathcal{D}(A)$ is the **Weyl algebra**

$$k\langle t, \frac{d}{dt}\rangle / \langle \left(\frac{d}{dt}\right) t - t \left(\frac{d}{dt}\right) - 1 \rangle, \quad D = \sum_i a_i \left(\frac{d}{dt}\right)^i, \quad a_i \in k[t].$$

2. Again: In general, the order is not a grading (even for smooth X)!

Conjecture (Nakai 1961, sort of)

$\mathcal{D}(k[X]) \cong U(k[X], \mathrm{Der}_k(k[X]))$ *holds if and only if* X *is smooth.*

3. By now, this is known for curves and a few more examples.

The Zariski-Lipman conjecture

1. If I is the kernel of the multiplication map $k[X] \otimes_k k[X] \to k[X]$ and $\Omega^1(X) = I/I^2$ is the $k[X]$-module of Kähler differentials, then $\mathrm{Der}_k(k[X]) \cong \mathrm{Hom}_A(\Omega^1(X), k[X])$, and we have:

Theorem (Hochschild–Kostant–Rosenberg 1962?)

X is smooth iff $\Omega^1(X)$ is a projective $k[X]$-module of rank $\dim(X)$.

2. In particular: If X is smooth, then $\mathrm{Der}_k(k[X])$ is a finitely generated projective $k[X]$-module.

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2. In particular: If X is smooth, then $\mathrm{Der}_k(k[X])$ is a finitely generated projective $k[X]$-module. The Nakai conjecture would imply:

Conjecture (Zariski, Lipman 1965)

This is an if and only if.

One motivation for our theorem

1. $U(A, L)$ is for all Lie-Rinehart algebras a Hopf algebroid.
2. There is an extension of Cartier-Milnor-Moore:

Theorem (Moerdijk, Mrčun 2010)

The cocommutative conilpotent left Hopf algebroids H that are graded projective as A-modules are precisely those of the form $U(A, L)$.

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Gutt feeling: For $D \in \mathcal{D}(A)$, there should be $E_i, F_i \in \mathcal{D}(A)$ with $D(ab) = \sum_i E_i(a)F_i(b)$, and this fit into a coproduct with $\Delta(D) = \sum_i E_i \otimes_A F_i$ so that

$$D(ab) = D_{(1)}(a)D_{(2)}(b).$$

But is this really true? And does $\mathcal{D}(A)$ have an antipode?

Another motivation

1. If H-**Mod** is closed monoidal with internal hom $\mathrm{Hom}_A(M, N)$, then

$$\mathrm{Hom}_H(M, N) \cong \mathrm{Hom}_H(A \otimes_A M, N) \cong \mathrm{Hom}_H(A, \mathrm{Hom}_A(M, N)).$$

2. Hence if M is A-projective, we have

$$\mathrm{Ext}^i_H(M, N) \cong \mathrm{Ext}^i_H(A, \mathrm{Hom}_A(M, N))$$

which answers a question from Tuesday.

Part 3: Formulas

The formulas: generators

1. Abbreviate from now on $A := k[t^2, t^3] \subseteq K := k[t, t^{-1}]$ and
$$\partial := \frac{d}{dt} \colon K \to K, \quad t^j \mapsto jt^{j-1}.$$
2. The following are differential operators of A:
$$D_0 := t\partial,\ D_1 := t^2\partial \in \mathcal{D}(A)^1,$$
$$E_{-1} := t\partial^2 - \partial,\ E_{-2} := \partial^2 - \frac{2}{t}\partial \in \mathcal{D}(A)^2,$$
$$E_{-3} := \partial^3 - \frac{3}{t}\partial^2 + \frac{3}{t^2}\partial \in \mathcal{D}(A)^3.$$

The formulas: relations

Proposition (Smith 1981)

The ring $\mathcal{D}(A)$ is generated as an algebra over k by the elements $x, y, D_0, E_{-2}, E_{-3}$, satisfying the following relations:

$$[x, y] = 0, \quad x^3 = y^2, \quad [E_{-2}, E_{-3}] = 0, \quad E_{-2}^3 = E_{-3}^2,$$
$$xE_{-2} = D_0(D_0 - 3), \quad E_{-2}x = (D_0 + 2)(D_0 - 1), \quad yE_{-2} = D_1(D_0 - 3),$$
$$E_{-2}y = D_1(D_0 + 3), \quad xE_{-3} = E_{-1}(D_0 - 4), \quad E_{-3}x = E_{-1}(D_0 + 2),$$
$$yE_{-3} = D_0(D_0 - 2)(D_0 - 4), \quad E_{-3}y = (D_0 + 3)(D_0 + 1)(D_0 - 1),$$
$$[D_0, x] = 2x, \ [D_0, y] = 3y, \ [D_0, E_{-2}] = -2E_{-2}, \ [D_0, E_{-3}] = -3E_{-3},$$

where $D_1 = y(D_0 - 1)E_{-2} - x^2E_{-3}$ *and* $E_{-1} = x(D_0 - 1)E_{-3} - yE_{-2}^2$.

The formulas: Δ and S

1. $\Delta : \mathcal{D}(A) \to \mathcal{D}(A) \times_A \mathcal{D}(A)$ is the morphism of A-rings such that

$$\Delta(D_0) = D_0 \otimes_A 1 + 1 \otimes_A D_0,$$

$$\Delta(E_{-2}) = E_{-2} \otimes_A 1 + 2D_0 \otimes_A (D_0 - 1)E_{-2} - 2D_1 \otimes_A E_{-3} + 1 \otimes_A E_{-2},$$

$$\begin{aligned}\Delta(E_{-3}) = {} & E_{-3} \otimes_A 1 + 3E_{-2} \otimes_A E_{-1} - 3E_{-1} \otimes_A E_{-2} \\ & + 6D_0 \otimes_A (D_0 - 1)E_{-3} - 6D_1 \otimes_A E_{-2}^2 + 1 \otimes_A E_{-3},\end{aligned}$$

2. $S : \mathcal{D}(A) \to \mathcal{D}(A)^{\mathrm{op}}$ is the involutive A-ring morphism such that

$$S(D_0) = 1 - D_0, \quad S(E_{-2}) = E_{-2}, \quad S(E_{-3}) = -E_{-3}.$$

Postludium: Symmetric numerical semigroups

Symmetric numerical semigroups

1. Consider $3\mathbb{N} + 8\mathbb{N}$ (underlined numbers are in):

$$\underline{0}, 1, 2, \underline{3}, 4, 5, \underline{6}, 7, \underline{8}, \underline{9}, 10, \underline{11}, \underline{12}, 13, \underline{14}, \underline{15}, \underline{16}, \underline{17}, \ldots$$

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Theorem (Sylvester 1884)

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4. Kunz: The semigroup is symmetric iff A is Gorenstein. In this case, $\mathcal{D}(A)$ is a full Hopf algebroid (has an antipode).

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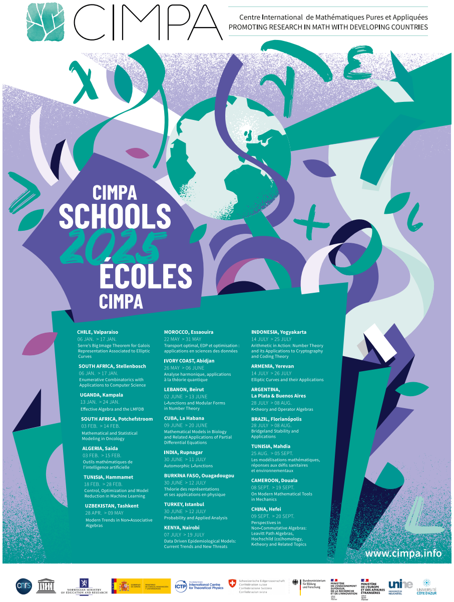
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