Cohomology for Some Unimodular Poisson Polynomial Algebras in Three Variables

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Definition of Poisson Algebras

Definition

Fix $\mathbb{k}$ to be a field of characteristic zero. A comm. $\mathbb{k}$-algebra A is called a Poisson algebra if there exists a skew-symmetric bilinear map $\{\cdot,\cdot\}\colon A \times A \longrightarrow A$ such that

$$\{ab, c\} = a\{b, c\} + b\{a, c\}$$

and

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$$

for all $a, b, c \in A$.

Examples of Poisson Algebras

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- For any $\Omega \in \Bbbk[x, y, z]$, $A_\Omega := \Bbbk[x, y, z]$ is a Poisson algebra under the following bracket

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\begin{aligned}
\{x, y\} &= \frac{\partial \Omega}{\partial z} = \Omega_z, \\
\{y, z\} &= \frac{\partial \Omega}{\partial x} = \Omega_x, \\
\{z, x\} &= \frac{\partial \Omega}{\partial y} = \Omega_y.
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- Set $A_{sing} = \Bbbk[x,y,z]/(\Omega_x, \Omega_y, \Omega_z)$. A potential Ω is said to have an isolated singularity at the origin if $\dim_{\Bbbk}(A_{sing}) < \infty$.

Poisson Cohomology Groups

For any Poisson algebra $(A, \{\cdot,\cdot\})$, denote by $\mathfrak{X}^\bullet(A) = \oplus_{i=0}^{\infty}\mathfrak{X}^i(A)$, the set of skew-symmetric multi-derivations of A. For $q \geq 0$, the *q-th Poisson cohomology* of A is defined to be the qth-cohomology of the cochain complex $(\mathfrak{X}^\bullet(A), \delta^\bullet)$ with $\delta^q(f)(a_0, \ldots, a_q)$ defined as

$$\sum_{i=0}^{q}(-1)^i\{a_i, f(a_0, \ldots, \widehat{a_i}, \ldots, a_q)\}$$
$$+ \sum_{0\leq i<j\leq q}(-1)^{i+j} f(\{a_i, a_j\}, a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_q)$$

for any $a_0, a_1, \ldots, a_q \in A$. That is, $PH^q(A) := \ker(\delta^q)/\mathrm{im}(\delta^{q-1})$.

Poisson Homology Groups

Let us denote by $\Omega^\bullet(A) := \oplus_{k=0}^{\infty}\Omega^k(A)$ the A-module of all (Kähler) differential forms with $\Omega^0(A) = A$. Denote by d the exterior differential and define the boundary operator $\partial_k\colon \Omega^k(A) \longrightarrow \Omega^{k-1}(A)$ as follows:

$$\partial_k(f_0 df_1 \wedge \cdots \wedge df_k) = \sum_{i=1}^{k}(-1)^{i+1}\{f_0, f_i\}df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k$$

$$+ \sum_{1\leq i<j\leq k} (-1)^{i+j} f_0 df(\{f_i, f_j\})df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{f_j} \wedge \cdots \wedge df_k.$$

The homology of the chain complex $(\Omega^\bullet(A), \partial_\bullet)$ is called the Poisson homology of A. There is a duality between the Poisson homology and cohomology when A is a unimodular (Luo-Wang-Wu, '15). Note that A_Ω is unimodular.

Cohomology for Poisson Polynomial Algebra A_Ω

The Hilbert series for the Poisson cohomology groups have been computed for A_Ω by

- Van den Bergh ('94): for $\Omega = x^3 + y^3 + z^3 + \lambda xyz$ with $\lambda^3 \neq -27$;

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- T-Wang-Zhang ('22): for $\Omega = x^3 + y^2 z$ and $x^3 + x^2 z + y^2 z$.
- When $\deg(x) = \deg(y) = \deg(z) = 1$, the only irreducible degree 3 potentials Ω are

$$x^3 + y^3 + z^3 + \lambda xyz (\lambda^3 \neq -27), \quad x^3 + y^2 z, \quad x^3 + x^2 z + y^2 z.$$

Classification of Weighted Potentials $\Omega \in \mathbb{k}[x, y, z]$

Theorem

(Huang-T-Wang-Zhang, '23) Let $A = \mathbb{k}[x, y, z]$ *with* $\deg(x) = a, \deg(y) = b, \deg(z) = c$ *for* $1 \leq a \leq b \leq c$. *Let* Ω *be a nonzero homogeneous polynomial of degree* $n := a + b + c$. *Up to a graded automorphism of A, all the potentials* Ω *can be explicitly listed.* A_Ω *is twist-rigid (that is,* $rgt(A) = 0$*) if and only if* Ω *is irreducible. Potentials with isolated singularity only exist in the case where* $(a, b, c) = (1, 1, 1)$ *or* $(1, 1, 2)$ *or* $(1, 2, 3)$.

Remark

The classification of Ω in the case $a = b = c$ is well-known (Dufour and Haraki, Donin-Makar and Limanov, Z.-J. Liu and Xu).

Some Notation

After making the identification:

$$\mathfrak{X}^0(A) \xrightarrow{\sim} A, \quad \mathfrak{X}^1(A) \xrightarrow{\sim} A^{\oplus 3}, \quad \mathfrak{X}^2(A) \xrightarrow{\sim} A^{\oplus 3},$$

we can write the differentials δ in the complex:

$$0 \longrightarrow \mathfrak{X}^0(A) \longrightarrow \mathfrak{X}^1(A)[w] \longrightarrow \mathfrak{X}^2(A)[2w] \longrightarrow \mathfrak{X}^3(A)[3w] \longrightarrow 0$$

where $w = \deg(\Omega) - a - b - c$ in a compact form

$$\begin{aligned}
\delta^0(f) &= \overrightarrow{\nabla} f \times \overrightarrow{\nabla}\Omega, \quad \text{for } f \in A, \\
\delta^1(\overrightarrow{f}) &= -\overrightarrow{\nabla}(\overrightarrow{f} \cdot \overrightarrow{\nabla}\Omega) + \mathrm{Div}(\overrightarrow{f})\overrightarrow{\nabla}\Omega, \quad \text{for } \overrightarrow{f} \in A^{\oplus 3}, \\
\delta^2(\overrightarrow{f}) &= -\overrightarrow{\nabla}\Omega \cdot (\overrightarrow{\nabla} \times \overrightarrow{f}) = -\mathrm{Div}(\overrightarrow{f} \times \overrightarrow{\nabla}\Omega), \quad \text{for } \overrightarrow{f} \in A^{\oplus 3}.
\end{aligned}$$

More Notation

For any graded vector space $M = \oplus_{i\in\mathbb{Z}} M_i$ that is locally finite, we use $h_M(t) = \sum_{i\in\mathbb{Z}} \dim_{\Bbbk}(M_i)\, t^i$ to denote the *Hilbert series* of M. The Hilbert series of A is given by

$$h_A(t) = \frac{1}{(1-t^a)(1-t^b)(1-t^c)}.$$

The Poisson cohomology groups $HP^\bullet(A)$ are graded vector spaces. Denote their Hilbert series as $h_{HP^\bullet(A)}(t)$, then we have

$$\sum_{i=0}^{3}(-t^{-w})^i h_{PH^i(A)}(t) = -\frac{1}{t^{3w+a+b+c}} \frac{(1-t^{w+a})(1-t^{w+b})(1-t^{w+c})}{(1-t^a)(1-t^b)(1-t^c)}.$$

Hilbert Series for Poisson Cohomology

Theorem

(Huang-T-Wang-Zhang, '23) Let A_Ω *be a connected graded Poisson algebra with* $\deg(x) = a$*,* $\deg(y) = b$*, and* $\deg(z) = c$ *defined by a potential* Ω *of degree* n*. Suppose that* $Z_P(A) = \Bbbk[\Omega]$ *and* A *is H-ozone and* A *has a degree zero Poisson derivation that is not ozone and* A *has no non-zero Poisson derivation of degree* $-n$*. The Hilbert series of the Poisson cohomology groups of* A_Ω *are given by*

(1) $h_{PH^0(A)(t)} = \frac{1}{1-t^n}$ *and* $h_{PH^1(A)}(t) = \frac{1}{1-t^n}$.

(2) $h_{PH^2(A)}(t) = \frac{1}{t^{a+b+c}} \left(\frac{(1-t^{n-a})(1-t^{n-b})(1-t^{n-c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1 \right)$.

(3) $h_{PH^3(A)}(t) = \frac{(1-t^{n-a})(1-t^{n-b})(1-t^{n-c})}{t^{a+b+c}(1-t^n)(1-t^a)(1-t^b)(1-t^c)}$.

Hilbert Series for Poisson Cohomology

Corollary

(Huang-T-Wang-Zhang, 23) If $\Omega \in \Bbbk[x, y, z]$ *of degree* $n = a + b + c$ *is an irreducible potential in the classification that is neither* $x^k + y^l$ *nor* $x^k + z^l$ *nor* $y^k + z^l$*, then the Hilbert series of Poisson cohomology of* A_Ω *are given by*

(1) $h_{PH^0(A)(t)} = \frac{1}{1-t^n}$.

(2) $h_{PH^1(A)}(t) = \frac{1}{1-t^n}$.

(3) $h_{PH^2(A)}(t) = \frac{1}{t^n}\left(\frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1\right)$.

(4) $h_{PH^3(A)}(t) = \frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{t^n(1-t^n)(1-t^a)(1-t^b)(1-t^c)}$.

Some Equivalent Conditions for A_Ω

Theorem

(Huang-T-Wang-Zhang, '23) Let $A := \mathbb{k}[x, y, z]$ *be a connected graded Poisson polynomial algebra. Denote by* Z *the Poisson center of* A_Ω*. Then, the following statements are equivalent.*

(1) $rgt(A) = 0$ *and any homogeneous Poisson derivation of* A *with negative degree is zero.*

(2) *Any graded twist of* A *is isomorphic to* A*, and any homogeneous Poisson derivation of* A *with a negative degree is zero.*

Some Equivalent Conditions for A_Ω

Theorem

(3) *The Hilbert series of the graded vector space of Poisson derivations of A is* $\frac{1}{(1-t^a)(1-t^b)(1-t^c)}$.

(4) $h_{PH^1(A)}(t)$ *is* $\frac{1}{1-t^n}$.

(5) $h_{PH^1(A)}(t)$ *is equal to* $h_Z(t)$.

(6) *Every Poisson derivation* ϕ *of A has a decomposition* $\phi = zE + H_a$, *where* $z \in Z$ *and* $a \in A$. *Here, z is unique, and a is unique up to a Poisson central element.*

(7) *Every Poisson derivation of A that vanishes on Z is Hamiltonian.*

Some Equivalent Conditions for A_Ω

Theorem

(8) *A is an unimodular Poisson algebra determined by an irreducible potential Ω that is balanced.*

(9) $h_{PH^3(A)}(t) - h_{PH^2(A)}(t) = t^{-n}$.

(10) *A is unimodular and* $h_{PH^2(A)}(t) = \frac{1}{t^n}\left(\frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{(1-t^n)(1-t^a)(1-t^b)(1-t^c)} - 1\right)$.

(11) *A is unimodular and* $h_{PH^3(A)}(t) = \frac{(1-t^{a+b})(1-t^{a+c})(1-t^{b+c})}{t^n(1-t^n)(1-t^a)(1-t^b)(1-t^c)}$.

(12) *A is uPH^2-vacant.*

Some Technical Conditions: K_1-sealed

Let Ω be a homogeneous element of degree $n > 0$ in the weighted polynomial ring $A := \Bbbk[x, y, z]$. Recall that the *Koszul complex* $K_\bullet(\vec{\nabla}\Omega)$ given by the sequence $\vec{\nabla}\Omega := (\Omega_x, \Omega_y, \Omega_z)$ in A is:

$$0 \to A \xrightarrow{\vec{\nabla}\Omega} \begin{matrix} A \\ \oplus A \\ \oplus A \end{matrix} \xrightarrow{\vec{\nabla}\Omega\times} \begin{matrix} A \\ \oplus A \\ \oplus A. \end{matrix} \xrightarrow{\vec{\nabla}\Omega\cdot} A \to A/(\Omega_x, \Omega_y, \Omega_z) \to 0.$$

Definition

We say Ω is *K_1-sealed* if, for any $\vec{f} \in A^{\oplus 3}$ with $\vec{f} \cdot \vec{\nabla}\Omega = 0$ in A and $\vec{\nabla} \cdot \vec{f} = 0$ when considered as an element in A_{sing}, then $\vec{f} = \vec{\nabla}\Omega \times \vec{g}$ for some $\vec{g} \in A^{\oplus 3}$.

Some Technical Definitions: uPH^2-vacant

Definition

Set $M^2(A) := \{f\pi_\Omega + \pi_g \mid f, g \in A\}$ which is a subspace of $\mathfrak{X}^2(A)$. Note that

$$\operatorname{im}(d^1_{\pi_\Omega}) \subseteq M^2(A) \subseteq \ker(d^2_{\pi_\Omega}).$$

(1) The *upper division of the second Poisson cohomology* of A_Ω is defined to be $uPH^2(A_\Omega) := \ker(d^2_{\pi_\Omega})/M^2(A)$.

(2) The *lower division of the second Poisson cohomology* of A_Ω is defined to be $lPH^2(A_\Omega) := M^2(A)/\operatorname{im}(d^1_{\pi_\Omega})$.

(3) We say A_Ω is *uPH^2-vacant* if $uPH^2(A_\Omega) = 0$, or equivalently $lPH^2(A_\Omega) = PH^2(A_\Omega)$.

Some Technical Definitions: H-ozone

Definition

Let $A = \Bbbk[x_1, \ldots, x_n]$ be a connected graded Poisson algebra with its Poisson center denoted by Z.

(1) $\delta \in Pd(A)$ is called *ozone* if $\delta(Z) = 0$.

(2) Let $Od(A)$ denote the Lie algebra of all ozone Poisson derivations of A.

(3) We say A is H-ozone if $Od(A) = Hd(A)$, namely, any ozone derivation is Hamiltonian.

(4) We say A is PH^1-minimal if $PH^1(A) \cong ZE$ as graded Z-modules where E is the Euler derivation.

Some Technical Definitions: Balanced-Potentials

Definition

We call an irreducible potential Ω in $\Bbbk[x, y, z]$ *balanced* if $\Omega_x\Omega_y\Omega_z \neq 0$ for any choice of graded generators (x, y, z); otherwise, we call it *non-balanced*.

We have the following:

1. Ω is K_1-sealed $\Rightarrow A_\Omega$ is uPH^2-vacant.

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We have the following:

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2. A_Ω is uPH^2-vacant $\Leftrightarrow A_\Omega$ is H-ozone.

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We have the following:

1. Ω is K_1-sealed $\Rightarrow A_\Omega$ is uPH^2-vacant.
2. A_Ω is uPH^2-vacant $\Leftrightarrow A_\Omega$ is H-ozone.
3. A_Ω is H-ozone $\Leftrightarrow \Omega$ is irreducible and balanced.

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THANK YOU!