Connected Hopf algebras of finite Gelfand-Kirillov dimension

Guisong Zhou (周贵松)

Ningbo University (宁波大学)

Poisson Geometry and Artin-Schelter Regular Algebras

IASM-BIRS

October 13-18, 2024

Outlines

Let us assume the base field $\mathbb{F}$ to be of characteristic 0.

A Hopf algebra is called

- *connected* if its coradical is of dimension one;
- *connected graded* if it is equipped with a $(\mathbb{N}-)$grading which is compatible with the algebra structure and the coalgebra structure, and of one-dimensional zeroth component.

An easy observation

For a Hopf algebra, we have

$$\text{connected graded} \Longrightarrow \text{connected}.$$

Theorem (Cartier-Milnor-Moore)

The assignments $\mathfrak{g} \mapsto U(\mathfrak{g})$ and $H \mapsto P(H)$ define mutually inverse equivalences between the category of Lie algebras and the category of cocommutative connected Hopf algebras. Moreover,

$$\mathrm{GKdim}\, U(\mathfrak{g}) = \dim \mathfrak{g}.$$

Connected Hopf algebras of finite GK dimension can be viewed as

- generalizations of the universal enveloping algebras of finite dimensional Lie algebras.

Theorem (Basic facts on algebraic groups)

Let U be an algebraic group. Then U is unipotent if and only if the coordinate ring $\mathcal{O}(U)$ is connected. Moreover,

$$\mathrm{GKdim}\,\mathcal{O}(U) = \dim U.$$

Connected Hopf algebras of finite GK dimension can be viewed as

- noncommutative counterparts of unipotent algebraic groups.

Hopf algebras arised from combinatoric are connected graded in nature.

They are always locally finite but of infinite GK dimension, and many of them are *neither commutative nor cocommutative*.

The principle of combinatorial Hopf algebras (Joni-Rota, 1979)

The assemble/disassemble of a discrete structure may be encoded in the multiplication/comultiplication of an appropriate Hopf algebra.

Notable examples:

1. the Hopf algebra of (quasi-)symmetric functions;
2. the Hopf algebra of permutations of Malvenuto and Reutenauer;
3. the Hopf algebra of rooted trees of Connes and Kreimer;
4. the Hopf algebra of planar binary trees of Loday and Ronco;
5. $\cdots$

For a connected Hopf algebra H, the associated graded Hopf algebra $\mathrm{gr}(H)$ is commutative.

By the well-known Hopf-Leray theorem, $\mathrm{gr}(H)$ is isomorphic as graded algebras to a weighted polynomial algebra.

Theorem (Zhuang, 2012)

Let H be a connected Hopf algebra of finite GK dimension n. Then

1. *n is an integer;*
2. *$\mathrm{gr}(H)$ is isomorphic as graded algebras to a weighted polynomial algebra in n variables;*
3. *H is a Noetherian domain of Krull dimension $\leq n$;*
4. *H is Artin-Schelter regular of dimension n;*
5. *H is twisted Calabi-Yau of dimension n;*
6. *H is Auslander regular and Cohen-Macaulay.*

Note that for a connected Hopf algebra H, if $\mathrm{GKdim}\, H \geq 2$ then $\dim P(H) \geq 2$ (Zhuang, 2011). So connected Hopf algebras of GK dimension $0, 1, 2$ are all cocommutative.

Theorem (Zhuang, 2012; Wang-Zhang-Zhuang, 2013)

Assume the base field $\mathbb{F}$ is algebraically closed.

1. *Connected Hopf algebras of GK dimension 3 which are not cocommutative are classified into two families;*
2. *Connected Hopf algebras of GK dimension 4 which are not cocommutative are classified into twelve families.*

In each of these fourteen families, the members

- are generically not commutative;
- are all isomorphic as algebras to the universal enveloping algebras of some Lie algebras.

There is a connected graded Hopf algebra of GK dimension 5 which is not isomorphic as algebras to the universal enveloping algebra of any Lie algebra.

Example (Brown-Gilmartin-Zhang, 2017)

Generators: a, b, c, z, w of degree $1, 2, 1, 3, 3$;

Relations:
$$[b, a] = 0, \quad [c, a] = -b, \quad [c, b] = 0,$$
$$[z, a] = [z, b] = [z, c] = 0,$$
$$[w, a] = [w, b] = [w, c] = 0, \quad [w, z] = -\tfrac{1}{3}b^3;$$

Comult.:
$$a \mapsto 1 \otimes a + a \otimes 1, \quad b \mapsto 1 \otimes b + b \otimes 1,$$
$$c \mapsto 1 \otimes c + c \otimes 1,$$
$$z \mapsto 1 \otimes z + z \otimes 1 + \underline{a \otimes b - b \otimes a},$$
$$w \mapsto 1 \otimes w + w \otimes 1 + \underline{c \otimes b - b \otimes c};$$

Counit: $a, b, c, z, w \mapsto 0$.

Let A be an algebra, $\sigma : A \to A$ an algebra automorphism of A, and $\delta : A \to A$ a left σ-derivation of A. We write

$$R = A[z; \sigma, \delta]$$

and say that R is an *Ore extention* of A provided that

1. R is an algebra and contains A as a subalgebra;
2. z is an element of R;
3. $za = \sigma(a)z + \delta(a)$ for all $a \in A$;
4. R is a free left A-module with basis $\{1, z, z^2, \dots\}$.

We simply write $R = A[z; \sigma]$ in the case that $\delta = 0$.

Let $R = A[z; \sigma, \delta]$ be an Ore extension.

- If A is a domain then so is R;
- If A is left (resp. right) Noetherian then so is R;
- If $\mathrm{gldim}\, A = d < \infty$ then $d \leq \mathrm{gldim}\, R \leq d + 1$;
- If A is affine and $\sigma = \mathrm{id}_A$ then $\mathrm{GKdim}\, R = \mathrm{GKdim}\, A + 1$;
- If A is Auslander regular then so is R;
- If A is twisted Calabi-Yau of dimension d then R is twisted Calabi-Yau of dimension $d + 1$;
- In the connected graded setting, if A is Artin-Schelter regular of dimension d then R is Artin-Schelter regular of dimension $d + 1$;
- $\cdots$

Example (Quantum affine spaces)

Let $q = (q_{ij})_{1 \leq i,j \leq n}$ be a matrix with $q_{ii} = 1$ and $q_{ij} q_{ji} = 1$. The multiparameter quantum affine n-space $\mathcal{O}_q(\mathbb{F}^n)$ has generators $x_1, \ldots, x_n$ and relations $x_i x_j = q_{ij} x_j x_i$ for all i, j.

$$\mathcal{O}_q(k^n) = \mathbb{F}[x_1][x_2; \sigma_2] \cdots [x_n; \sigma_n].$$

Example (Weyl algebras)

The n-th Weyl algebra $A_n = A_n(\mathbb{F})$ has generators $x_1, y_1, \ldots, x_n, y_n$ and relations $x_i x_j = x_j x_i$, $y_i y_j = y_j y_i$, $x_i y_j = y_j x_i - \delta_{ij}$ for all i, j.

$$A_n = A_{n-1}[x_n][y_n; \mathrm{id}, \partial/\partial x_n].$$

Let $(\mathfrak{g}, \mathfrak{h})$ be a semisimple f.d. Lie algebra. Let $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ be a basis of the root system. For any $w \in W$, choose a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_N}$. Then for $r = 1, \ldots, N$, let

$$\beta_r := (s_{i_1} \cdots s_{i_{r-1}})(\alpha_{i_r}) \quad \text{and} \quad E_{\beta_r} = (T_{i_1} \cdots T_{i_{r-1}})(E_{i_r}),$$

where $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the Lusztig's automorphism. Let

$$U_q^+[w] := \operatorname{span}\{E_{\beta_1}^{r_1} \cdots E_{\beta_N}^{r_N} \mid r_j \in \mathbb{N}\}.$$

It is well-known that $U_q^+[w]$ is a subalgebra of $U_q^+(\mathfrak{g})$ and

$$U_q^+[w] = \mathbb{F}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \cdots [E_{\beta_N}; \sigma_N, \delta_N].$$

Moreover, it is independent of the choice of the reduced expression.

Theorem (Heckenberger-Schneider, 2013)

The assignment $w \mapsto U_q^+[w]$ defines an isomorphism between the poset W and the poset of homogeneous right coideal subalgebras of $U_q^+(\mathfrak{g})$, provided that q is not a root of unity.

Let H be a Hopf algebra.

An Ore extension $K = H[z;\sigma,\delta]$ of H is called a *Hopf-Ore extension* if in addition K is a Hopf algebra that contains H as a Hopf subalgebra.

Theorem (Brown-O'Hagan-Zhang-Zhuang, 2015)

Let $K = H[z;\sigma,\delta]$ be a Hopf-Ore extension of a connected Hopf algebra H. Then K is also connected and

$$\Delta_K(z) \in 1 \otimes z + z \otimes 1 + H \otimes H.$$

Moreover, $\mathrm{GKdim}\, K = \mathrm{GKdim}\, H + 1$.

Brown-O'Hagan-Zhang-Zhuang (2015) asks: after changing z, does

$$\Delta_K(z) \in a \otimes z + z \otimes 1 + H \otimes H$$

for some group-like element a of H? Hongdi Huang (2020) gave it a positive answer when R is Noetherian and $R \otimes R$ is a domain.

Theorem (Lu-Shen-Z., 2020)

Let H be a connected graded Hopf algebra of finite GK dimension n. Then there exists a chain of homogeneous Hopf subalgebras

$$\mathbb{F} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = H$$

such that $H_i = H_{i-1}[z_i; \mathrm{id}, \delta_i]$, in graded sense, for $i = 1, \ldots, n$.

More examples of iterated Hopf-Ore extension of $\mathbb{F}$:

- commutative connected Hopf algebras of finite GK dimension;
- connected Hopf algebras of GK dimension ≤ 4, when $\mathbb{F} = \overline{\mathbb{F}}$.

Connected Hopf algebras of finite GK dimension are NOT necessary iterated Hopf-Ore extensions of $\mathbb{F}$. Counterexamples:

- the universal enveloping algebra of simple Lie algebras of rank ≥ 2.

Theorem (Z., 2024)

Let H be a connected graded Hopf algebra. Let $A \subseteq B$ be homogeneous left (resp. right) coideal subalgebras of H. Then there exists a family $\{z_\xi\}_{\xi\in\Xi}$ of homogeneous elements of B_+ and a total order $<$ on Ξ s.t.

1. *the set $\{z_{\xi_1}^{r_1}\cdots z_{\xi_n}^{r_n} \mid n \geq 0,\ \xi_1 < \cdots < \xi_n,\ r_i \geq 0\}$ is a basis of B as a left A-module as well as a right A-module;*
2. *for each $\xi \in \Xi$, the left and right A-submodule of B spanned by*
$$\{z_{\xi_1}^{r_1}\cdots z_{\xi_n}^{r_n} \mid n \geq 0,\ \xi_1 < \cdots < \xi_n \leq \xi,\ r_i \geq 0\}$$
are equal; denote this common subspace by $H[\xi]$;
3. *$H[\xi]$ is a left (resp. right) coideal subalgebra of H, and it is a Hopf subalgebra if A and B are both so, for each $\xi \in \Xi$;*
4. *$[z_\mu, z_\nu] \in \bigcup_{\xi<\mu} H[\xi]$ for each pair $\mu, \nu \in \Xi$ with $\mu > \nu$.*

Moreover, $\operatorname{GKdim} B = \operatorname{GKdim} A + \#(\Xi)$.

Karchenko's argument[1] (with some improvements):

1. Choose a specific set of homogeneous generators X of H and a specific well order on X. They both depend on (A, B).
2. By combinatorial features of Lyndon words, one may associate to each Lyndon word ξ on X an element $z_\xi \in H$.
3. By the noncommutative Gröbner basis theory, one may figure out a set Ξ of Lyndon words on X.
4. Let $<$ be the restriction of the lexicographic order to Ξ.
5. The behavior of z_ξ under Δ assures that the pair $(\{z_\xi\}_{\xi\in\Xi}, <)$ has the desired properties listed as above.

[1] V. K. Kharchenko, A quantum analogue of Poincaré-Birkhoff-Witt theorem, Alg. Log., vol 38, (1999) 259-276.

Theorem (Z., 2024)

Let H be a connected graded Hopf algebra. Let $A \subseteq B$ be homogeneous left (resp. right) coideal subalgebras of H. Assume that A and B are of finite GK dimension m and n respectively. Then there exists a chain of homogeneous left (resp. right) coideal subalgebras

$$A = H_0 \subset H_1 \subset \cdots \subset H_{n-m} = B$$

such that $H_i = H_{i-1}[z_i; \mathrm{id}, \delta_i]$, in graded sense, for each i. Moreover, H_i can be chosen to be Hopf subalgebras if A and B are both so.

Theorem (Brown-Gilmartin, 2016; Z., 2024)

Let H be a connected Hopf algebra. Let A be a left or right coideal subalgebra of H of finite GK dimension n. Then

1. *n is an integer;*
2. *$\mathrm{gr}(A)$ is isomorphic as graded algebras to a weighted polynomial algebra in n variables;*
3. *A is a Noetherian domain of Krull dimension $\leq n$;*
4. *A is Artin-Schelter regular of dimension n;*
5. *A is twisted Calabi-Yau of dimension n;*
6. *A is Auslander regular and Cohen-Macaulay.*

Brown and Gilmartin assume $\mathbb{F} = \overline{\mathbb{F}}$ and their argument is geometric, using some facts from the theory of algebraic groups.

Theorem (Z., 2024)

Let H be a connected Hopf algebra. Let $A \subseteq B$ be commutative left (resp. right) coideal subalgebras of H.

1. *B is a polynomial algebra over A.*
2. *Assume A and B are of finite GK dimension m and n resp.. Then there exists a chain of left (resp. right) coideal subalgebras*

$$A = H_0 \subset H_1 \subset \cdots \subset H_{n-m} = B$$

such that $H_i = H_{i-1}[z_i]$ for each i. Moreover, H_i can be chosen to be Hopf subalgebras if A and B are both so.

Connected Hopf algebras of finite GK dimension are not necessary iterated Hopf-Ore extensions of $\mathbb{F}$. But all known counterexamples are the universal enveloping algebra of certain Lie algebras.

Question 1

Let H be any connected Hopf algebra of finite GK dimension. Is it necessary an iterated Hopf-Ore extension of the universal enveloping algebra of its primitive space $P(H)$?

The answer is positive in the following cases:

1. H is commutative or cocommutative;
2. H is connected graded;
3. $\dim P(H) = \mathrm{GKdim}\, H - 1$;
4. $\mathrm{GKdim}\, H \leq 4$ and $\mathbb{F} = \overline{\mathbb{F}}$.

Let H be a connected Hopf algebra. Define $e_H : H \to H$ to be

$$e_H := \log(\mathrm{id}_H) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\mathrm{id}_H - \eta\varepsilon)^{*n}}{n}.$$

If H is commutative or cocommutative, then e_H is an idempotent and it is called the (first) Eulerian idempotent of H.

Theorem (Hopf-Leray)

Let H be a commutative connected Hopf algebra. The inclusion map $\mathrm{Im}(e_H) \to H$ extends to an isomorphism of algebras

$$\mathrm{Sym}(\mathrm{Im}(e_H)) \to H.$$

Moreover, $\mathrm{GKdim}\, H = \dim \mathrm{Im}(e_H)$.

Let H be connected Hopf algebra of finite GK dimension over $\mathbb{C}$.

Let $V := \mathrm{Im}(e_{\mathrm{gr}(H)})$. Then $\dim V = \mathrm{GKdim}\, H < \infty$ and

$$\mathbb{C}[V^*] = \mathrm{Sym}(V) \cong \mathrm{gr}(H)$$

is naturally a Poisson algebra. It makes V^* a Poisson manifold.

For $H = U(\mathfrak{g})$, there is a canonical linear isomorphism $\mathfrak{g} \xrightarrow{\cong} V$. Its dual is an isomorphism of Poisson manifolds

$$V^* \xrightarrow{\cong} \mathfrak{g}^*.$$

The symplectic leaves of $\mathfrak{g}^*$ are exactly the coadjoint orbits.

Question 2

What can we say about the symplectic leaves of V^* and their relations to the representations of H?

Thanks for Your Attention!