Poisson Valuations

Xingting Wang

Louisiana State University

Joint work with Hongdi Huang, Xin Tang and James Zhang

BIRS-IASM Station, Hangzhou, China

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Table of Contents

Noncommutative projective algebraic geometry

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- These nononcommutative connected graded algebras nowadays are called Artin-Schelter (AS) regular algebras of dimension n, which can be thought as the coordinate rings of noncommutative $\mathbb{P}^{n-1}$.
- AS-regular algebras are twisted graded Calabi-Yau algebras, which were introduced by Ginzburg to transport the geometry of a Calabi-Yau manifold to noncommutative algebraic geometry.

Atin's conjecture

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- Artin suggested a list of function fields - division rings with GK-transcendence degree 2 - that could be the complete list of all birationally equivalent classes of noncommutative projective surfaces.

Artin's conjecture

Today, Artin's conjecture on the birational classification of noncommutative projective surfaces is (arguably) the most important open problem in noncommutative projective geometry. Various algebraic and geometric methods have been proposed for Artin's conjecture, but an ultimate solution remains elusive.

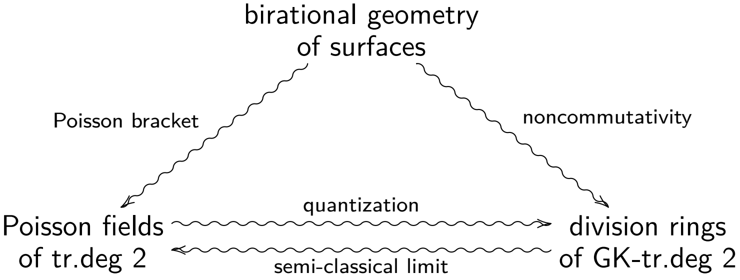
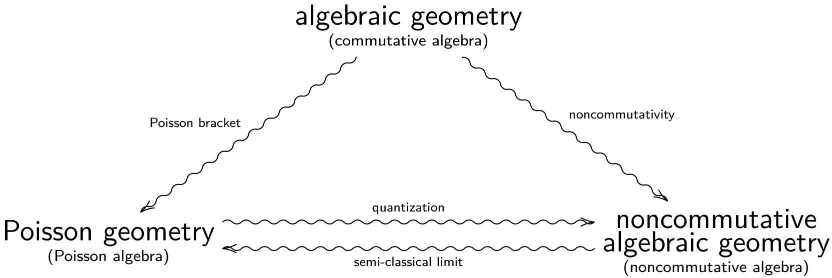


Table of Contents

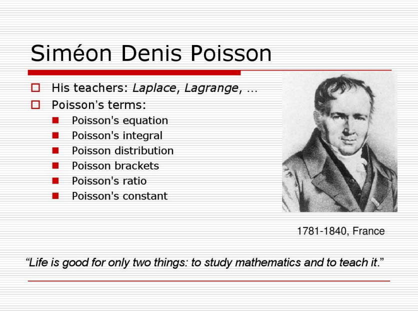
Poisson bracket

- The notion of Poisson bracket was introduced by French mathematician Siméon Denis Poisson in the search for integrals of motion in Hamiltonian mechanics.
- The relations among algebraic geometry, noncommutative algebraic geometry, and Poisson geometry can be summarized in the following diagram.



Poisson Origin Story

The notion of a Poisson bracket (or fish bracket, if translated) was first introduced by French mathematician Siméon Denis Poisson, a student of Lagrange and Laplace, in his 1809 essay, roughly translated as "Essay on the Variation of Arbitrary Constants in Questions of Mechanics".



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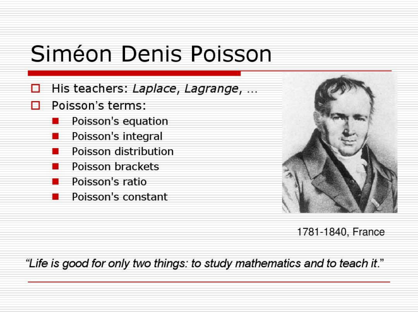

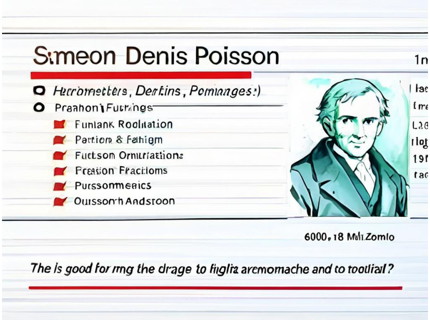

Jacobian structure

Example

Let $A = \Bbbk[x, y, z]$. For any polynomial $\Omega \in A$, define the Poisson bracket on A:

$$\{f, g\} = \det \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ \Omega_x & \Omega_y & \Omega_z \end{pmatrix}, \quad f, g \in A.$$

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- We denote this Poisson algebra by (A_Ω, π_Ω) and its quotient Poisson algebra $P_\Omega = A_\Omega/(\Omega)$

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- We denote this Poisson algebra by (A_Ω, π_Ω) and its quotient Poisson algebra $P_\Omega = A_\Omega/(\Omega)$
- If Ω is homogeneous, both A_Ω and P_Ω are graded Poisson algebras.

Discrete valuations

Definition (Discrete Valuation)

Let K be any field. A (discrete) valuation on K is a map

$$\boxed{\nu : K \to \mathbb{Z} \cup \{\infty\}}$$

which satisfies the following properties: for all $a, b \in K$,

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- $\nu(a + b) = \min\{\nu(a), \nu(b)\}$ if $\nu(a) \neq \nu(b)$.
- $R = \{a \in K \mid \nu(a) \geq 0\}$ is a discrete valuation ring (DVR).

Examples of discrete valuations

Example

- Let $\mathbb{Q}$ be the rationals and p be any prime number. The p-adic valuation ν_p on $\mathbb{Q}$ is given by, for any $a \in \mathbb{Q}$, we write $a = p^i \frac{m}{n}$ where $p \nmid mn$ and so

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- Let $X \subset \mathbb{P}^2$ be a complex projective curve with function field K. For any smooth point $p \in X$, the local ring $(\mathcal{O}_p, \mathfrak{m})$ is a DVR. Choose any uniformizer π for $\mathfrak{m}$. This gives a discrete valuation ν on K such that, for any $a \in K$, we write $a = \pi^i b$ where $b \in \mathcal{O}_p \setminus \mathfrak{m}$ and so

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- We denote by $\mathbb{V}(K) = \{\text{Poisson valuations on } K\}$.

Discrete valuation versus good filtrations

Definition (Filtration)

Let A be an (Poisson) algebra over $\Bbbk$. Let $\mathbb{F} = \{F_i \mid i \in \mathbb{Z}\}$ be a descending chain of $\Bbbk$-subspaces of A. We say $\mathbb{F}$ is a filtration of A if it satisfies

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namely, $\{(\mathrm{gr}_{\mathbb{F}} A)_i, (\mathrm{gr}_{\mathbb{F}} A)_j\} \subseteq (\mathrm{gr}_{\mathbb{F}} A)_{i+j}$ for all $i, j \in \mathbb{Z}$.
- We call $\mathbb{F}$ good if $\mathrm{gr}_{\mathbb{F}} A$ is a domain.

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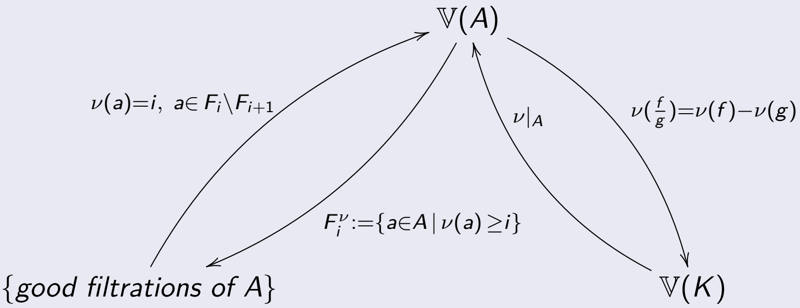
Theorem (Huang-Tang-Zhang-W. 2023)

Let A be a Poisson domain with Poisson fractional field K. We have the following natural bijections.



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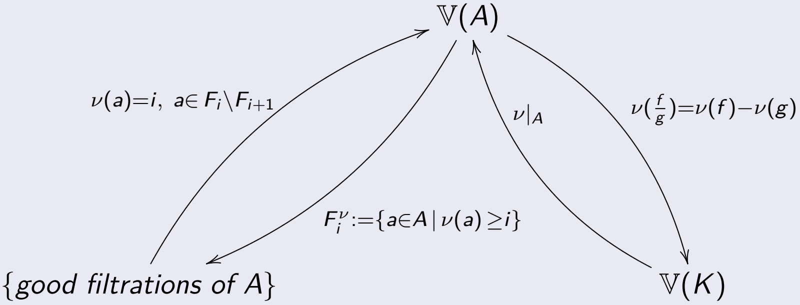

- We denote $\mathrm{gr}_{\nu} A := \mathrm{gr}_{\mathbb{F}^{\nu}} A$.

Discrete valuations versus good filtrations

Corollary (Huang-Tang-Zhang-W. 2023)

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded Poisson domain. There are two canonical Poisson discrete valuations on A, denoted by $\nu^{\pm Id}$, such that

$$\boxed{\nu^{Id}(a) = n \qquad \nu^{-Id}(a) = -m}$$

where $a = a_n + a_{n+1} + \cdots + a_m$ with $a_i \in A_i$, $a_n, a_m \neq 0$, and $n \leq m$.

Table of Contents

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- We denote by $\mathbb{V}_f(K) = \{\text{faithful Poisson valuations on } K\}$.
- We say a valuation on a Poisson domain is faithful if it is true on its Poisson fractional field.

Discretion valuation versus good filtration

Example (Computing faithful Poisson valuations)

Let $A = \Bbbk[x, y, z]$, and Ω be a cubic potential with an isolated singularity. Let $P_\Omega = A_\Omega/(\Omega)$ be its quotient Poisson algebra and consider $V_f(P_\Omega)$.

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Let $A = \Bbbk[x, y, z]$, and Ω be a cubic potential with an isolated singularity. Let $P_\Omega = A_\Omega/(\Omega)$ be its quotient Poisson algebra and consider $V_f(P_\Omega)$.

- By changing variables, $\Omega = x^3 + y^3 + z^3 + 3\lambda xyz$ with $\lambda^3 \neq -1$.

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$$a + b \leq \nu\left(\{x, y\} - 3\lambda xy\right) = \nu(3z^2) = 2c.$$

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- So $\{\nu^{\pm Id}\} = \mathbb{V}_f(P_\Omega)$.

Table of Contents

Poisson fields of tr.deg 2 and their valuations

Poisson fields of tr.deg 2	fields	Poisson brackets	$\#\mathbb{V}_f(K)$
Weyl field	$\Bbbk(x, y)$	$\{x, y\} = 1$	∞ uncountable
Skew field	$\Bbbk(x, y)$	$\{x, y\} = qxy$	∞ countable
Graded version of elliptic type	$Q(P_\Omega)$	π_Ω	2
Elliptic type-Ω	$Q_{\mathrm{gr}}(A_\Omega)_0$	π_Ω	1
Smooth projective curve Ω with genus $g \geq 3$	$Q(P_\Omega)$	π_Ω	0

Table: Poisson fields of tr.deg 2 and their valuations

Atin's conjecture via deformation quantization

Poisson fields of tr.deg 2	Division rings of GK-tr.deg 2	types	$\#\{$prime divisors$\}$
Weyl field	$Q(A_1(\mathbb{k}))$	q-ruled	∞ uncountable
Skew field	$\mathbb{k}_q(x, y), q \notin \sqrt[n]{1}$	q-ruled	∞ countable
Graded version of elliptic type	$\mathbb{k}(E)(t, \sigma)$ $\sigma \in \mathrm{Aut}_{\mathbb{k}}(E)$ with $\lvert\sigma\rvert = \infty$	q-ruled	2
Elliptic type	$Q_{\mathrm{gr}}(\mathrm{Skly}_{a,b,c})_0$ generic $[a : b : c] \in \mathbb{P}^2$	q-rational	1
?	finite-module over center	birationally PI	?

Table: Deformation quantizations of Poisson fields of tr.deg 2

Table of Contents

Automorphism Problem

Poisson Automorphism Problem:

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- $Aut_P(A_\Omega) \cong C_3 \ltimes G,\ G = \{(a, b, c) \in (\Bbbk^\times)^3 \mid a^3 = b^3 = c^3 = abc\}$.

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- *For any* $\xi \neq 0$, *there is an exact sequence of automorphism groups*

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- $Aut_P(A_\Omega)$ *is a finite subgroup of* $GL_3(\Bbbk)$ *of order bounded above by* $42(\deg \Omega)(\deg \Omega - 3)^2$.

Poisson discrete valuations

Definition (Huang-Tang-Zhang-W. 2023)

Let K be a Poisson field. For any integer w, a Poisson w-valuation on K is discrete valuation

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(4) $\nu(\{a, b\}) \geq \nu(a) + \nu(b) - w$ for all $a, b \in K$.

We denote by $\mathbb{V}_w(K) = \{\text{Poisson } w\text{-valuations on } K\}$.

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2. The Γ^w-cap of K is defined to be

$$\boxed{\Gamma^w(K) := \bigcap_{\nu \in \mathbb{V}_w(K)} F_0^\nu(K)}$$

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(3) (*Controlling theorem*) *If* A *is a noetherian normal Poisson domain, then* $\Gamma^1(Q(A)) \subseteq A$.

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Theorem (Huang-Tang-Zhang-W. 2023)

Let $A = \Bbbk[x, y, z]$ *and* Ω *a homogeneous polynomial of degree* ≥ 5 *with an isolated singularity. Let* $P_\xi := A_\Omega/(\Omega - \xi)$ *for some* $\xi \in \Bbbk$.

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1. *There is a unique faithful $(\deg \Omega - 3)$-valuation ν on P_ξ such that $\nu(x), \nu(y), \nu(z) < 0$. In particular, $F^{\nu}_{-1} = \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$.*

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2. *Then, for every w between 1 and $\deg \Omega - 4$, $\Gamma^w(Q(A_\Omega)) = A_\Omega$ and $\Gamma^w(Q(P_\xi)) = P_\xi$.*

Sketch of the proof of the main theorem

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Lemma (Lifting Method)

Let $A = \mathbb{k}[x_1, \dots, x_n]$ and Ω a homogeneous polynomial with an isolated singularity. Suppose $\varphi \in Aut(\mathbb{k}[x_1, \dots, x_n])$ satisfies

- $\varphi(\Omega) = \Omega$,
- $\varphi = \mathrm{id}$ *in* $A/(\Omega)$,
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- $1 \to \mu_{d-3} \to Aut_P(A_\Omega/(\Omega)) \xrightarrow{\pi} Aut(\mathrm{Proj}(A/(\Omega)))$ gives the bound of $|Aut_P(A_\Omega)|$ by Hurwitz's automorphism theorem.

Your Questions

Thank you for taking the time out of your busy schedules to be here !