Noncommutative del Pezzo surfaces as Artin-Schelter regular algebras

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§ Introduction / Background

X: smooth projective algebraic variety over k: field.

Noncommutative deformation of X = (flat) deformation of $D^b\mathrm{coh}\,X$ ($\mathrm{coh}\,X$)

Lowen - Van den Bergh

Deformation theory is described by the Hochschild dgla $R\mathrm{End}_{X\times X}(\mathcal{O}_{\Delta_X})[1]$

$\rightsquigarrow$ ∘ 1st order defos (tangent space) $\simeq HH^2(X) \overset{\text{HKR}}{\simeq} H^0(\wedge^2 T_X) \oplus H^1(T_X) \oplus H^2(\mathcal{O}_X)$

categorical deformation quantization $\Longleftrightarrow$ Poisson 小平-Spencer gerby

∘ an obstruction space $HH^3(X) \simeq \bigoplus_{p+q=3} H^q(\wedge^p T_X)$

- 1st order defos (tangent space) $\simeq HH^2(X) \overset{\text{HKR}}{\simeq} \underset{\text{Poisson}}{\underline{H^0(\Lambda^2 T_X)}} \oplus \underset{\text{小平-Spencer}}{\underline{H^1(T_X)}} \oplus \underset{\text{gerby}}{\underline{H^2(\mathcal{O}_X)}}$

- an obstruction space $\quad HH^3(X) \simeq \bigoplus_{p+q=3} H^q(\Lambda^p T_X)$

$\dim X = 1 \Rightarrow HH^2(X) \simeq H^1(T_X) \Rightarrow \nexists$ nc defos.

$\dim X = 2$: the first non-trivial dimension. $\Lambda^2 T_X \simeq \omega_X^{-1}$: anticanonical bundle

<u>Def</u> X is a (weak) del Pezzo surface if $\dim X = 2$, ω_X^{-1} is ample
(nef big)

<u>Lem</u> X : weak dP $\Rightarrow \begin{cases} HH^2(X) \simeq H^0(\omega_X^{-1}) \oplus H^1(T_X) \\ HH^3(X) = 0 \Rightarrow \text{unobstructed!} \end{cases}$

cf) nc defos of $\mathbb{P}^n$ $(n \geq 3)$ and $\Sigma_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$ $(d \geq 4)$ are obstructed.

In this talk we

- Introduce classes of AS-regular $\mathbb{Z}$-algebras A

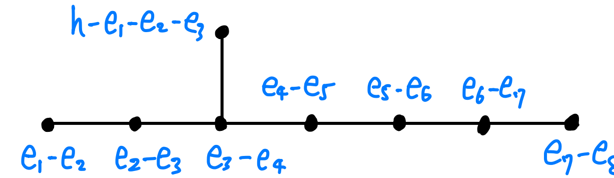
  such that $\mathrm{qmod}\,A$ exhaust nc del Pezzo surfaces

- Explain conjectures on these $\mathbb{Z}$-algebras

  (partial results, possible approaches)

based on a joint project w/ Kazushi Ueda, Tarig Abdelgadir, ...

§ del Pezzo surfaces & root system



Classification of del Pezzo surfaces (10 defo types)

X	$\mathbb{P}^2$	$\mathbb{P}^1\times\mathbb{P}^1$	$\Sigma_1=Bl_1\mathbb{P}^2$	$Bl_2\mathbb{P}^2$	$Bl_3\mathbb{P}^2$	$Bl_4\mathbb{P}^2$	$Bl_5\mathbb{P}^2$	$Bl_6\mathbb{P}^2$	$Bl_7\mathbb{P}^2$	$Bl_8\mathbb{P}^2$
$d=\omega_X^2$	9	8	8	7	6	5	4	3	2	1
R	$\emptyset$	$\mathbb{A}_1$	NA	NA	$\mathbb{A}_1+\mathbb{A}_2$	$\mathbb{A}_4$	$\mathbb{E}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$

$(\mathrm{Pic}X, \otimes) \simeq \left(H^2(X^{an}, \mathbb{Z}), \cup\right)$ $\quad R := \omega_X^{\perp} \left(\subseteq \mathrm{Pic}(X)\right)$: root lattice

Def Fix a deformation type of X. $\Lambda_d := \left(\mathbb{Z}^{10-d}, \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix}\right)$ $\quad \omega := \begin{bmatrix} 3 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \in \Lambda$

A marking of X is an isometry $\varphi : \Lambda_d \xrightarrow{\sim} \mathrm{Pic}(X)$

$$\text{s.t.} \quad \begin{matrix} \Lambda_d & & \mathrm{Pic}(X) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \omega & \longmapsto & \omega_X \end{matrix}$$

Prop

- $\{\varphi : \Lambda_d \xrightarrow{\sim} \mathrm{Pic}X \mid \text{marking}\} \curvearrowleft W(R) = \langle s_\alpha \mid \alpha : \text{roots of } R\rangle$ is free & transitive. (Weyl group)
- $[\varphi : \Lambda_d \xrightarrow{\sim} \mathrm{Pic}X] \Longleftrightarrow [E_1, E_2, \ldots, E_{9-d} \subseteq X \text{ disjoint } (-1)\text{-curves}]$

§ AS-regular algebras

<u>Some History</u>

- [Artin-Schelter 1987] initiated the classification of 3-dimensional AS-regular graded algebras generated in degree 1.

  ⇝ quadratic / cubic

  quadratic $\Updownarrow$ nc $\mathbb{P}^2$

  cubic $\Updownarrow$ (part of) nc $\mathbb{P}^1 \times \mathbb{P}^1$ (2-dim'l moduli)

- [Artin-Tate-Van den Bergh 1990] establish 1-to-1 correspondence between these algebras and geometric data (point scheme)

triangulated categorical point of view ⤵

- [Bondal-Polishchuk 1993] $\mathbb{Z}$-algebra version of the quadratic case
- [Van den Bergh 2011] cubic $\mathbb{Z}$-algebra version of [BP].

  cubic $\mathbb{Z}$-algebra $\Updownarrow$ all nc $\mathbb{P}^1 \times \mathbb{P}^1$ (3-dim'l moduli)

Key observation • (AS-regular 3-dim'l) quadratic / cubic $\mathbb{Z}$-algebras are generalizations of certain generators of $D^b\mathrm{coh}\,\mathbb{P}^2$ / $D^b\mathrm{coh}\,\mathbb{P}^1\times\mathbb{P}^1$

• Starting with similar generators of $D^b\mathrm{coh}\,X$, we obtain **definitions** of classes of AS-regular 3-dim'l $\mathbb{Z}$-algebras by which we can exhaust nc defos of X.

Prototypical example (3-dim'l Sklyanin algebras = generic AS-regular 3-dim'l quadratic algebras)

$$S_{a,b,c} := k\langle x,y,z\rangle \Big/ \left(\begin{array}{l} cz^2+axy+byx, \\ cx^2+ayz+bzy, \\ cy^2+azx+bxz \end{array} \right)$$

$[a:b:c]\in\mathbb{P}^2$ $\quad \mathrm{qgr}\,S := \dfrac{\mathrm{grmod}\,S}{\mathrm{tors}\,S}$. $\quad \mathrm{coh}\,\mathbb{P}^2 \simeq \mathrm{qgr}\,S_{1,-1,0} \overset{\text{defo}}{\rightsquigarrow} \underset{\text{nc }\mathbb{P}^2}{\mathrm{qgr}\,S_{a,b,c}}$

Prototypical example (continued) (3-dim'l Sklyanin algebras = generic AS-regular 3-dim'l quadratic algebras)

$$S_{a,b,c} := k\langle x,y,z\rangle \Big/ \begin{pmatrix} cz^2+axy+byx, \\ cx^2+ayz+bzy, \\ cy^2+azx+bxz \end{pmatrix}$$

$[a:b:c]\in\mathbb{P}^2$ $\quad \mathrm{qgr}\, S := \dfrac{\mathrm{grmod}\, S}{\mathrm{tors}\, S}$. $\quad \mathrm{coh}\,\mathbb{P}^2 \simeq \mathrm{qgr}\, S_{1,-1,0} \overset{\text{defo}}{\rightsquigarrow} \underset{\text{nc } \mathbb{P}^2}{\mathrm{qgr}\, S_{a,b,c}}$

- $S_{a,b,c}$ is the deformation quantization of the Poisson structure $\beta_{a,b,c}$, where

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^2}(3) & \longleftrightarrow & \omega_{\mathbb{P}^2}^{-1} \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ abc(x^3+y^3+z^3)-(a^3+b^3+c^3)xyz & \longleftrightarrow & \beta_{a,b,c} \end{array}$$

"$E \overset{?}{\hookrightarrow} \mathrm{Proj}\, S$"

$\mathrm{coh}\,\mathrm{Proj}\, S_{a,b,c}$ "$\wr$"

"$\|$"

- $E_{a,b,c} := \mathrm{Zero}(\beta_{a,b,c}) \hookrightarrow \mathbb{P}^2$ "stays" in "$\mathrm{Proj}\, S_{a,b,c}$" : $\exists$adjoint $\mathrm{qgr}\, S_{a,b,c} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{coh}\, E_{a,b,c}$ ($i^* \dashv i_*$)

Point scheme (moduli of "points")

§ Artin-Schelter regular I-algebras & pure helices

$\Bbbk$: field, I: poset

Def: An I-algebra (over $\Bbbk$) is a $\Bbbk$-linear category A + a bijection $I \xrightarrow[\sim]{\varphi} \mathrm{obj}(A)$

$\Bbbk$-algebroid

"marking" or "polarization"

Write $A_{ij} = \mathrm{Hom}_A(\varphi(j), \varphi(i))$

$e_i = \mathrm{id}_{\varphi(i)} \in A_{ii}$ (local unit)

米田 : $A \hookrightarrow \mathrm{Fun}(A^{op}, \mathrm{Vect}_\Bbbk) =: \mathrm{Mod}\,A$ ← right modules

$$\begin{array}{ccc} A & \hookrightarrow & \mathrm{Fun}(A^{op}, \mathrm{Vect}_\Bbbk) \\ \Psi & & \Psi \\ i & \longmapsto: & P_i \end{array}$$

← the i-th projective module

- A is $\begin{cases} \text{connected} \iff A_{ii} = \Bbbk e_i \\ \text{positively graded} \iff [A_{ij} \neq 0 \Rightarrow i \geq j] \end{cases}$
- $P_i \to S_i \to 0$ ← the i-th simple module, $S_i = \Bbbk e_i$

Def $\mathrm{qmod}\,A := \dfrac{\mathrm{mod}\,A}{\mathrm{tors}\,A}$

$$\begin{array}{ccc} \pi : \mathrm{mod}\,A & \longrightarrow & \mathrm{qmod}\,A \\ \Psi & & \Psi \\ P_i & \longmapsto: & \mathcal{O}_A(i) \end{array}$$

Rem ($I = \mathbb{Z}$ - algebras vs $\mathbb{Z}$ -graded algebras)

$R = \bigoplus_{i \in \mathbb{Z}} R_i \rightsquigarrow \check{R}$: $\mathbb{Z}$ - algebra defined by $(\check{R})_{ij} := R_{i-j}$

Lem $\operatorname{Grmod} R \simeq \operatorname{Mod} \check{R}$ $\quad$ $R(n) \longleftrightarrow P_n$

$\rotatebox{90}{$\in$} \qquad \rotatebox{90}{$\in$}$ $\quad$ $k(n) \longleftrightarrow S_n$

$M = \bigoplus_{i \in \mathbb{Z}} M_i \mapsto \left[\check{M} : j \longmapsto M_{-j}\right]$

Prop $A \simeq \check{R}$ $\exists R \iff A(1) \simeq A$ (1-periodic), where $A(1)_{ij} = A_{i+1, j+1}$.

Def A : connected & positively graded I-algebra

A is **Artin-Schelter Gorenstein**

if $\forall i \in I$ $\exists (j_0, l_0) \in I \times \mathbb{Z}$ s.t. $\operatorname{Ext}^l_{\operatorname{Mod} A}(S_i, P_j) = \begin{cases} k & \text{if } (j, l) = (j_0, l_0) \\ 0 & \text{otherwise} \end{cases}$

Def $\mathcal{D}$: triangulated k-linear category.

A helix is $(\mathcal{E}_i)_{i\in\mathbb{Z}} \subseteq \mathrm{Obj}\,\mathcal{D}$

s.t. $(\mathcal{E}_{j+1}, \mathcal{E}_{j+2}, \dots, \mathcal{E}_{j+r})$ is a full exceptional collection of $\mathcal{D}$ $\forall j\in\mathbb{Z}$.

$$K_0(\mathcal{D}) \simeq \mathbb{Z}^{\oplus r}$$
$$\Uparrow$$

Rem $S_{\mathcal{D}} \curvearrowright (\mathcal{E}_i)_{i\in\mathbb{Z}}$, FEC is a "fundamental domain".

Serre functor

Def $(\mathcal{E}_i)_{i\in\mathbb{Z}}$ is pure if $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[l]) = 0$ if $i\leq j$ & $l\neq 0$.

- geometric (Bridgeland-Stern)
- acyclic
- very strong (VdB)

Ex $\pi: \mathrm{Tot}(\omega_X) = \mathrm{Spec}\,\mathrm{Sym}\,\omega_X^{-1} \longrightarrow X$

$(\mathcal{E}_i)_{i\in\mathbb{Z}} \subseteq \mathrm{coh}\,X$ is pure $\Leftrightarrow$ $T := \pi^*\left(\bigoplus_{i=1}^{r} \mathcal{E}_i\right) \in \mathrm{coh}\,\mathrm{Tot}(\omega_X)$ is tilting:

i.e. $R\mathrm{End}_{\mathrm{Tot}(\omega_X)}(T) \simeq \mathrm{End}_{\mathrm{Tot}(\omega_X)}(T)$.

$$\Gamma := \mathrm{End}_{\mathrm{Tot}(\omega_X)}(T) = \mathrm{Hom}\Big(\pi^*\bigoplus_{i=1}^{r}\mathcal{E}_i,\ \pi^*\bigoplus_{i=1}^{r}\mathcal{E}_i\Big) \underset{PF}{\simeq} \underbrace{\mathrm{Hom}_X\Big(\bigoplus_{i=1}^{r}\mathcal{E}_i,\ \bigoplus_{i=1}^{r}\mathcal{E}_i \otimes \Big(\bigoplus_{d\geq 0}\omega_X^{-d}\Big)\Big)}_{\mathbb{Z}\text{-graded by } d}$$

<u>Prop</u> • $D^b\mathrm{coh}\,X \simeq D^b\mathrm{qgr}\,\Gamma$

• $\check{\Gamma} \simeq (\mathcal{E}_i)_{i\in\mathbb{Z}}$ as $\mathbb{Z}$-algebras

<u>Assume</u> $\dim X = 2 \Rightarrow \Gamma$: graded 3-CY algebra $\overset{\text{Ginzburg}}{\Longrightarrow}$ $\Gamma \cong \mathrm{Jac}(\check{Q}, W)$

rolled-up quiver $\to \check{Q}$; superpotential $\to W$

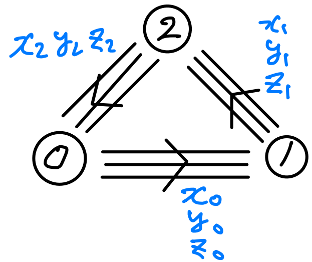
<u>Cor</u> Explicit projective resolutions of simple modules $\in \mathrm{grmod}\,\Gamma \simeq \mathrm{mod}((\mathcal{E}_i)_{i\in\mathbb{Z}})$:

$$\boxed{0 \to P_{i-r} \to \bigoplus_{i<a\leq r} P_{a-r}^{\oplus m_{ai}} \xrightarrow{\ \mathrm{Hess}(W)\ } \bigoplus_{1\leq b<i} P_b^{\oplus m_{ib}} \to P_i \to S_i \to 0}$$

<u>Def</u> AS-regular algebra of type $\check{Q}$ is an AS-Gorenstein $I = I_{\check{Q}}$-algebra whose simples admit projective resolutions of the shape determined by $\bigstar_i$.

Ex 1 $\mathbb{P}^2$, $(\mathcal{O}(i))_{i\in\mathbb{Z}}$: pure helix

$\check{Q}$:



$$W = \Omega(x_2 y_1 z_0 + y_2 z_1 x_0 + z_2 x_1 y_0) - \Omega(z_2 y_1 x_0 + x_2 z_1 y_0 + y_2 x_1 z_0)$$

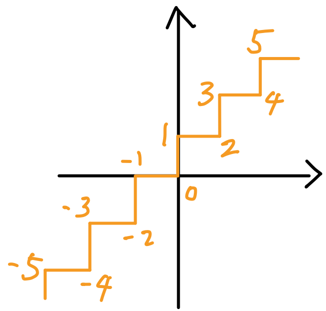
$$\rightsquigarrow \operatorname{Jac}(\check{Q}, W) \simeq \operatorname{End}_{\omega_{\mathbb{P}^2}}\Big(\pi^* \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^2}(i)\Big)$$

Resolutions: $0 \to P_{i-3} \to P_{i-2}^{\oplus 3} \longrightarrow P_{i-1}^{\oplus 3} \longrightarrow P_i \longrightarrow S_i \to 0 \quad \sim ⛤_i$

AS-regular $I\check{Q} = \mathbb{Z}$-algebra of type $\check{Q}$

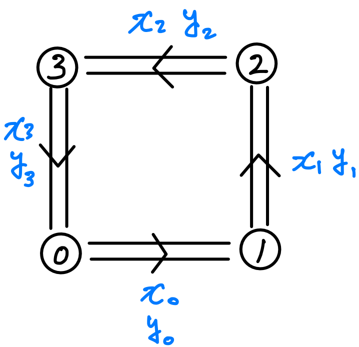
= AS-regular 3-dimensional quadratic $\mathbb{Z}$-algebra!

Ex 2 $\mathbb{P}^1\times\mathbb{P}^1$,



$\subseteq \operatorname{coh} \mathbb{P}^1\times\mathbb{P}^1$

$\check{Q}$:



$$W = \circlearrowright(x_3x_2y_1y_0 + y_3y_2x_1x_0) - \circlearrowright(x_3y_2y_1x_0 + y_3x_2x_1y_0)$$

Gorenstein parameter

Resolutions: $0 \to P_{i-4} \to P_{i-3}^{\oplus 2} \to P_{i-1}^{\oplus 2} \to P_i \to S_i \to 0$ $\sim \circledast_i$

AS-regular $\mathbb{Z}$-algebra of type $\check{Q}$

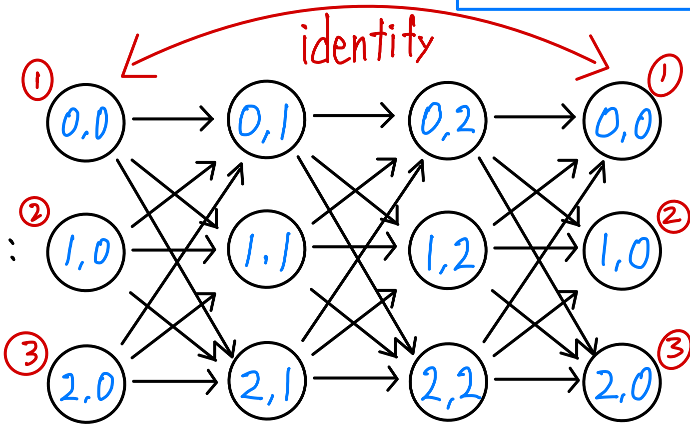
= AS-regular 3-dimensional cubic $\mathbb{Z}$-algebra!

Ex 3 $Bl_{6pts}\mathbb{P}^2$ $(d=3)$

3-block helix gen. by:

$\mathcal{O}_X\left(-3H+\sum_{i=1}^{6}E_i+E_1\right)$	$-2H+\sum_{i=1}^{6}E_i-E_4$	$-2H+\sum_{i=1}^{6}E_i$
$-3H+\sum_{i=1}^{6}E_i+E_2$	$-2H+\sum_{i=1}^{6}E_i-E_5$	$-H+\sum_{i=1}^{3}E_i$
$-3H+\sum_{i=1}^{6}E_i+E_3$	$-2H+\sum_{i=1}^{6}E_i-E_6$	0

$\check{Q}$:



$I = I_{\check{Q}} = \{0,1,2\} \times \mathbb{Z}$

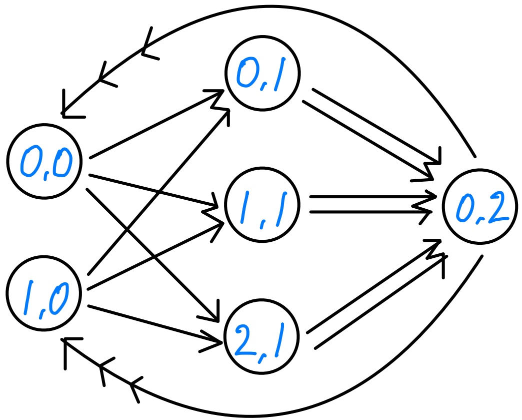
Resolution of S_i

$$0 \to P_{i-(0,3)} \to \bigoplus_{a\in\{0,1,2\}} P_{i-(a,2)} \to \bigoplus_{b\in\{0,1,2\}} P_{i-(b,1)} \to P_i \to S_i \to 0$$

Ex 4 $Bl_{3pts}\mathbb{P}^2$ $(d=6)$ 3-block helix gen by:

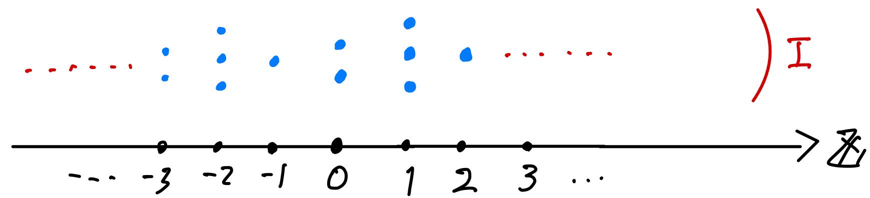
$\mathcal{O}_X(-2H+\sum_{i=1}^{3}E_i)$	$\mathcal{O}_X(-H+E_1)$	$\mathcal{O}_X$
$\mathcal{O}_X(-H)$	$\mathcal{O}_X(-H+E_2)$	
	$\mathcal{O}_X(-H+E_3)$	

$\check{Q}$:



$i=(i_1,i_2)$

$\hat{I}$ = the poset obtained by unfolding $\check{Q}$.



Resolutions of S_i

$\boxed{i_1\equiv 2 \pmod 3}$ $\quad 0\to P_{i-(0,3)}\to \bigoplus_{a\in\{0,1\}} P_{(a,i_1-2)}^{\oplus 3}\to \bigoplus_{b\in\{0,1,2\}} P_{(b,i_1-1)}^{\oplus 2}\to P_i\to S_i\to 0$

$\boxed{i_1\equiv 1 \pmod 3}$ $\quad 0\to P_{i-(0,3)}\to P_{(0,i_1-2)}^{\oplus 2}\to \bigoplus_{b\in\{0,1\}} P_{(b,i_1-1)}\to P_i\to S_i\to 0.$

$\boxed{i_1\equiv 0 \pmod 3}$ $\quad 0\to P_{i-(0,3)}\to \bigoplus_{a\in\{0,1,2\}} P_{(a,i_1-2)}\longrightarrow P_{(0,i_1-1)}^{\oplus 3}\to P_i\to S_i\to 0.$

$\boxed{\text{Outlook}}$ ∘ Any del Pezzo surface X admits a pure helix

$\rightsquigarrow$ Get definition of AS-regular $I=I_{\check{Q}}$ algebras A
s.t. $\operatorname{qmod} A$ exhaust defos of $\operatorname{coh} X$ (VdB, de Deken-Lowen)

∘ Any X except Σ_1 & $Bl_{2\,pts}\mathbb{P}^2$ admits 3-block helix.

∘ Each X admits ∞'ly many pure helices (ex helices on $\mathbb{P}^2$
$\longleftrightarrow$ solutions of $x^2+y^2+z^2=3xyz$)

$\rightsquigarrow$ Get ∞'ly many $\check{Q}$

$\rightsquigarrow$ Get ∞'ly many classes of AS-regular algebras of type $\check{Q}$
for the same defo type of del Pezzos.

Q1 How are they related? $\boxed{\text{Q2}}$ Classification via geometric data?

Q3 Affine Weyl group action $\widetilde{W}(R) = R \rtimes W(R) \curvearrowright \{A\}/\mathrm{iso}.$

s.t. $\mathrm{qmod}\, A \simeq \mathrm{qmod}\, A' \iff \widetilde{W}(R)A = \widetilde{W}(R)A'$

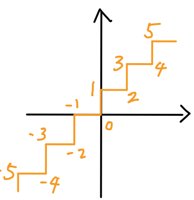
- A = nc del Pezzo surface + marking / polarization

Q3 is true for
- quadratic $\mathbb{Z}$-algs (Stafford - VdB) $(\widetilde{W}(\phi) = \{1\})$
- cubic $\mathbb{Z}$-algs (Kitamura - O) $(\widetilde{W}(A_1) = D_\infty)$

§ Towards classification of AS-regular algebras of type $\check{Q}$ by geometric data

Known cases

1) 3-dim'l quadratic $\mathbb{Z}$-algs $\left(\Longleftrightarrow \mathbb{P}^2, (\mathcal{O}(i))_i\right)$

2) 3-dim'l cubic $\mathbb{Z}$-algs $\left(\Longleftrightarrow \mathbb{P}^1\times\mathbb{P}^1,\right.$  $\left.\right)$

1) $A \longleftrightarrow (Y, L_0, L_1)$

(Artin-Tate-Van den Bergh, Bondal-Polishchuk)

- $Y \simeq \mathbb{P}^2$ or $Y \in |-K_{\mathbb{P}^2}|$
- $L_i \in \mathrm{Pic}(Y)$, $\deg(L_i) = 3$, very ample
- $L_0 \otimes L_1^{-1} \in \mathrm{Pic}^0(Y)$

2) $A \longleftrightarrow (Y, L_0, L_1, L_2)$

(Van den Bergh)

- $Y \simeq \mathbb{P}^1\times\mathbb{P}^1$ or $Y \in |-K_{\mathbb{P}^1\times\mathbb{P}^1}|$
- $L_i \in \mathrm{Pic}(Y)$, $\deg(L_i) = 2$, $L_0\otimes L_1$, $L_1 \otimes L_2$ very ample
- $L_0 \otimes L_2^{-1} \in \mathrm{Pic}^0(Y)$, $L_0 \not\simeq L_2$

Goal Generalize these results to all AS-regular algebras of type $\check{Q}$.

- For this, we have to find more conceptual proofs of BP thms & VdB thms.

Geometric data $\longrightarrow$ Algebra cf) arXiv: 2007.07620 (Ueda - O)

geometric data $\Leftrightarrow$ pure spherical helix $(\mathcal{S}_i)_{i\in\mathbb{Z}} \subseteq \operatorname{coh} Y$ 1-spherical objects

$\rightsquigarrow$ define A as the "ambient surface" s.t. "$\exists\, \iota : Y \hookrightarrow \operatorname{Proj} A$"

$$\mathcal{S}_i \xleftarrow[\iota^*]{} \mathcal{O}(i)$$

Concretely $\mathcal{S}_1, \dots, \mathcal{S}_r \in \operatorname{perf}(Y)$: sequence of 1-spherical objects

$\mathcal{B} := (\mathcal{S}_i)_{i=1}^r \subseteq \operatorname{perf}(Y)$: dg category $\rightsquigarrow \mathcal{B} \supseteq \mathcal{A} = (\mathcal{E}_i)_{i=1}^r$: directed dg subcategory

$\rightsquigarrow$ define $(\mathcal{S}_i)_{i\in\mathbb{Z}}$ and $(\mathcal{E}_i)_{i\in\mathbb{Z}}$ by iterated spherical twists and mutations, respectively.

(OU) $(\mathcal{S}_i)_{i\in\mathbb{Z}}$ pure $\Rightarrow (\mathcal{E}_i)_{i\in\mathbb{Z}}$ pure

(work in progress) make [Bondal-Polishchuk] conceptual from this perspective + [Ginzburg]

Algebra $\longrightarrow$ Geometric data

FSEC: Assume $\mathcal{E}_r = \mathcal{O}$.

Idea • $A = (\mathcal{E}_i)_{i\in\mathbb{Z}} \supseteq \underline{(\mathcal{E}_i)_{i=1}^{r}} =: B \Rightarrow R\mathrm{Hom}\left(\bigoplus_{i=1}^{r} \mathcal{E}_i, -\right) : D^b \mathrm{qmod}\, A \xrightarrow{\simeq} D^b \mathrm{mod}\, B$

• Define the moduli space of "skyscraper sheaves" on "Proj A" as stable (in the sense of A. King) B-modules of dimension vector $\underline{\alpha := (\mathrm{rk}\, \mathcal{E}_i)_{i=1}^{r}} \in \mathbb{Z}^r$.

$\odot$ $R\mathrm{Hom}\left(\bigoplus_i \mathcal{E}_i, \mathcal{O}_x\right) \simeq \bigoplus_i \mathcal{E}_i^\vee|_x$

Issues • Which stability condition should we choose?

• what kind of curves Y do we get? $\in |-K_X|$?

Solution Define the Hilbert scheme of 1-point as the moduli of representations of the chipped collection $(\mathcal{E}_i)_{i=1}^{r-1} =: B'$.

Observation

$$\boxed{(\mathcal{E}_i)_{i=1}^{r-1} =: B'}$$

$x \in$ "Proj A" a point $\to$ $[0 \to I_x \to \mathcal{O} \to \mathcal{O}_x \to 0]$

$\to [\mathcal{O}_A \to \mathcal{O}_x \to I_x[1] \xrightarrow{+1} \mathcal{O}_A[1]] \to I_x[1] \in \mathcal{O}^\perp \cap \operatorname{mod} B$

$\simeq \operatorname{mod} B'$.

Note $\operatorname{RHom}(\bigoplus_{i=1}^{r-1} \mathcal{E}_i, -) : \mathcal{O}^\perp \xrightarrow{\sim} D^b \operatorname{mod} B'$

$$\begin{array}{ccc} \mathcal{O}^\perp & \xrightarrow{\sim} & D^b \operatorname{mod} B' \\ \Psi & & \cup \\ I_x[1] & \longmapsto & {:}M' \in \operatorname{mod} B' \end{array}$$

Let X : comm. del Pezzo $\Rightarrow j : X \hookrightarrow \mathrm{Rep}(B', \alpha') :=$ moduli stack of B'-modules of dim vector

$$\begin{array}{ccc} X & \hookrightarrow & \mathrm{Rep}(B', \alpha') \\ \Psi & & \Psi \\ x & \longmapsto & M' \end{array}$$

$\alpha' = (\operatorname{rk} \mathcal{E}_i)_{i=1}^{r-1}$.

Main Observation j is an open immersion s.t. $j^* : \operatorname{Pic}(\mathrm{Rep}(B', \alpha')) \xrightarrow{\sim} \operatorname{Pic}(X)$.

$$\begin{array}{ccc} \operatorname{Pic}(\mathrm{Rep}(B', \alpha')) & \xrightarrow{\sim} & \operatorname{Pic}(X) \\ \Psi & & \Psi \\ \mathcal{O}' & \longmapsto & \omega_X^{-1} \end{array}$$

"independent" of X. $\longrightarrow \mathcal{O}'$

Define $\mathrm{Hilb}_A^{[1]}$ as moduli of $\mathcal{O}'$-stable B'-modules of dim vector α'.

Prop $\mathrm{Hilb}_A^{[1]}$ is a commutative weak dP surface

Conj • $\exists$ stability condition θ of B, indep of the choice of A, s.t. the natural rational map

$$\begin{tikzcd}
\theta\text{-stable modules} \arrow[r] & \mathrm{Rep}(B,\alpha,\theta) \arrow[r, dashed] & \mathrm{Rep}(B',\alpha',\theta') & \theta'\text{-stable modules} \arrow[l] \\
& M \arrow[r, mapsto] \arrow[u, phantom, "\in"] \arrow[d, leftrightarrow] & M' := L_{\mathcal{O}} M[1] \arrow[u, phantom, "\in"] \arrow[d, leftrightarrow] & \\
& \text{"}\mathcal{O}_x\text{"} \arrow[r, mapsto] & \text{"}I_x[1]\text{"} &
\end{tikzcd}$$

is a closed immersion. $\mathrm{Rep}(B,\alpha,\theta) :=$ the moduli of point representations of A.

- $A \longleftrightarrow \mathrm{Rep}(B,\alpha,\theta)$ + the universal B-module

is a 1-to-1 correspondence.

Note Conj is known to be the case for AS-regular quadratic / cubic $\mathbb{Z}$-algebras.

$\theta = (-2,1,1) \,/\, (-3,1,1,1)$