When is a Koszul algebra a domain?

(joint with D. Rogalski)

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Zero-divisors and homological properties

Ring structure and homological properties

We expect that if a ring is "homologically nice," then it is should be "ring-theoretically nice" as well. A famous example:

Theorem (Serre, Auslander-Buschsbaum-Nagata): Let A be a commutative noetherian local ring. If $\operatorname{gldim} A < \infty$, then A is a unique factorization domain.

This is common in commutative algebra, but much more difficult to realize in noncommutative algebra. This is why we still have open problems like:

Question: If R is a noncommutative local noetherian ring of finite global dimension, is R a domain?

Ring structure and homological properties

Famous graded analogue of the question:

Conjecture (Artin-Tate-Van den Bergh 1991): Every Artin-Schelter regular algebra is a domain.

Special cases are known, mostly by classification:

- Dimension 2: written down by Artin-Schelter, all noetherian domains.
- Dimension 2 with $\mathsf{GKdim}(A) = \infty$ (non-noetherian): classified by Zhang, all domains.
- Dimension 3: classified by ATV, all noetherian domains.
- Dimension 4: ATV showed that noetherian implies domain.

Remains open in general, even in the case where A is Koszul.

Fundamental problem:

How can we deduce good ring-theoretic properties from good homological properties?

Greatest hits: “homologically nice” $\Longrightarrow$ domain or prime

Here are some of the best noncommutative results of this type.

Say R is “nearly local” if $R/\operatorname{rad}(R)$ is simple artinian.

Brown-Hajarnavis-MacEachern ’82: R noetherian and nearly local, then $R/\sqrt{0}$ is a matrix ring over a local domain.

Levasseur ’92: Connected graded, noetherian, “Auslander regular” algebras are domains.

Stafford-Zhang ’94: Connected graded PI algebras of finite global dimension are domains.

Teo ’97: Nearly local fully bounded noetherian rings of finite global dimension are matrix rings over local domains.

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Very nice results! Yet each has a strong restriction.

Piecewise domains

The rings above enjoy a stronger property than being prime.

Def (Gordon-Small '72): A ring R is a piecewise domain with respect to orthogonal idempotents $1 = e_1 + \cdots + e_r$ if

$$x \in e_i R e_j,\ y \in e_j R e_\ell,\ \text{and } xy = 0 \implies x = 0 \text{ or } y = 0.$$

Remarks:

- prime rings need not be piecewise domains
- piecewise domains need not be prime: $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$
- $\mathbb{M}_n$(domain) is a prime piecewise domain.

Today's topic

This talk: Joint work with Dan Rogalski providing one method to deduce that a ring is prime or a (piecewise) domain using homological information.

Advantage: We don't need to impose strong ring-theoretic hypotheses. The only homological assumption is that we have a *Koszul algebra*.

Trade-off: The homological requirement is more complex. We need to understand the structure of an *Ext algebra* associated to the ring.

Graded algebras

We're interested in *locally finite*, $\mathbb{N}$-*graded* algebras: $A = \bigoplus_{n=0}^{\infty} A_n$ with each $\dim_k A_i < \infty$. In this case:

- The *graded* Jacobson radical is $J(A) = \operatorname{rad} A_0 \oplus A_{\geq 1}$
- Denote $S = A/J(A)$, a semisimple f.d. algebra.
- $\operatorname{gr.gldim}_l(A) = \operatorname{pdim}({}_A S) = \operatorname{pdim}(S_A) = \operatorname{gr.gldim}_r(A)$
- If S is separable, then $\operatorname{gldim}(A) = \operatorname{gr.gldim}(A)$.

Special case: A is connected if $A_0 = k$ (e.g., AS regular algebras).

Then $S = k$ is evidently separable, so all global dimensions are equal to length of a minimal resolution $P_\bullet \to k_A \to 0$.

The Ext algebra

If A is a connected graded algebra, then the Ext algebra

$$E(A) = \mathrm{Ext}_A^{\bullet}(k, k)$$

encodes important homological information about A:

- The minimal resolution $P_\bullet \to k \to 0$ computes $E(A) \cong \mathrm{Hom}_A(P_\bullet, k)$
- If $d = \mathrm{gldim}(A) < \infty$, then d is maximal s.t. $E(A)_d \neq 0$.

Most significantly for us:

Thm (Smith, Lu-Palmieri-Wu-Zhang): A connected graded algebra A is AS regular if and only if $E(A)$ is Frobenius.

Koszul algebras

A class algebras A for which $E(A)$ is easier to compute.

Def (Priddy, '70): A connected graded algebra A is Koszul if the minimal resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to k_A \to 0$$

has each P_i generated in degree i.

Fact: A Koszul algebra is always *quadratic:* can write $A \cong k\langle x_1, \ldots, x_d \rangle / (r_1, \ldots, r_n)$ for homogeneous r_i of degree 2.

Koszul duality

Viewing free algebras as tensor algebras $k\langle x_1, \ldots, x_d \rangle = T(V)$ ($\dim_k V = d$), quadratic means $A = T(V)/(L)$ for $L \subseteq V \otimes V$.

Def: The quadratic dual is $A^! = T(V^*)/(L^\perp)$ where $L^\perp \subseteq V^* \otimes V^* = (V \otimes V)^*$ vanishes on L.

This gives a simple method to compute the Ext algebra:

Theorem (Priddy): If A is Koszul, then $E(A) \cong A^!$.

Corollary: We also have $E(A^!) \cong (A^!)^! \cong A$.

Koszul examples

Polynomial algebras: $A = k[x_1, \ldots, x_d]$ is Koszul, whose dual is famously the exterior algebra on $V = k^d$:

$$E(A) \cong A^! \cong \Lambda(V)$$

Skew polynomials: $A = k_q[x, y]$ are Koszul with

$$A^! = k\langle \widehat{x}, \widehat{y} \rangle / (\widehat{x}^2, \widehat{y}^2, \widehat{y}\widehat{x} + q\widehat{x}\widehat{y}),$$

and similarly for $k_{\mathbf{q}}[x_1, \ldots, x_n]$.

An approach to proving primeness

Guo's approach

In the paper [special thanks to Jason Gaddis for sharing!]:

- Jin Yun Guo, *On the primeness of Artin-Schelter regular Koszul algebra*, 2005

it was claimed that Koszul AS regular algebras are prime.

Unfortunately, there is an issue with the proof. But the idea can be adapted to give interesting information.

Orbital algebras

Fix the following data:

- $\mathcal{C}$ = additive category.
- Object $M \in \mathcal{C}$.
- $F\colon \mathcal{C} \to \mathcal{C}$ an additive endofunctor.

The orbital algebra is a graded ring

$$O(F, M) = \bigoplus_{i=0}^{\infty} \mathrm{Hom}(F^i M, M)$$

with multiplication defined on $f \in \mathrm{Hom}(F^i M, M)$ and $g \in \mathrm{Hom}(F^j M, M)$ by composing with $F^i(g) : F^{i+j}M \to F^i M$,

$$f * g = f \circ F^i(g) \in \mathrm{Hom}(F^{i+j}M, M).$$

Orbital algebras—examples

Certain constructions are secret examples of orbital algebras.

(Note: In these cases, $\mathcal{C}$ is the *opposite* of a familiar category, as we need $\mathrm{Hom}(M, F^i M) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(F^i M, M)$...)

Skew polynomial rings: Fix a ring A and an automorphism σ:

- $\mathcal{C} = \{\text{f.g. projective } A\text{-modules}\}^{\mathrm{op}}$ with object $M = A$
- $F\colon \mathcal{C} \to \mathcal{C}$ is the twist $F(P) = {}^{\sigma}P$

Then

$$\begin{aligned} O(F, M) &= \bigoplus \mathrm{Hom}(A, F^i(A)) \\ &= \bigoplus \mathrm{Hom}(A, {}^{\sigma^i}A) \\ &\cong A[x; \sigma] \end{aligned}$$

Orbital algebras—examples

Twisted homogeneous coordinate rings: X = scheme with an automorphism σ and $\mathcal{L}$ an invertible sheaf

- $\mathcal{C} = \mathsf{Qcoh}(X)^{\mathrm{op}}$ with object $M = \mathcal{O}_X$
- $F\colon \mathcal{C} \to \mathcal{C}$ is pullback by σ composed with tensoring: $F(\mathcal{M}) = \sigma^*(\mathcal{L} \otimes \mathcal{M})$

Then

$$\begin{aligned} O(F, \mathcal{O}_X) &= \bigoplus \mathsf{Hom}(\mathcal{O}_X, F^i(\mathcal{O}_X)) \\ &\cong \bigoplus \Gamma(X, \mathcal{L}^{\sigma^i} \otimes \cdots \otimes \mathcal{L}^{\sigma} \otimes \mathcal{L}) \\ &= B(X, \sigma, \mathcal{L}) \end{aligned}$$

Orbital algebras—examples

Ext algebras: $\mathcal{A}$ = abelian category with enough injectives or projectives

- $\mathcal{C} = \mathcal{D}^{\pm}(\mathcal{A})^{\mathrm{op}}$ with any object $M \in \mathcal{A}$
- $F = \Sigma\colon \mathcal{C} \to \mathcal{C}$ is translation

Then

$$\begin{aligned} O(\Sigma, M) &= \bigoplus \mathrm{Hom}(M, \Sigma^i M) \\ &\cong \bigoplus \mathrm{Ext}^i_A(M, M) \\ &= \mathrm{Ext}^\bullet_A(M, M). \end{aligned}$$

A sufficient condition to be a domain

Theorem

Let $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{C}$ be as above. Suppose $M \in \mathcal{C}$ satisfies:

1. *Every nonzero $f\colon F^i(M) \to M$ is an epimorphism;*
2. *If $g\colon F^i(M) \to M$ is an epimorphism, then each $F^j(g)$ is also an epimorphism.*

Then $O(F, M)$ is a domain.

Proof: Let $0 \neq f, g \in O(F, M)$ be homogeneous with $d = \deg f$ and $e = \deg g$.

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$$f * g = f \circ F^d(g) \neq 0.$$

Special cases as classical results

Special cases of this theorem recover some well-known results:

Example: For $\mathcal{C} = \{\text{f.g. projective } A\text{-modules}\}^{\mathrm{op}}$ with object $M = A$ and $F = {}^{\sigma}(-)$, we get

$$A \text{ domain}, \ \sigma \text{ injective} \implies A[x;\sigma] \text{ is a domain.}$$

Example: For $\mathcal{C} = \mathbf{Qcoh}(X)^{\mathrm{op}}$ with object $M = \mathcal{O}_X$ and $F = \sigma^*(\mathcal{L} \otimes -)$, we get

$$X \text{ integral}, \ \sigma \text{ auto.} \implies B(X,\sigma,\mathcal{L}) \text{ is a domain.}$$

This isn't a "better" proof, but it has the potential to generalize in new directions...

A sufficient condition to be prime

Theorem (à la Guo)

Let $\mathcal{C}$ and $F\colon \mathcal{C} \to \mathcal{C}$ be as above. Suppose that M is an object of $\mathcal{C}$ with a decomposition $M = S_1 \oplus \cdots \oplus S_r$ in $\mathcal{C}$, and consider the conditions:

1. *every nonzero $f\colon F^d(S_j) \to S_\ell$ is an epimorphism;*
2. *if $g\colon F^e(S_j) \to S_\ell$ is an epimorphism, then so is $F^d(g)$ for any $d \geq 0$;*
3. *for $j \neq \ell$, there exists a nonzero morphism $h\colon F^i(S_j) \to S_\ell$ for some $i \geq 0$.*

If (1) and (2) hold, then $O(F, M)$ is a piecewise domain. If (3) also holds, then $O(F, M)$ is prime.

Putting the condition to work

Syzygy modules

Guo's idea: Take $F = \Omega$ to be the syzygy construction from a projective cover:

$$0 \to \Omega(M) \to P \to M \to 0.$$

Then $\mathrm{Ext}^i_\Lambda(M, N) \cong \mathrm{Hom}_\Lambda(\Omega^i(M), N)$.

Problem: This is not well-defined on morphisms of graded modules! Different lifts of f can produce different morphisms...

Example: Take the exterior algebra $\Lambda = \Lambda(V)$ for $V = kx \oplus ky$.

For $M = k \oplus k(-1)$, define $f : M \to M$ by $f(a, b) = (\lambda a, \mu b)$.

If we try to lift this to a commuting diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \longrightarrow & \Lambda \oplus \Lambda(-1) & \longrightarrow & k \oplus k(-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Omega(f)} & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & \Omega(M) & \longrightarrow & \Lambda \oplus \Lambda(-1) & \longrightarrow & k \oplus k(-1) & \longrightarrow & 0
\end{array}$$

there are many choices of $\widehat{f}$: for any $z \in \Lambda_1$ we can take

$$\widehat{f}_z(a, b) = (\lambda a, \mu b + za).$$

But then if $w \in \Lambda_1$ we get $(w, 0) \in \Omega M$ and $g(w, 0) = (\lambda w, zw)$ depends on $\widehat{f}_z$!

Syzygy *functor*

So when *can* we define a functor from Ω? It works well if we can guarantee that f lifts uniquely:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Omega(f)} & & \Big\downarrow{\scriptstyle \widehat{f}} & & \Big\downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \Omega(N) & \longrightarrow & P_N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Lemma: If $f : M \to N$ is a graded morphism with M and N both generated in degree 0, then f lifts uniquely to projective covers and $\Omega(f)$ is well-defined.

Syzygy *functor*

A module is Koszul if $P_\bullet \to M \to 0$ has P_i generated in degree i. Then the resolution $Q_\bullet = P_{\bullet+1}$ of $\Omega(M)$

$$\cdots \to P_2 \to P_1 \to \Omega(M) \to 0$$

has its ith term is generated in degree $i+1$.

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$$\cdots \to P_2(1) \to P_1(1) \to \Omega(M)(1) \to 0.$$

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Theorem

The assignment $F(M) = \Omega(M)(1)$ defines an endofunctor of the category $\mathcal{C} = \mathcal{K}(\Lambda)$ of Koszul modules.

Furthermore, if $0 \to L \to M \to N \to 0$ is an exact sequence of Koszul modules, then so is $0 \to F(L) \to F(M) \to F(N) \to 0$.

When the Ext algebra is prime

Theorem

Let $M \in \mathcal{K}(\Lambda)$ be Koszul and semisimple, with simple decomposition $M = M_1 \oplus \cdots \oplus M_r$. Consider the conditions:

1. *For every graded $f : F^i(M_j) \to M_\ell$ for $i \geq 0$ and j, ℓ arbitrary, $\ker f$ is Koszul.*
2. *For any j, ℓ, there exists $i = i(j, \ell)$ such that $\mathrm{Ext}^i_{\text{Gr-}\Lambda}(M_j(i), M_\ell) \neq 0$.*

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Then we have

$$(1) \iff \mathrm{Ext}^\bullet_{\text{Gr-}\Lambda}(M, M) \text{ is a piecewise domain,}$$
$$(1) \text{ and } (2) \implies \mathrm{Ext}^\bullet_{\text{Gr-}\Lambda}(M, M) \text{ is prime.}$$

A few applications

Koszul rings

We want to apply this to Koszul algebras, but the tool is general enough to do a little better.

Defintion (Beilinson-Ginzburg-Soergel '96): A graded ring R with $S = R_0$ semisimple is a Koszul ring if the minimal projective resolution $P_\bullet \to S_R \to 0$ has P_i generated in degree i.

Fact: As before, can write $R = T_S(V)/(L)$ for $V = R_1$ and $L \subseteq V \otimes_S V$. Can define left and right quadratic duals $R^! = T_S(V^*)/(L^\perp)$ and ${}^!R = T_S({}^*V)/({}^\perp L)$.

Generalized Koszul duality

Standing assumption: $V = R_1$ has both ${}_SV$ and V_S f.g.

Beilinson-Ginzburg-Soergel showed that if R as above is Koszul, then

$$\mathrm{Ext}^{\bullet}_{R\text{-}\mathbf{Gr}}(S,S) \cong R^!,$$
$$\mathrm{Ext}^{\bullet}_{\mathbf{Gr}\text{-}R}(S,S) \cong {}^!R.$$

Also, $R \cong ({}^!R)^! \cong {}^!(R^!)$, so that we can recover R as an Ext algebra itself: setting $\Lambda = R^!$, we get

$$R \cong {}^!\Lambda \cong \mathrm{Ext}_{\mathbf{Gr}\text{-}\Lambda}(S,S).$$

First key condition

Continue to denote $S = \Lambda_0$ and $F = \Omega(-)(1)$ as above.

Definition

Let Λ be a Koszul ring, and fix a simple decomposition

$$S = S_1 \oplus \cdots \oplus S_r.$$

Then Λ satisfies the Koszul syzygy condition if, for all $i \geq 0$ and all graded morphisms

$$f \colon F^i(S_j) \to S_\ell,$$

the module $\ker f$ is Koszul.

Second key condition

If R is graded with $S = R_0$ semisimple, write $S = S_1 \oplus \cdots \oplus S_r$. This corresponds to an idempotent decomposition

$$1 = e_1 + \cdots + e_r.$$

Each $D_i = e_i S e_i$ is a division ring.

Def: The underlying quiver of R has vertices $\{1, \ldots, r\}$ and

$$\#\{\text{arrows } i \to j\} = \dim_{D_i} e_j R_1 e_i.$$

Def: A quiver Q is strongly connected if there exists a path between any two vertices of Q.

When a Koszul ring is prime

Since a Koszul ring R can be viewed as $R \cong \mathrm{Ext}^\bullet_{\mathrm{Gr}\text{-}\Lambda}(S, S)$ for $\Lambda = R^! \cong \mathrm{Ext}^\bullet_R(S, S)$, we can specialize to the following.

Theorem

For R a Koszul ring, consider the conditions:

1. $\Lambda = R^!$ *satisfies the Koszul syzygy condition.*
2. *The underlying quiver of R is strongly connected.*

Then we have:

$$\begin{aligned} (1) &\iff R \text{ is a piecewise domain}, \\ (1) \text{ and } (2) &\iff R \text{ is a prime piecewise domain}. \end{aligned}$$

When a Koszul ring is a domain

What condition implies that a Koszul ring is a domain? We have a similar equivalent characterization.

Theorem

Suppose R is a Koszul ring with $S = R_0$. The following are equivalent:

1. *R is a domain;*
2. *S is a division ring and $\Lambda = R^!$ satisfies the Koszul syzygy condition.*

Some applications

Some situations where these results apply:

Theorem

Let A be a (connected gradded) Koszul algebra with $\Lambda = E(A) \cong A^!$. Then A is a domain if and only if Λ satisfies the Koszul syzygy condition.

Theorem

Let R be a local ring with $D = R/J(R)$, and set $\Lambda = \operatorname{Ext}^{\bullet}_{R}(D, D)$. If Λ is a Koszul ring and satisfies the Koszul syzygy condition, then R is a domain.

Non-connected application: twisted CY-2 algebras

There is a non-connected generalization of AS regular algebras. The *enveloping algebra* of A is $A^e = A \otimes A^e$. (A^e-modules = (A, A)-bimodules)

Def: An algebra A is twisted Calabi-Yau of dimension d if:

- There is a finite type projective resolution of A in **Mod**-A^e;
- $\operatorname{Ext}^i_{A^e}(A, A^e) = 0$ for $i \neq d$ and $\operatorname{Ext}^d_{A^e}(A, A^e) = U$ for an invertible bimodule U.

If we can take $U \cong A$, then A is Calabi-Yau.

Fact: (R.-Rogalski-Zhang '14) For connected graded algebras,

$$\text{AS regular} \iff \text{twisted CY.}$$

Structure of twisted CY-2 algebras

Some properties known about graded twisted CY-2 algebras:

Thm [R-Rogalski '22]: A graded twisted CY-2 algebra A is noetherian if and only if $\mathrm{GKdim}(A) < \infty$.

Thm [R-Rogalski '19]: Let $A = kQ/I$ be a graded twisted CY-2 algebra, where kQ has the ordinary grading.

1. A is an algebra with "mesh relations" ($I = (\omega)$ for $\omega = \sum_{a \in Q_1} \tau(a)a$).
2. The incidence matrix M of Q has spectral radius $\rho(M) \geq 2$.
3. $\mathrm{GKdim}(A) < \infty$ if and only if $\rho(M) = 2$, in which case $\mathrm{GKdim}(A) = 2$.

A non-connected application

We can now say in nice cases that such algebras are prime piecewise domains.

Theorem

Let $A = kQ/I$ be graded twisted Calabi-Yau algebra of dimension 2, and assume that A is a Koszul ring. Suppose each vertex of Q is the source of at least two arrows. Then A is a piecewise domain, and

$$A \text{ is prime} \iff Q \text{ is connected} \iff Q \text{ strongly connected.}$$

Note: Here $S = k^{Q_0}$, and the underlying quiver is just Q.

Elements of the proof

The main work is to verify the Koszul syzygy condition for $\Lambda = \mathrm{Ext}^\bullet_A(S,S)$. We also have $\Lambda \cong A^! \cong kQ^{\mathrm{op}}/(I_2^\perp)$.

Theorem [Li-Wu '23]: For Koszul $A = kQ/I$, A is twisted CY if and only if Λ is Frobenius.

Recall our favorite functor $F = \Omega(-)(1)$.

For Frobenius $\Lambda = S \oplus \Lambda_1 \oplus \Lambda_2$ with enough arrows at each vertex, we can roughly "classify" the F-orbits of kernels of maps $F^i(S) \to S$, which shows they are all Koszul.

Special case: preprojective algebras

For a quiver Γ, its *double* $\overline{\Gamma}$ has the same vertices, all arrows of Γ, and a new "double" α^* arrow corresponding to each arrow α (i.e., $\overline{\Gamma}_1 = \Gamma_1 \sqcup \Gamma_1^*$).

Def: The preprojective algebra of a quiver Γ is

$$\Pi(\Gamma) = k\overline{\Gamma}/(\sum_{\alpha \in \Gamma_1} \alpha\alpha^* - \alpha^*\alpha).$$

These are known to be Calabi-Yau algebras of dimension 2 whenever Γ is not Dynkin.

Thm (Baer-Geigle-Lenzing '87): For Γ connected representation-finite (Euclidean, or "extended Dynkin"), $\Pi(\Gamma)$ is prime and noetherian.

The preprojective algebra for "most" Γ won't be noetherian. But the twisted CY-2 result lets us extend this to many more.

Theorem

If Γ is connected and every vertex of its underlying graph has degree ≥ 2, then $\Pi(\Gamma)$ is prime.

Idea: Γ connected implies $Q = \overline{\Gamma}$ strongly connected

Remaining questions

In dimension 2, if we remove the restriction on the number of arrows at every vertex, the twisted CY algebra may not be a piecewise domain.

Ex: If Γ has type $\widetilde{D}$ or $\widetilde{E}$ then $\Pi(\Gamma)$ is not a piecewise domain, but is still prime by [BGL].

Conjecture: If Q is (strongly) connected, any twisted CY-2 algebras of the form $A = kQ/I$ is prime.

Remaining questions

Of course, the biggest question is whether this can help us resolve if Koszul AS regular algebras are domains!

Theorem: A Koszul algebra is a domain if and only if $E(A)$ satisfies the Koszul syzygy condition.

This translates the Koszul case of ATV's conjecture to:

all Koszul AS regular algebras are domains $\iff$ all Koszul Frobenius algebras satisfy the Koszul syzygy condition.

Thank you!