The adjunction map associated to a semisimple Hopf algebra action

Jiwei He

(Hangzhou Normal University)

Poisson Geometry and Artin-Schelter regular algebras
Oct. 13-18, 2024

Outline

(I) Morita context associated to a semisimple Hopf algebra action

(II) Cokernel of the adjunction map

(III) Recollements and injectivity of the adjunction map

(I) Morita context associated to a semisimple Hopf algebra action

Settings

- Let $\Bbbk$ be an algebraically closed field of characteristic zero.
  - Let H be a semisimple Hopf $\Bbbk$-algebra with coproduct Δ and counit ε.
  - Let $e \in H$ be the integral of H such that $\varepsilon(e) = 1$.
  - Let A be an $\Bbbk$-algebra, and H acts on A from the left.
  - Invariant subalgebra: $A^H = \{a \in A | h \cdot a = \varepsilon(h)a, \forall h \in H\}$.

Settings

- Let $\Bbbk$ be an algebraically closed field of characteristic zero.
  - Let H be a semisimple Hopf $\Bbbk$-algebra with coproduct Δ and counit ε.
  - Let $e \in H$ be the integral of H such that $\varepsilon(e) = 1$.
  - Let A be an $\Bbbk$-algebra, and H acts on A from the left.
  - Invariant subalgebra: $A^H = \{a \in A | h \cdot a = \varepsilon(h)a, \forall h \in H\}$.
- smash product
  $A\#H = A \otimes H$ with product:

$$(a\#h)(b\#g) = a(h_{(1)} \cdot b)\#h_{(2)}g,$$

where $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

Basic properties

- We view A and H as subalgebras of $A\#H$ through the natural inclusion maps. Then $e \in H$ is also an idempotent element of $A\#H$.

Basic properties

- We view A and H as subalgebras of $A\#H$ through the natural inclusion maps. Then $e \in H$ is also an idempotent element of $A\#H$.
- There is an algebra isomorphism $A^H \cong e(A\#H)e$.

Basic properties

- We view A and H as subalgebras of $A\#H$ through the natural inclusion maps. Then $e \in H$ is also an idempotent element of $A\#H$.
- There is an algebra isomorphism $A^H \cong e(A\#H)e$.
- A can be viewed as an $A\#H$-A^H-bimodule, and as an A^H-$A\#H$-bimodule.

Basic properties

- We view A and H as subalgebras of $A\#H$ through the natural inclusion maps. Then $e \in H$ is also an idempotent element of $A\#H$.
- There is an algebra isomorphism $A^H \cong e(A\#H)e$.
- A can be viewed as an $A\#H$-A^H-bimodule, and as an A^H-$A\#H$-bimodule.
- As a left (or right) $A\#H$-module, $A \cong (A\#H)e$ (resp. $A \cong e(A\#H)$).

Basic properties

- We view A and H as subalgebras of $A \# H$ through the natural inclusion maps. Then $e \in H$ is also an idempotent element of $A \# H$.
- There is an algebra isomorphism $A^H \cong e(A \# H)e$.
- A can be viewed as an $A \# H$-A^H-bimodule, and as an A^H-$A \# H$-bimodule.
- As a left (or right) $A \# H$-module, $A \cong (A \# H)e$ (resp. $A \cong e(A \# H)$).
- If A is noetherian, then so is A^H, and A is finitely generated both as a left and a right A^H-module.

Basic properties

- We view A and H as subalgebras of $A\#H$ through the natural inclusion maps. Then $e \in H$ is also an idempotent element of $A\#H$.
- There is an algebra isomorphism $A^H \cong e(A\#H)e$.
- A can be viewed as an $A\#H$-A^H-bimodule, and as an A^H-$A\#H$-bimodule.
- As a left (or right) $A\#H$-module, $A \cong (A\#H)e$ (resp. $A \cong e(A\#H)$).
- If A is noetherian, then so is A^H, and A is finitely generated both as a left and a right A^H-module.

※ **S. Montgomery**, Hopf algebras and their actions on rings, CBMS 82, Amer. Math. Soc., 1993.

Morita context

- An A^H-bimodule map

$$\alpha_{A,H} : A \otimes_{A\#H} A \longrightarrow A^H, \quad a \otimes_{A\#H} b \mapsto e(ab);$$

An $A\#H$-bimodule map

$$\beta_{A,H} : A \otimes_{A^H} A \longrightarrow A\#H, \quad a \otimes_{A^H} b \mapsto aeb.$$

- An A^H-bimodule map

$$\alpha_{A,H} : A \otimes_{A\#H} A \longrightarrow A^H, \quad a \otimes_{A\#H} b \mapsto e(ab);$$

An $A\#H$-bimodule map

$$\beta_{A,H} : A \otimes_{A^H} A \longrightarrow A\#H, \quad a \otimes_{A^H} b \mapsto aeb.$$

- **Remark.**
  - $({}_{A^H}A_{A\#H}, {}_{A\#H}A_{A^H}, \alpha_{A,H}, \beta_{A,H})$ forms a Morita context;
  - The A^H-bimodule map $\alpha_{A,H}$ is always surjective;
  - We call $\beta_{A,H}$ the adjunction map associated to the H-action on A.

Surjectivity of the adjunction map

- If $\beta_{A,H}$ is surjective, then A^H and $A\#H$ are Morita equivalent. In this case, the extension A/A^H is called a Hopf Galois extension.

Surjectivity of the adjunction map

- If $\beta_{A,H}$ is surjective, then A^H and $A\#H$ are Morita equivalent. In this case, the extension A/A^H is called a Hopf Galois extension.

Theorem

Let H be a semisimple Hopf algebra, and A a left H-module algebra. Then TFAE:

1. A/A^H is a Hopf Galois extension;
2. A_{A^H} is projective, and $A\#H \cong \mathrm{End}(A_{A^H})$.

⋇ **M. Cohen, D. Fishman, S. Montgomery**, Hopf Galois extensions, smash products, and Morita equivalence, J. Algebra 133 (1990), 351–372.

(II) Cokernel of the adjunction map

Cofinite case

- Recall $\beta_{A,H} : A \otimes_{A^H} A \longrightarrow A\#H, \quad a \otimes b \mapsto aeb.$

Cofinite case

- Recall $\beta_{A,H} : A \otimes_{A^H} A \longrightarrow A\#H, \quad a \otimes b \mapsto aeb.$
- Note that $\mathrm{Im}(\beta_{A,H})$ is an ideal of $A\#H$.

  Question 1. When is the algebra $A\#H/(\mathrm{Im}\beta_{A,H})$ finite dimensional?

Cofinite case

- Recall $\beta_{A,H} : A \otimes_{A^H} A \longrightarrow A\#H, \quad a \otimes b \mapsto aeb.$
- Note that $\mathrm{Im}(\beta_{A,H})$ is an ideal of $A\#H$.

  Question 1. When is the algebra $A\#H/(\mathrm{Im}\beta_{A,H})$ finite dimensional?

- **Notation:**
  Assume B is a noetherian algebra.

  mod B the category of finitely generated right B-modules

  tors B the category of finite dimensional right B-modules

  qmod B := mod B/ tors B.

- **Notation:**
  Let M_B be a B-module. An element $x \in M$ is a torsion element if xB is finite dimensional.

  Let $\tau(M) = \{x \in M | x \textit{ is torsion}\}$.

  Then we obtain a functor $\tau : \mathrm{Mod}\, B \longrightarrow \mathrm{Mod}\, B$.

  Define depth$(M) = \min\{i | R^i\tau(M) \neq 0\}$.

Cofinite case

Theorem

Let H be a semisimple Hopf algebra, and let A be a noetherian left H-module algebra. Then TFAE:

1. $A\#H/\mathrm{Im}\beta_{A,H}$ is finite dimensional;
2. there is an equivalence of abelian categories qmod $A\#H \cong$ qmod A^H.

Moreover, if the above equivalent conditions are satisfied and $\mathrm{depth} A_A \geq 2$, then we have an isomorphism

$$A\#H \cong \mathrm{End}(A_{A^H}).$$

⋇ **J.-W. He, F. Van Oystaeyen, Y. Zhang**, Hopf dense Galois extensions with applications, J. Algebra 476 (2017), 134–160.

Cofinite case: graded algebras

- **Notation.**
  Suppose B is a noetherian $\mathbb{Z}$-graded algebra.

  $\mathsf{gr}\, B$ the category of f.g. right graded B-modules

  $\mathsf{tors}\, B$ the subcategory of $\mathsf{gr}\, B$ consisting of f.d. graded modules.

  $\mathsf{qgr}\, B := \mathsf{gr}\, B / \mathsf{tors}\, B$.

Cofinite case: graded algebras

Theorem

Let A be an Artin-Schelter regular algebra of global dimension 2. Suppose a semisimple Hopf algebra H acts on A inner faithfully. Then

1. $A\#H/\mathrm{Im}\beta_{A,H}$ is finite dimensional;
2. $A\#H \cong \mathrm{End}(A_{A^H})$;
3. $\mathrm{qgr}\, A\#H \cong \mathrm{qgr}\, A^H$.

※ **K. Chan, E. Kirkman, C. Walton, J.J. Zhang**, McKay correspondence for semisimple Hopf actions on regular graded algebras, I, J. Algebra 508 (2018), 512–538.

Cofinite case: graded algebras

- Let $V_n = \Bbbk_{-1}[x_1, \ldots, x_n]$ be the skew polynomial algebra.

Theorem

Let G be one of the following permutation group, and A the corresponding skew polynomial algebra. Then $A \# \Bbbk G/(\mathrm{Im}\beta_{A,\Bbbk G})$ is finite dimensional.

1. $G = \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ and $A = V_4$;
2. $G = \langle (1\ 2)(3\ 4) \cdots (2n-1\ 2n) \rangle$ and $A = V_{2n}$;
3. $G = \langle (1\ 2\ \cdots\ 2^n) \rangle$ and $A = V_{2^n}$;

※ **J. Gaddis, E. Kirkman, W. Moore, W. Frank**, Auslander's theorem for permutation actions on noncommutative algebras, Proc. Amer. Math. Soc. 147 (2019), 1881–1896.

- **Notation.**
  Let B be a noetherian graded algebra of $\mathrm{GKdim} B = n$.

  $\mathsf{gr}\, B$ the category of f.g. right graded B-modules

  $\mathsf{gr}_k\, B$ the subcategory of $\mathsf{gr}\, B$ consisting of graded modules of $\mathrm{GKdim} \le k$.

  $\mathsf{qgr}_k\, B := \mathsf{gr}\, B / \mathsf{gr}_k\, B$.

Infinite dimensional case

Theorem

Let A be an Artin-Schelter regular and Cohen-Macaulay of GKdim$= n \geq 2$. A semisimple Hopf algebra H acts on A inner faithfully. TFAE

1. $\mathsf{GKdim}(A\#H/\mathrm{Im}\beta_{A,H}) = d \leq n-2$;
2. $A\#H \cong \mathsf{End}(A_{A^H})$.

Moreover, in this case, for any $k \leq n-d$,

$$\mathsf{qgr}_{n-k}\, A\#H \cong \mathsf{qgr}_{n-k}\, A^H.$$

※ **Y. Bao, J.-W. He, J.J. Zhang**, Pertinency of Hopf actions and quotient categories of Cohen - Macaulay algebras, J. Noncommut. Geom. 13 (2019), 667–710.

(III) Injectivity of the adjunction map

- **Question 2.** When is the adjunction map $\beta_{A,H}$ injective?

Restriction to finite group action

- From now on, G is a finite group

  A is a quantum polynomial algebra, i.e.,

  A is an Artin-Schelter regular algebra, and

  Hilbert series $H_A(t) = \sum_{i \geq 0} t^i \dim(A_i) = \frac{1}{(1-t)^n}$ for some n.

- From now on, G is a finite group

  A is a quantum polynomial algebra, i.e.,

  A is an Artin-Schelter regular algebra, and

  Hilbert series $H_A(t) = \sum_{i \geq 0} t^i \dim(A_i) = \dfrac{1}{(1-t)^n}$ for some n.

- Let $g \in Aut_{gr}(A)$ be an automorphism.

Definition

The trace of g is defined to be

$$Tr_A(g, t) := \sum_{n \geq 0} tr(g|_{A_n}) t^n,$$

where $tr(g|_{A_n})$ is the usual trace function.

※ **P. Jørgensen, J.J. Zhang**, Gourmet's guide to Gorensteinness, Adv. Math. 151 (2000), 313–345.

Quasi-reflections

Definition

Let A be a quantum polynomial algebra of global dimension n. An automorphism $g \in Aut_{gr}(A)$ is called a quasi-reflection if

$$Tr_A(g,t) = \frac{1}{(1-t)^{n-1}(1-\lambda t)}, \quad \lambda \neq 1.$$

※ **E. Kirkman, J. Kuzmanovich, J.J. Zhang**, Rigidity of graded regular algebras, T. AMS 360 (2008), 6331-6369.

Restriction to finite group action

Theorem

Let A be a quantum polynomial algebra of global dimension n. Assume g is a quasi-reflection of A. If g is of finite order, then g is in one of the following cases:

1. There is a basis $\{x_1, \ldots, x_n\}$ of A_1 such that
$$g(x_1) = \lambda x_1, g(x_j) = x_j \text{ for } j \geq 2.$$
2. The order of g is 4, and there is a basis $\{x_1, \ldots, x_n\}$ of A_1 such that
$g(x_1) = ix_1, g(x_2) = -ix_2, g(x_j) = x_j$ for $j \geq 3$.

※ **E. Kirkman, J. Kuzmanovich, J.J. Zhang**, Rigidity of graded regular algebras, T. AMS 360 (2008), 6331-6369.

Injectivity of adjunction maps

- Let A be a quantum polynomial algebra, and let g be a quasi-reflection of A. Let $G = \langle g \rangle$.

Proposition

The adjunction map $\beta_{A,\Bbbk G}$ is injective.

- Let A be a quantum polynomial algebra, and let g be a quasi-reflection of A. Let $G = \langle g \rangle$.

Proposition

The adjunction map $\beta_{A,\Bbbk G}$ is injective.

Theorem

Let A be a quantum polynomial algebra, and let $G \subseteq \mathrm{Aut}_{gr}(A)$ be a finite abelian subgroup. If G is generated by quasi-reflections, then the adjunction map $\beta_{A,\Bbbk G}$ is injective.

⁕ **J.-W. He**, The injectivity of the adjunction map associated to a finite group action, in preparation.

- Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be triangulated categories. A recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is a diagram of triangulated categories and triangle functors:

$$\mathcal{T}' \underset{\overleftarrow{\quad i^! \quad}}{\overset{\overleftarrow{\quad i^* \quad}}{\xrightarrow{\quad i_*=i_! \quad}}} \mathcal{T} \underset{\overleftarrow{\quad j_* \quad}}{\overset{\overleftarrow{\quad j_! \quad}}{\xrightarrow{\quad j^*=j^! \quad}}} \mathcal{T}'',$$

  such that

  (i) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^* = j^!, j_!)$ are adjoint triple, i.e, i^* is left adjoint to i_*, and i_* is left adjoint to $i^!$ etc.;

  (ii) $i^! j_* = 0$;

  (iii) $i_*, j_*, j_!$ are full embeddings;

  (iv) any object T in $\mathcal{T}$ admits triangles

$$i_! i^! T \longrightarrow T \longrightarrow j_* j^* T \longrightarrow i_! i^! T[1], \quad j_! j^! T \longrightarrow T \longrightarrow i_* i^* T \longrightarrow j_! j^! T[1].$$

⋇ **A.A. Beilinson, J. Berstein, P. Deligne**, Faisceaux pervers, in: Analyse et topologie sur les espaces singuliers, Astérisque **100** (1982), 1–172.

Recollements obtained from an idempotent

Theorem

Let B be an algebra, and let $e \in B$ be an idempotent. Then there is a differential graded algebra R and a recollement of derived categories

$$\mathcal{D}(R) \underset{i^!}{\overset{i^*}{\leftleftarrows}} \xrightarrow{i_*=i_!} \mathcal{D}(B) \underset{j_*}{\overset{j_!}{\leftleftarrows}} \xrightarrow{j^*=j^!} \mathcal{D}(eBe),$$

such that

1. R is negative, i.e., $R^i = 0$ for $i \geq 1$;
2. $H^0(R) \cong B/BeB$.

※ **M. Kalck, D. Yang**, Relative singularity categories I: Auslander resolutions, Adv. Math. 301 (2016), 973–1021.

Infinite dimensional case

- **Question 3.** When do we have $R \cong B/BeB$?

Stratifying action

Definition

Let A be a noetherian connected graded algebra, and let G be a finite subgroup acting on A homogeneously. We say that the G-action is stratifying if the following conditions are satisfied:

1. $\mathrm{Tor}_n^{A^G}(A, A) = 0$ for all $n \geq 1$,
2. the adjunction map $\beta_{A,G}$ is injective.

Stratifying action

Definition

Let A be a noetherian connected graded algebra, and let G be a finite subgroup acting on A homogeneously. We say that the G-action is stratifying if the following conditions are satisfied:

1. $\mathrm{Tor}_n^{A^G}(A, A) = 0$ for all $n \geq 1$,
2. the adjunction map $\beta_{A,G}$ is injective.

Proposition

Let A be a quantum polynomial algebra. Let $G \subseteq \mathrm{Aut}_{gr}(A)$ be a finite abelian subgroup. If G is generated by quasi-reflections, then the G-action is stratifying.

Recollement

- Let $e = \frac{1}{|G|}\sum_{g\in G} g \in \Bbbk G$.

Theorem

Let A be a quantum polynomial algebra, and let $G \subseteq \mathrm{Aut}_{gr}(A)$ be a finite abelian subgroup. Set $B = A\#\Bbbk G$ and $C = B/AeA$. Then the following are equivalent.

(i) *G is generated by quasi-reflections.*

(ii) *There is a recollement*

$$D^b(\mathrm{Gr}\,C)\ \begin{matrix} \xleftarrow{\quad i^* \quad} \\ \xrightarrow{\quad i_*=i_! \quad} \\ \xleftarrow{\quad i^! \quad} \end{matrix}\ D^b(\mathrm{Gr}\,B)\ \begin{matrix} \xleftarrow{\quad j_! \quad} \\ \xrightarrow{\quad j^*=j^! \quad} \\ \xleftarrow{\quad j_* \quad} \end{matrix}\ D^b(\mathrm{Gr}\,A^G).$$

※ **J.-W. He**, The injectivity of the adjunction map associated to a finite group action, in preparation.

Remark

- **Remark.** Shephard-Todd-Chevalley Theorem

  Let A be a quantum polynomial algebra, and let G be an **abelian** finite subgroup of $\mathsf{Aut}_{gr}(A)$. Then A^G is regular if and only if G is generated by quasi-reflections.

※ **Kirkman, J. Kuzmanovich, J.J. Zhang**, Shephard-Todd-Chevalley Theorem for skew polynomial rings, Algebra Represent. Theory 13 (2010), 127–158.

Example

- Let $A = \Bbbk\langle x, y\rangle/(x^2 - y^2)$, and let $G = \langle g\rangle$, where
$$g(x) = \sqrt{-1}x, g(y) = -\sqrt{-1}y.$$
Then
$$A^G = \Bbbk[xy, yx].$$

Example

- Let $A = \Bbbk\langle x, y\rangle/(x^2 - y^2)$, and let $G = \langle g\rangle$, where

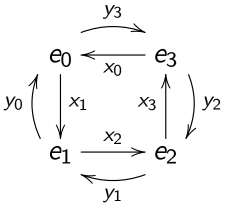
$$g(x) = \sqrt{-1}x, g(y) = -\sqrt{-1}y.$$

  Then

$$A^G = \Bbbk[xy, yx].$$

- The McKay quiver of $B := A\#\Bbbk G$ is the following:

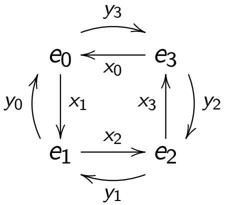
Γ :



Example

- Let $A = \Bbbk\langle x, y\rangle/(x^2 - y^2)$, and let $G = \langle g\rangle$, where
$$g(x) = \sqrt{-1}x, g(y) = -\sqrt{-1}y.$$
Then
$$A^G = \Bbbk[xy, yx].$$
- The McKay quiver of $B := A\#\Bbbk G$ is the following:

Γ :

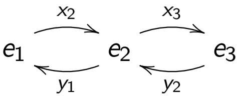


- B is the path algebra of Γ subject to the relations:
$$x_j x_{j+1} - y_{j+2} y_{j+1}, \text{ for all } 0 \le j \le 3.$$

Eample

- $C = B/BeB$ is the following algebra:

Q :



$C \cong \Bbbk Q/(x_2x_3, y_2y_1)$.

Thank you for your attention!