Non-formal deformation quantization and 3-associativity

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Overview

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Quantization may require partial algebras.

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- What about non-formal deformation quantization?

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Various approaches to non-formal $\star_\hbar$-products:

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Our approach: use **semi-classical analysis** (closely related to the *symplectic micro-category* of A. Cattaneo, B. Dehrin and A. Weinstein)

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(iii) $\exists W \subset T^*M$ such that if $\mathrm{WF}_{\mathfrak{sc}}(f_i) \Subset W$: $\quad f_1 \star_\hbar (f_2 \star_\hbar f_3) \underset{\mu}{=} (f_1 \star_\hbar f_2) \star_\hbar f_3$.

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(iv) $\exists$ linear maps $L, R : C^\infty_c(M) \to \Psi^0_{\mathfrak{sc}}(1, M)$: $\quad \begin{cases} f \star_\hbar - \underset{\mu}{=} L_f \\ - \star_\hbar f \underset{\mu}{=} R_f \end{cases}$

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It is called of SCFI-type if

$$f_1 \star_\hbar f_2(x) = \int_M K_\hbar(x, x_1, x_2) f(x_1) f(x_2) \mathrm{d}x_1 \, \mathrm{d}x_2,$$

with $K_\hbar$ a semi-classical Fourier integral distribution.

Main results

Theorem 1

If $\star_\hbar$ is non-formal of SCFI-type quantizing $(M, \{\cdot, \cdot\})$ then its underlying Lagrangian $\Lambda \subset T^*M \times T^*M \times \overline{T^*M}$ is the graph of multiplication of a local symplectic groupoid integrating $(M, \{\cdot, \cdot\})$.

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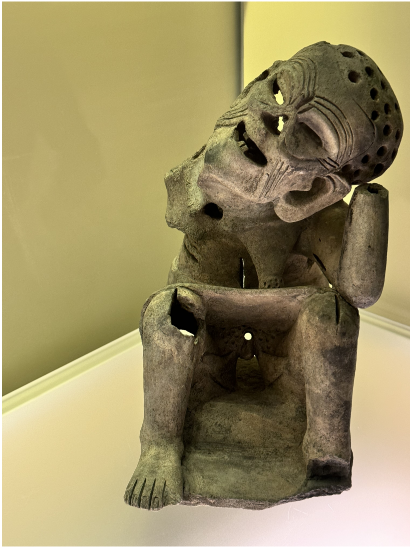
Theorem 3

If $(\mathcal{A}_W, \star_\hbar)$ is a global associative partial algebra, then $(M, \{\cdot,\cdot\})$ integrates to a symplectic groupoid.



¡Thank you!