Polynomial integrable systems and cluster structures

Jiang-Hua Lu The University of Hong Kong (Joint work with Yanpeng Li and Yu Li)

Zhejiang University, IASM, October 15, 2024

- §1: Introduction to polynomial integrable systems;
- §2: A general construction of polynomial integrable systems;
- §3: Polynomial integrable systems from cluster structures;
- §4: Three examples.

Assumptions:

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Lemma

If ϕ_1, \ldots, ϕ_m are holomorphic such that $d\phi_1 \wedge \cdots \wedge d\phi_m \neq 0$ and

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then

$$m \le n - \frac{1}{2} \operatorname{rk}(\pi) = \underbrace{(n - \operatorname{rk}(\pi))}_{\# \text{ of Casimirs}} + \frac{1}{2} \operatorname{rk}(\pi)$$

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An integrable system on (P, π) is a set (ϕ_1, \ldots, ϕ_m) of holomorphic functions on P such that

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$$\{\phi_i, \phi_j\}_{\pi} = 0$$
, for all $i, j = 1, ..., n$.

• g: (finite dimensional) Lie algebra over field **k** with $char(\mathbf{k}) = 0$;

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Example: Use the isomorphism

$$\mathfrak{gl}_n \simeq (\mathfrak{gl}_n)^*, \ x \mapsto \left(y \mapsto \operatorname{tr}(xy) \right)$$

to regard π_0 on $(\mathfrak{gl}_n)^*$ as on \mathfrak{gl}_n with entry coordinates $\{x_{ij}\}$.

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$$\{x_{ij}, x_{kl}\}_{\pi_0} = \delta_{jk} x_{il} - \delta_{li} x_{kj}, \quad i, j, k, l = 1, \dots, n.$$

Mishchenko-Fomenko Conjecture (1981), proved by S. Sadetov (2004):

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• How to construct concrete polynomial integrable systems on (\mathfrak{g}^*, π_0) ?

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<u>Our work</u>: A new method for a class of (\mathfrak{g}^*, π_0) .

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An example

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n := {strictly upper triangular matrices of size n × n}
n_ := {strictly lower triangular matrices of size n × n}
n ≃ (n_)*, x ↦ (y ↦ tr(xy));
For n = 4:

$$x = \begin{pmatrix} 0 & \overline{x_{12}} & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_1 = x_{12}, \quad f_2 = x_{13}, \quad f_3 = x_{14}, \quad f_4 = x_{13}x_{24} - x_{14}x_{23}$$

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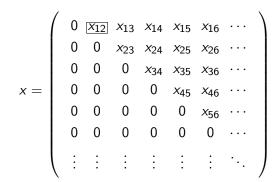
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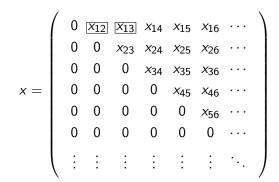
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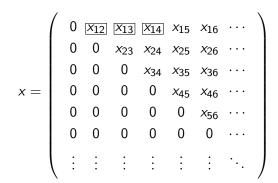
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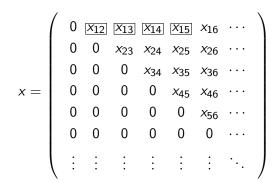
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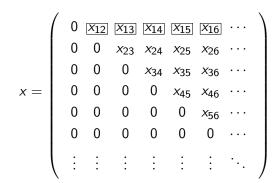
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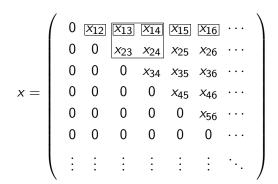
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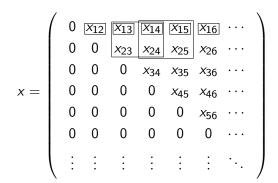
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The minors indicated above form an integrable system on (n, π_0) .

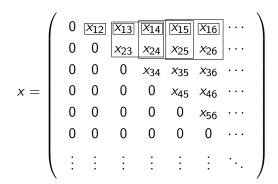
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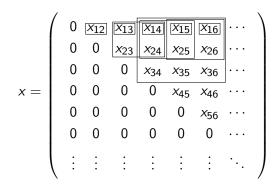


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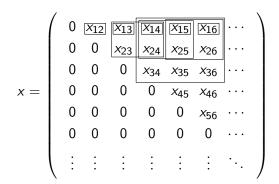
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<u>Goal</u>: Construct integrable systems on $(T_{p_0}P, \pi_0)$ using (P, π) .

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• Choose local holomorphic coordinates $z = (z_1, \ldots, z_n)$ s.t. $z(p_0) = 0$.

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- Define

 $\phi^{\text{low}} = \text{lowest order term of } \phi$ in the Tyalor expansion of ϕ at 0.

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Lemma

 ϕ^{low} is independent of the choice of local coordinates (z_1, \ldots, z_n) .

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Lemma

If ϕ and ψ are local holomorphic functions near \mathbf{p}_0 such that

 $\{\phi,\psi\}_\pi\in\mathbb{C}\phi\psi,$

then $\{\phi^{\text{low}}, \psi^{\text{low}}\}_{\pi_0} = 0.$

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Proof: Compare lowest order terms of expansions of both sides of

$$\{\phi,\psi\}_{\pi} = \lambda\phi\psi.$$

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 - **1** Φ is π -log-canonical;
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We have proved

Lemma

If
$$(\phi_1, \ldots, \phi_n)$$
 is π -log-canonical, then

$$\{\phi_i^{\mathrm{low}},\phi_j^{\mathrm{low}}\}_{\pi_0}=0, \quad \forall i,j\in 1,\ldots,n.$$

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<u>Recall</u>: To form a polynomial integrable system on $(T_{p_0}P, \pi_0)$, need

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algebraically independent Poisson commuting polynomials on $T_{p_0}P$.

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A natural question:

• Given log-canonical system $\Phi = (\phi_1, \ldots, \phi_n)$ on (P, π) at p_0 , when does

$$\Phi^{\text{low}} \stackrel{\text{def}}{=} (\phi_1^{\text{low}}, \ldots, \phi_n^{\text{low}})$$

contain $n - \frac{1}{2}rk(\pi_0)$ algebraically independent elements?

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Preparation:

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• use Taylor expansion in local coordinates $z = (z_1, \ldots, z_n)$ to define

$$\mu^{\text{low}} = \sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k} dz_{i_1} \wedge \cdots \wedge dz_{i_k}$$

with all $a_{i_1,...,i_k}$ homogeneous polynomials in z of degree m.

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- μ^{low} is a polynomial *k*-form on $T_{\rho_0}P$;
- μ^{low} is well-defined, i.e., independent of choice of (z_1, \ldots, z_n) .

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• introduce meromorphic *n*-form on *P*

$$\mu_{\Phi} = \frac{d\phi_1 \wedge \cdots \wedge d\phi_n}{\phi_1 \cdots \phi_n} = d(\log \phi_1) \wedge \cdots \wedge d(\log \phi_n),$$

and call μ_{Φ} the log-volume form of Φ ;

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and call μ_{Φ} the log-volume form of Φ ;

• define rational *n*-form on $T_{p_0}P$

$$\mu_{\Phi}^{\text{low}} = \frac{(d\phi_1 \wedge \cdots \wedge d\phi_n)^{\text{low}}}{\phi_1^{\text{low}} \cdots \phi_n^{\text{low}}};$$

§2.2. Main theorem: a sufficient condition

<u>Definition</u>: Given log-canonical system $\Phi = (\phi_1, \dots, \phi_n)$ on (P, π) at p_0 ,

• introduce meromorphic *n*-form on *P*

$$\mu_{\Phi} = \frac{d\phi_1 \wedge \cdots \wedge d\phi_n}{\phi_1 \cdots \phi_n} = d(\log \phi_1) \wedge \cdots \wedge d(\log \phi_n),$$

and call μ_{Φ} the log-volume form of Φ ;

• define rational *n*-form on $T_{p_0}P$

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define

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Given log-canonical system Φ on (P,π) at $p_0,$ if

$$\begin{split} & \deg(\mu_{\Phi}^{\mathrm{low}}) \leq \frac{1}{2} \mathrm{rk}(\pi_0), \quad \textit{equivalently}, \quad \deg(\mu_{\Phi}^{\mathrm{low}}) = \frac{1}{2} \mathrm{rk}(\pi_0), \\ & \textit{then } \Phi^{\mathrm{low}} \textit{ contains a (polynomial) integrable system on } (T_{p_0}P, \pi_0). \end{split}$$

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Cluster theory is a source of log-canonical systems!

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Suppose that (P, π) carries a compatible cluster structure C. If

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There are typically infinitely many extended clusters in \mathcal{C} !

Let (P, π) be a smooth affine Poisson variety with $\pi(p_0) = 0$.

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Explain compatible cluster structures and the Wonderful Lemma;

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Rest of the talk:

- Explain compatible cluster structures and the Wonderful Lemma;
- Look at three examples from Lie theory.

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3. Definitions of cluster algebras/cluster structures:

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() Given a mutation equivalence class C of seeds in $\mathbb{C}(P)$ and

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Implication: all extended clusters in C consist of regular functions on P.

Finally! Let π be a Poisson structure on P.

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- Having a cluster structure on a given P is a miracle;
- Given a cluster structure on P, checking compatibility with π is an easy linear algebra check.
- Cluster structure are typically of infinite type, i.e., having infinitely many extended clusters.

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If \mathcal{C} is a cluster structure on P compatible with π , then

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Since $\phi_k \phi'_k = a_1 + a_2$, where a_1 and a_2 do not contain ϕ_k , we have
 $\mu_{\Phi'} = \pm \alpha \wedge d(\log \phi'_k) = \pm \alpha \wedge (d(-\log \phi_k) + d(\log(a_1 + a_2)))$
 $= \pm \alpha \wedge d(\log \phi_k) = \pm \mu_{\Phi}.$

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$$\mu_{\mathcal{C}} = \mu_{\Phi} = d(\log \phi_1) \wedge \cdots \wedge d(\log \phi_n).$$

• Assume C has Property \mathcal{I} at p_0 , i.e., $\deg(\mu_{\mathcal{C}}^{\text{low}}) = \frac{1}{2} \operatorname{rk}(\pi_0)$.

Recap: Let (P, π) be a smooth affine Poisson variety with $\pi(p_0) = 0$.

- Suppose that C is a cluster structure on P compatible with π;
- Use any extended cluster $\Phi = (\phi_1, \dots, \phi_n)$ in $\mathcal C$ to define

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- Assume C has Property \mathcal{I} at p_0 , i.e., $\operatorname{deg}(\mu_{\mathcal{C}}^{\operatorname{low}}) = \frac{1}{2}\operatorname{rk}(\pi_0)$.
- Then for every extended cluster $\Phi = (\phi_1, \dots, \phi_n)$, the set $\Phi^{\text{low}} = (\phi_1^{\text{low}}, \dots, \phi_n^{\text{low}})$

contains a polynomial integrable system on $(T_{p_0}P, \pi_0)$.

<u>Remark</u>: We may need to modify μ_C to $\tilde{\mu}_C$ by modifying the frozen variable part to ensure

$$\mathsf{deg}(\widetilde{\mu}^{\mathrm{low}}_{\mathcal{C}}) = rac{1}{2}\mathrm{rk}(\pi_0).$$

In this case we also say that C Property \mathcal{I} at p_0 .

- G: connected complex semi-simple Lie group with $Lie(G) = \mathfrak{g}$;
- (B, B_{-}) : pair of opposite Borel sub-groups of G;
- $N \subset B$ and $N_{-} \subset B_{-}$: unipotent radicals;
- $T = B \cap B_-$: maximal torus of G;
- Respective Lie algebras $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{b}_{-}, \mathfrak{n}, \mathfrak{n}_{-};$
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One then has

• the standard complex semi-simple Poisson Lie group $(G, \pi_{\rm st})$ with

$$\pi_{\rm st} = r_{\rm st}^L - r_{\rm st}^R,$$

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• (G, π_{st}) is the semi-classical limit of the quantum group $U_q \mathfrak{g}$.

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Example 1: Integrable systems on linearization of (G, π_{st}) at $e \in G$.

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Theorem (Li-Li-L. 2024)

The BFZ cluster structure on G has Property \mathcal{I} at $e \in G$.

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linearization (\mathfrak{g}, π_0) of (G, π_{st}) at $e \in G$.

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§4.1: Integrable systems on linearization of $(G, \pi_{\rm st})$

linearization (\mathfrak{g}, π_0) of (G, π_{st}) at $e \in G$.

• The linear Poisson structure is defined by the Lie algebra

$$\mathfrak{g}^* = \{(y_- + y_0, -y_0 + y_+) : y_- \in \mathfrak{n}_-, y_0 \in \mathfrak{t}, y_+ \in \mathfrak{n}\} \subset \mathfrak{b}_- \oplus \mathfrak{b}$$

with the non-degenerate bilinear pairing between $\mathfrak g$ and $\mathfrak g^*$ given by

$$\langle x, (y_1, y_2) \rangle = \langle x, y_1 - y_2 \rangle_{\text{Killingform}}.$$

 Examples of polynomial integrable systems on (g, π₀) consist of generalized minors.

Example 2: Integrable systems on linearization of $(BwB/B, \pi)$ at w.B.

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- π_{st} on G projects to a well-defined Poisson structure π on G/B;
- For every $w \in W$, the Schubert cell

$$C^w := BwB/B \subset G/B$$

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 Each C^w has a standard cluster structure compatible with π (Berenstein-Fomin-Zelevinsky, Geiss-Leclerc-Schroder, Goodearl-Yakimov, ...);

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Theorem (Li-Li-L. 2024)

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Linearization of (C^w, π) at $w B \in C^w$.

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Linearization of (C^w, π) at $w B \in C^w$.

For $w \in W$, let $N^w = N \cap \overline{w} N_- \overline{w}^{-1}$ and $N_-^w = N_- \cap \overline{w} N \overline{w}^{-1}$, and

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- Thus every extended cluster in the standard cluster structure on C^w gives a polynomial integrable system on (n^w, π₀).
- When $w = w_0$, we have $\mathfrak{n}^{w_0} = \mathfrak{n}$.
- Example in §1.2 is an integrable system on (\mathfrak{n}, π_0) for $\mathfrak{g} = \mathfrak{sl}_n$.

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Example 3: Integrable systems on linearization of (GL_n^*, π) at $\mathbb{1}_n$.

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Theorem (Li-Li-L. 2024)

The GGV generalized cluster structure on GL_n^* has Property \mathcal{I} at $\mathbb{1}_n$.

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Linearization $(\mathfrak{gl}_n^*, \pi_0)$ of (GL_n^*, π) at $\mathbb{1}_n$

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Identify

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$$\mathfrak{gl}_n \simeq (\mathfrak{gl}_n)^*, \ x \mapsto \left(y \mapsto \operatorname{tr}(xy) \right)$$

and regard π_0 as a linear Poisson structure on \mathfrak{gl}_n given by

$$\{x_{ij}, x_{kl}\}_{\pi_0} = \delta_{jk} x_{il} - \delta_{li} x_{kj}, \ \forall 1 \leq i, j, k, l \leq n.$$

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• Define $F : \mathfrak{gl}_n \longrightarrow \mathfrak{gl}_n$ by

$$x \longmapsto \begin{bmatrix} xe_1 & x^2e_1 & \cdots & x^ne_1 \end{bmatrix}.$$

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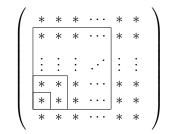
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 $x \longmapsto \begin{bmatrix} xe_1 & x^2e_1 & \cdots & x^ne_1 \end{bmatrix}$.

<u>**Remark</u>**: *F* does not preserve π_0 on \mathfrak{gl}_n .</u>

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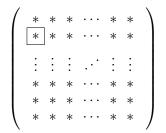
Consider the following minors of F(x):

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as well as the following Casimir functions:

$$c_j := \sum_{\{i_1, \cdots, i_j\} \in \binom{n}{j}} \Delta_{\{i_1, \cdots, i_j\}, \{i_1, \cdots, i_j\}}(x)$$

= sum of all principal $j \times j$ minors of $x, j \in [1, n]$.

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The functions defined above form integrable system on (\mathfrak{gl}_n, π_0) .

Remark: This integrable system is not the Gelfand-Zeitlin system.

For n = 3: dim $\mathfrak{gl}_3 = 9$,

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For n = 3: dim $\mathfrak{gl}_3 = 9$, magic number is 6.

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$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$F(x) = \begin{bmatrix} x_{11} & x_{11}x_{11} + x_{12}x_{21} + x_{13}x_{31} & * \\ x_{21} & x_{21}x_{11} + x_{22}x_{21} + x_{23}x_{31} & * \\ x_{31} & x_{31}x_{11} + x_{32}x_{21} + x_{33}x_{31} & * \end{bmatrix}$$

The minors of F(x) constructed above are

 $f_1 = x_{31}, \quad f_2 = x_{21}^2 x_{32} + x_{21} x_{31} x_{33} - x_{21} x_{22} x_{31} - x_{23} x_{31}^2, \quad f_3 = x_{21}.$

The Casimirs are $c_1 = tr(x)$, $c_3 = det(x)$, and

 $c_2 = x_{11}x_{22} + x_{11}x_{33} - x_{12}x_{21} - x_{13}x_{31} + x_{22}x_{33} - x_{23}x_{32}.$

Thank you!

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