Polynomial integrable systems and cluster structures

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(Joint work with Yanpeng Li and Yu Li)

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- §1: Introduction to polynomial integrable systems;
- §2: A general construction of polynomial integrable systems;
- §3: Polynomial integrable systems from cluster structures;
- §4: Three examples.

§1.1. Polynomial integrable systems

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If $\phi_1, \ldots, \phi_m$ are holomorphic such that $d\phi_1 \wedge \cdots \wedge d\phi_m \neq 0$ and

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then

$$m \leq n - \frac{1}{2}\mathrm{rk}(\pi) = \overbrace{(n - \mathrm{rk}(\pi))}^{\#\ \textit{of Casimirs}} + \frac{1}{2}\mathrm{rk}(\pi).$$

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Definition

An integrable system on (P, π) is a set $(\phi_1, \ldots, \phi_m)$ of holomorphic functions on P such that

1. $d\phi_1 \wedge \cdots \wedge d\phi_m \neq 0$;
2. $\{\phi_i, \phi_j\}_\pi = 0$, for all $i, j = 1, \ldots, n$.

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Example: Use the isomorphism

$$\mathfrak{gl}_n \simeq (\mathfrak{gl}_n)^*, \; x \mapsto \Big(y \mapsto \mathrm{tr}(xy)\Big)$$

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$$\{x_{ij}, x_{kl}\}_{\pi_0} = \delta_{jk} x_{il} - \delta_{li} x_{kj}, \qquad i, j, k, l = 1, \ldots, n.$$

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Mishchenko-Fomenko Conjecture (1981), proved by S. Sadetov (2004):

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Our work: A new method for a class of $(\mathfrak{g}^*, \pi_0)$.

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For $n = 4$:

$$x = \begin{pmatrix} 0 & \boxed{x_{12}} & x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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The minors indicated above form an integrable system on $(\mathfrak{n}, \pi_0)$.

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Goal: Construct integrable systems on $(T_{p_0}P, \pi_0)$ using (P, π).

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Lemma

ϕ^{low} *is independent of the choice of local coordinates* $(z_1, \ldots, z_n)$.

Lemma

If ϕ and ψ are local holomorphic functions near p_0 such that

$$\{\phi, \psi\}_\pi \in \mathbb{C}\phi\psi,$$

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Lemma

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Proof: Compare lowest order terms of expansions of both sides of

$$\{\phi, \psi\}_\pi = \lambda\phi\psi.$$

Q.E.D.

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We have proved

Lemma

If $(\phi_1, \ldots, \phi_n)$ is π-log-canonical, then

$$\{\phi_i^{\mathrm{low}}, \phi_j^{\mathrm{low}}\}_{\pi_0} = 0, \quad \forall\, i, j \in 1, \ldots, n.$$

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Recall: To form a polynomial integrable system on $(T_{p_0}P, \pi_0)$, need

$$m = n - \frac{1}{2}\mathrm{rk}(\pi_0) \quad \text{(the magic number)}$$

algebraically independent Poisson commuting polynomials on $T_{p_0}P$.

§2.1. A key observation

Recall: To form a polynomial integrable system on $(T_{p_0}P, \pi_0)$, need

$$m = n - \frac{1}{2}\mathrm{rk}(\pi_0) \quad \text{(the magic number)}$$

algebraically independent Poisson commuting polynomials on $T_{p_0}P$.

A natural question:

- Given log-canonical system $\Phi = (\phi_1, \ldots, \phi_n)$ on (P, π) at p_0, when does

$$\Phi^{\mathrm{low}} \stackrel{\mathrm{def}}{=} (\phi_1^{\mathrm{low}}, \ldots, \phi_n^{\mathrm{low}})$$

  contain $n - \frac{1}{2}\mathrm{rk}(\pi_0)$ algebraically independent elements?

§2.2. Main theorem: a sufficient condition

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with all $a_{i_1,\ldots,i_k}$ homogeneous polynomials in z of degree m.

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- μ^{low} is a polynomial k-form on $T_{p_0}P$;
- μ^{low} is well-defined, i.e., independent of choice of $(z_1, \ldots, z_n)$.

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$$\mu_\Phi = \frac{d\phi_1 \wedge \cdots \wedge d\phi_n}{\phi_1 \cdots \phi_n} = d(\log \phi_1) \wedge \cdots \wedge d(\log \phi_n),$$

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Theorem (Li-Li-L 2024)

Given log-canonical system Φ on (P, π) at p_0, if

$$\deg(\mu_{\Phi}^{\mathrm{low}}) \leq \frac{1}{2}\mathrm{rk}(\pi_0), \quad \textit{equivalently}, \quad \deg(\mu_{\Phi}^{\mathrm{low}}) = \frac{1}{2}\mathrm{rk}(\pi_0),$$

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- Property $\mathcal{I} \Longrightarrow \Phi^{\mathrm{low}}$ contains an integrable system on $(T_{p_0}P, \pi_0)$.

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contains a polynomial integrable system on $(T_{p_0}P, \pi_0)$.

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There are typically infinitely many extended clusters in $\mathcal{C}$!

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Rest of the talk:

- Explain compatible cluster structures and the Wonderful Lemma;
- Look at three examples from Lie theory.

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- Numerator in (2) contains no ϕ_k because $m_{k,k} = 0$.
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- Mutation equivalent seeds share the same set of frozen variables.

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Implication: all extended clusters in $\mathcal{C}$ consist of regular functions on P.

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Finally! Let π be a Poisson structure on P.

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- Having a cluster structure on a given P is a miracle;
- Given a cluster structure on P, checking compatibility with π is an easy linear algebra check.

§3.1: Compatible cluster structures on Poisson varieties

Finally! Let π be a Poisson structure on P.

Definition [Gekhtman-Shapiro-Vainshtein]: A cluster structure $\mathcal{C}$ on P is compatible with π if every extended cluster in $\mathcal{C}$ is π-log-canonical.

Remarks:

- Having a cluster structure on a given P is a miracle;
- Given a cluster structure on P, checking compatibility with π is an easy linear algebra check.
- Cluster structure are typically of infinite type, i.e., having infinitely many extended clusters.

§3.2: Proof of Wonderful Lemma

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Since $\phi_k \phi'_k = a_1 + a_2$, where a_1 and a_2 do not contain ϕ_k, we have

$$\begin{aligned} \mu_{\Phi'} &= \pm\alpha \wedge d(\log \phi'_k) = \pm\alpha \wedge (d(-\log \phi_k) + d(\log(a_1 + a_2)) \\ &= \pm\alpha \wedge d(\log \phi_k) = \pm\mu_{\Phi}. \end{aligned}$$

Q.E.D.

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$$\Phi^{\mathrm{low}} = (\phi_1^{\mathrm{low}}, \ldots, \phi_n^{\mathrm{low}})$$

contains a polynomial integrable system on $(T_{p_0}P, \pi_0)$.

§3.2: Proof of Wonderful Lemma

Remark: We may need to modify $\mu_{\mathcal{C}}$ to $\widetilde{\mu}_{\mathcal{C}}$ by modifying the frozen variable part to ensure

$$\deg(\widetilde{\mu}_{\mathcal{C}}^{\mathrm{low}}) = \frac{1}{2}\mathrm{rk}(\pi_0).$$

In this case we also say that $\mathcal{C}$ Property $\mathcal{I}$ at p_0.

§4: Three examples from Lie theory

Notation: Fix

- G: connected complex semi-simple Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$;
- (B, B_-): pair of opposite Borel sub-groups of G;
- $N \subset B$ and $N_- \subset B_-$: unipotent radicals;
- $T = B \cap B_-$: maximal torus of G;
- Respective Lie algebras $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{b}_-, \mathfrak{n}, \mathfrak{n}_-$;
- $W = N_G(T)/T$: the Weyl group; $w_0 \in W$ the longest element.

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- the standard complex semi-simple Poisson Lie group (G, π_{st}) with
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where $r_{\mathrm{st}} \in \mathfrak{g} \otimes \mathfrak{g}$ is the standard classical quasi-triangular r-matrix;
- (G, π_{st}) is the semi-classical limit of the quantum group $U_q\mathfrak{g}$.

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Theorem (Li-Li-L. 2024)

The BFZ cluster structure on G has Property $\mathcal{I}$ at $e \in G$.

linearization $(\mathfrak{g}, \pi_0)$ of $(G, \pi_{\rm st})$ at $e \in G$.

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- The linear Poisson structure is defined by the Lie algebra

$$\mathfrak{g}^* = \{(y_- + y_0,\ -y_0 + y_+) : y_- \in \mathfrak{n}_-,\ y_0 \in \mathfrak{t},\ y_+ \in \mathfrak{n}\} \subset \mathfrak{b}_- \oplus \mathfrak{b}$$

  with the non-degenerate bilinear pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by

$$\langle x,\ (y_1, y_2)\rangle = \langle x, y_1 - y_2\rangle_{\rm Killingform}.$$

- Examples of polynomial integrable systems on $(\mathfrak{g}, \pi_0)$ consist of generalized minors.

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$$C^w := BwB/B \subset G/B$$

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- Example in §1.2 is an integrable system on $(\mathfrak{n}, \pi_0)$ for $\mathfrak{g} = \mathfrak{sl}_n$.

§4.3: Integrable systems on linearization of (GL_n^*, π)

Example 3: Integrable systems on linearization of (GL_n^*, π) at $\mathbb{1}_n$.

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- Identify

$$\mathfrak{gl}_n \simeq (\mathfrak{gl}_n)^*, \; x \mapsto \Big(y \mapsto \mathrm{tr}(xy)\Big)$$

and regard π_0 as a linear Poisson structure on $\mathfrak{gl}_n$ given by

$$\{x_{ij}, x_{kl}\}_{\pi_0} = \delta_{jk} x_{il} - \delta_{li} x_{kj}, \; \forall 1 \le i, j, k, l \le n.$$

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- Define $F : \mathfrak{gl}_n \longrightarrow \mathfrak{gl}_n$ by
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Remark: F does not preserve π_0 on $\mathfrak{gl}_n$.

§4.3: Integrable systems on linearization of (GL_n^*, π)

Consider the following minors of $F(x)$:

$$\begin{pmatrix} * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \end{pmatrix}$$

§4.3: Integrable systems on linearization of (GL_n^*, π)

$$\begin{pmatrix} * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \end{pmatrix}$$

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as well as the following Casimir functions:

$$
\begin{aligned}
c_j &:= \sum_{\{i_1,\cdots,i_j\}\in\binom{n}{j}} \Delta_{\{i_1,\cdots,i_j\},\{i_1,\cdots,i_j\}}(x) \\
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Fact:

The functions defined above form integrable system on $(\mathfrak{gl}_n, \pi_0)$.

as well as the following Casimir functions:

$$c_j := \sum_{\{i_1,\cdots,i_j\}\in\binom{n}{j}} \Delta_{\{i_1,\cdots,i_j\},\{i_1,\cdots,i_j\}}(x)$$
$$= \text{ sum of all principal } j\times j \text{ minors of } x, \quad j\in[1,n].$$

Fact:

The functions defined above form integrable system on $(\mathfrak{gl}_n, \pi_0)$.

Remark: This integrable system is not the Gelfand-Zeitlin system.

For $n = 3$: $\dim \mathfrak{gl}_3 = 9$,

§4.3: Integrable systems on linearization of (GL_n^*, π)

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$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$F(x) = \begin{bmatrix} x_{11} & x_{11}x_{11} + x_{12}x_{21} + x_{13}x_{31} & * \\ x_{21} & x_{21}x_{11} + x_{22}x_{21} + x_{23}x_{31} & * \\ x_{31} & x_{31}x_{11} + x_{32}x_{21} + x_{33}x_{31} & * \end{bmatrix}$$

The minors of $F(x)$ constructed above are

$$f_1 = x_{31}, \quad f_2 = x_{21}^2 x_{32} + x_{21}x_{31}x_{33} - x_{21}x_{22}x_{31} - x_{23}x_{31}^2, \quad f_3 = x_{21}.$$

The Casimirs are $c_1 = \mathrm{tr}(x)$, $c_3 = \det(x)$, and

$$c_2 = x_{11}x_{22} + x_{11}x_{33} - x_{12}x_{21} - x_{13}x_{31} + x_{22}x_{33} - x_{23}x_{32}.$$

Thank you!