# Skew Calabi-Yau algebras and Poisson algebras via filtered deformations

Q. -S. Wu (吴泉水)

(Joint with Ruipeng Zhu)
School of Mathematical Sciences, Fudan University

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# Outline

## Smooth varieties and Calabi-Yau algebras

Let $R = k[X]$, where $X$ is an affine smooth variety of dimension $d$. Then $R^e = R \otimes R = k[X \times X]$ is ($R$ is *homologically*) smooth, and

- $\mathrm{Tor}_n^{R^e}(R, R) = \mathrm{H}_n(R, R) \overset{HKR}{\cong} \wedge^n \Omega^1_{R|k} \cong \Omega^n_{R|k}$.
- $\mathrm{Ext}^n_{R^e}(R, R) = \mathrm{H}^n(R, R) \overset{HKR}{\cong} \wedge^n \mathrm{Der}_k(R) \cong (\Omega^n_{R|k})^*$.

## Smooth varieties and Calabi-Yau algebras

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- $\mathrm{Ext}^n_{R^e}(R,R) = \mathrm{H}^n(R,R) \overset{HKR}{\cong} \wedge^n \mathrm{Der}_k(R) \cong (\Omega^n_{R|k})^*$.

- $R$ is called to have **trivial canonical bundle** $\overset{\text{def}}{\Leftrightarrow} \Omega^d_{R|k} \cong R$
  $\overset{\text{def}}{\Leftrightarrow} X$ is a Calabi-Yau variety.
- In general, $\Omega^d_{R|k}$ is an invertible $R$-$R$-bimodule.
  ($\Rightarrow$ $R$ has "Van den Bergh duality" of dimension $d$.)

## A fact about smooth algebras

Suppose that $R$ is a smooth domain of dimension $d$. Then

- $\mathrm{Ext}^i_{R^e}(R, R^e) = \mathrm{Ext}^i_{R^e}(R, R) = 0$ for all $i < d$.
- $\mathrm{Ext}^d_{R^e}(R, R^e) \cong \mathrm{Ext}^d_{R^e}(R, R)$ as $R$-modules, the isomorphism is induced by the multiplication $m : R^e \to R$ ($R^e$-morphism).

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The proof follows from:

- $\ker(m : R \otimes R \to R)$ is a locally complete intersection, and
- Koszul complex for regular sequences.

**Definition (Van den Bergh duality)**

An algebra $A$ is said to have Van den Bergh duality of dim. $d$, if

- $A$ is **homologically smooth**, that is, ${}_{A^e}A$ has a finite resolution by finitely generated projective $A^e$-modules;
- $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq d$, and ${}_AU_A := \mathrm{Ext}^d_{A^e}(A, A^e)$ is an **invertible $A$-$A$-bimodule**.

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In this case, there is a **twisted Poincaré duality**, i.e., for any ${}_AN_A$,

- $\mathrm{H}^n(A, N) \cong \mathrm{H}_{d-n}(A, U \otimes_A N)$;
- $\mathrm{H}_n(A, N) \cong \mathrm{H}^{d-n}(A, U^{-1} \otimes_A N)$, $U^{-1}$ is the inverse of ${}_AU_A$.

## Definition [Gin]

A $k$-algebra $A$ is called **skew Calabi-Yau** of dimension $d$, if

(i) $A$ is **homologically smooth**;

(ii) $\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d \\ A^{\mu_A}, & i = d \end{cases}$ as $A^e$-modules, for some automorphism $\mu_A \in \mathrm{Aut}_k(A)$.

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Graded skew Calabi-Yau algebras are defined similarly in the category of graded bimodules.

**V. Ginzburg**, Calabi-Yau algebras, arXiv:math.AG/0612139.

This $\mu_A$ is unique up to an inner automorphism; it is called a **Nakayama automorphism** of $A$.

If $\mu_A$ is inner, then $A$ is called **Calabi-Yau**.

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**Nakayama automorphisms** are important and useful invariants for genuing **skew Calabi-Yau algebras**.

## Filtered algebras

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- $\operatorname{gr} A := \oplus_{n\geq 0} F_nA/F_{n-1}A$ is the **associated graded algebra**, with the multiplication given by

$$(a + F_{n-1}A)(b + F_{m-1}A) := ab + F_{n+m-1}A$$

for any $a \in F_nA, b \in F_mA$.

**Theorem**. Let $A$ be a positively filtered algebra.

- If $\operatorname{gr} A$ has Van den bergh duality, then so has $A$.
- If $\operatorname{gr} A$ is skew Calabi-Yau of dim $d$, then so is $A$.
  If $\mu_{\operatorname{gr} A}$ is a Nakayama automorphism of $\operatorname{gr} A$, then there is a Nakayama automorphism $\mu_A$ of $A$ such that $\mu_{\operatorname{gr} A} = \operatorname{gr} \mu_A$.

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$\operatorname{gr} A$ is CY "$\Rightarrow$" $\operatorname{gr} \mu_A = \mathrm{id}_{\operatorname{gr} A}$ for some $\mu_A$.

It may happen that $\operatorname{gr} A$ is Calabi-Yau, but $A$ is not Calabi-Yau.

**M. Van den Bergh**, A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. 126 (1998), 1345–1348.

**Q.-S. Wu, R.-P. Zhu**, Nakayama automorphisms and modular derivations in filtered quantizations, J. Algebra 572 (2021), 381–421.

**Example**. Both $A_n(\mathbb{C})$ and $\mathcal{U}(\mathfrak{g})$ are **filtered deformations** of polynomial algebras, which are Calabi-Yau.
(1) $A_n(\mathbb{C})$ is Calabi-Yau of dim $2n$.
(2) Let $\mathfrak{g}$ be an $n$-dim Lie algebra.

- $\mathcal{U}(\mathfrak{g})$ is skew Calabi-Yau, with a Nakayama automorphism $\mu$ such that $\mu(x) = x + \mathrm{tr}([x, -]|_{\mathfrak{g}})$ for all $x \in \mathfrak{g}$.
- $\mathcal{U}(\mathfrak{g})$ is Calabi-Yau $\Leftrightarrow$ $\mathrm{tr}(ad_{\mathfrak{g}}(x)) = 0$ for all $x \in \mathfrak{g}$.

**A. Yekutieli**, The rigid dualizing complex of a universal enveloping algebra, J. Pure Appl. Alg. 150 (2000), 85–93.

**Q.-S. Wu, C. Zhu**, PBW deformation of Koszul Calabi-Yau algebras, Algebra and Representation Theory 16 (2013), 405-420.

## Filtered deformation

If $\operatorname{gr} A$ is commutative, then $\operatorname{gr} A$ has a **Poisson algebra** structure:

$$\{\bar{a}, \bar{b}\} := ab - ba + F_{n+m-2}A \in F_{n+m-1}A/F_{n+m-2}A$$

for any $a \in F_nA, b \in F_mA$. In this case, $A$ is called a **filtered deformation** of $\operatorname{gr} A$. ($[F_nA, F_mA] \subseteq F_{m+n-1}A$)

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In fact, to get a **nontrivial Poisson structure** by taking maximal integer $\ell \geqslant 1$ such that $[F_nA, F_mA] \subseteq F_{m+n-\ell}A$ for all $m, n$, and

$$\{\bar{a}, \bar{b}\} := ab - ba + F_{n+m-\ell-1}A \in F_{n+m-\ell}A / F_{n+m-\ell-1}A$$

for any $a \in F_nA, b \in F_mA$.

**O. Gabber**, The integrability of the characteristic variety, Amer. J. Math. 103 (1981), 445–468.

## Hypothesis

$A$ is a filtered deformation with $\mathrm{gr}\, A$ is a commutative $d$-dim affine smooth algebra with a trivial canonical bundle $\Omega^d(\mathrm{gr}\, A) = (\mathrm{gr}\, A)\,\eta$.

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$$\mathrm{H}^d(\operatorname{gr} A, \operatorname{gr} A^e) \cong \mathrm{H}^d(\operatorname{gr} A, \operatorname{gr} A) \overset{HKR}{\cong} (\Omega^d(\operatorname{gr} A))^*$$

$\Rightarrow \operatorname{gr} A$ is d-dim Calabi-Yau

$\Rightarrow A$ is d-dim skew Calabi-Yau with $\mu_A$

$\operatorname{gr} A$ has a **modular derivation** $\phi_\eta$, which will be defined in a moment.

## Main purpose of this talk

Discuss the relation between

the **Nakayama automorphism** $\mu_A$ of $A$

$\Updownarrow$ (using homo. determinants)

the **modular derivation** $\phi_\eta$ of $\mathrm{gr}\, A$

**Q.-S. Wu, R.-P. Zhu**, Nakayama automorphisms and modular derivations in filtered quantizations, J. Algebra 572 (2021), 381–421.

**J. Luo, S.-Q. Wang and Q.-S. Wu**, Poincaré duality for smooth Poisson algebras and BV structure on Poisson cohomology, J. Algebra 649 (2024), 169–211.

Let $R$ be a smooth Poisson algebra of dimension $d$ with trivial canonical bundle $\Omega^d(R) = R\,\eta$, where $\eta$ is a volume form.

**Definition**. The **modular derivation** of $R$ with respect to $\eta$ is defined as the map $\phi_\eta : R \to R : f \mapsto \frac{L_{H_f}(\eta)}{\eta}$, where

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**Definition**. The **modular derivation** of $R$ with respect to $\eta$ is defined as the map $\phi_\eta : R \to R : \ f \mapsto \frac{L_{H_f}(\eta)}{\eta}$, where

- $H_f := \{f, -\} : R \to R$ is the Hamiltonian derivation associated to $f$
- $\iota_{H_f} : \Omega^d(R) \to \Omega^{d-1}(R), a_0\, \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_d \mapsto$
  $\sum_i (-1)^{i-1} a_0 \{f, a_i\}\, \mathrm{d}a_1 \wedge \cdots \widehat{\mathrm{d}a_i} \cdots \wedge \mathrm{d}a_d$
- The Lie-derivation $L_{H_f} = [\mathrm{d}, \iota_{H_f}]$ is of degree 0 on $\Omega^d(R)$.

**J. Luo, S.-Q. Wang and Q.-S. Wu**, Twisted Poincaré duality between Poisson homology and Poisson cohomology, J. Algebra 442 (2015), 484–505.

$\phi_\eta : R \to R$ is both a derivation and Poisson derivation.

**Example**. Let $R = k[x_1, x_2, \cdots, x_d]$ be a polynomial Poisson algebra with Poisson bracket $\{-,-\}$. Then $\Omega^1(R) = \oplus_{i=1}^{d} R\, dx_i$.

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**Example**. Let $R = k[x_1, x_2, \cdots, x_d]$ be a polynomial Poisson algebra with Poisson bracket $\{-,-\}$. Then $\Omega^1(R) = \oplus_{i=1}^{d} R\,\mathrm{d}x_i$.

$\Omega^d(R) = R\eta$ where $\eta = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d$ is a volume form. The **modular derivation** $\phi_\eta$ is given by

$$\phi_\eta(f) = \sum_{j=1}^{d} \frac{\partial\{f, x_j\}}{\partial x_j}, \ \forall f \in R.$$

**J. Luo, S.-Q. Wang and Q.-S. Wu**, Twisted Poincaré duality between Poisson homology and Poisson cohomology, J. Algebra 442 (2015), 484–505.

Let $\{\mathrm{d}x_i; (\mathrm{d}x_i)^*\}_{i=1,2,\cdots,r}$ be a **dual basis** of the finitely generated projective module $\Omega^1(R)$. In general, $r \geqslant d$.

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If $r = d$, the set $S$ has only one element $I = \{1 < 2 < \cdots < d\}$.

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To simplify the notation, for any $I = \{i_1 < i_2 < \cdots < i_d\} \in S$, let

$$\mathrm{d}x_I := \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_d} \text{ and } \mathrm{d}x_I^* := \mathrm{d}x_{i_1}^* \wedge \mathrm{d}x_{i_2}^* \wedge \cdots \wedge \mathrm{d}x_{i_d}^*.$$

Then $\{\mathrm{d}x_I, \mathrm{d}x_I^*\}_{I\in S}$ is a **dual basis** for the projective module $\Omega^d(R)$.

Since $(\eta, \eta^*)$ is a dual basis of $\Omega^d(R)$,

$$\mathrm{d}x_I = \eta^*(\mathrm{d}x_I)\,\eta = b_I\,\eta, \qquad b_I := \eta^*(\mathrm{d}x_I),$$
$$\mathrm{d}x_I^* = \mathrm{d}x_I^*(\eta)\,\eta^* = a_I\,\eta^*, \qquad a_I := \mathrm{d}x_I^*(\eta).$$

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Similarly, since $\{\mathrm{d}x_I, \mathrm{d}x_I^*\}_{I\in S}$ is a dual basis of $\Omega^d(R)$,

$$\eta = \sum_{I\in S} \mathrm{d}x_I^*(\eta)\,\mathrm{d}x_I = \sum_{I\in S} a_I\,\mathrm{d}x_I,$$
$$\eta^* = \sum_{I\in S} \eta^*(\mathrm{d}x_I)\,\mathrm{d}x_I^* = \sum_{I\in S} b_I\,\mathrm{d}x_I^*$$

**J. Luo, S.-Q. Wang and Q.-S. Wu**, Poincaré duality for smooth Poisson algebras and BV structure on Poisson cohomology, J. Algebra 649 (2024), 169–211.

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**Theorem**. For any $a \in R$,

$$\phi_\eta(a) = \sum_{1 \le i \le r} \mathrm{d}x_i^*(\{a, x_i\}) + \sum_{I \in S} \{a, a_I\} b_I,$$

where $a_I = \mathrm{d}x_I^*(\eta)$ and $b_I = \eta^*(\mathrm{d}x_I)$.

**J. Luo, S.-Q. Wang and Q.-S. Wu**, Poincaré duality for smooth Poisson algebras and BV structure on Poisson cohomology, J. Algebra 649 (2024), 169–211.

**Theorem**. Let $R$ be a Poisson algebra which is smooth with trivial canonical bundle $\Omega^d(R) = R\eta$.

- The **modular derivation** $\phi_\eta; R \to R$ is a Poisson derivation.
- For any Poisson $R$-module $M$, the $R$-module M with $\{x, a\}_{\phi_\eta} := \{x, a\} + \phi_\eta(a)x$ is a Poisson $R$-module, which is denoted by $M_{\phi_\eta}$ (the twisted Poisson module by $\phi_\eta$).
- There is a twisted Poincaré duality for Poisson (co)homology:

$$\mathrm{PH}^n(R, M) \cong \mathrm{PH}_{d-n}(R, M_{\phi_\eta}).$$

**J. Luo, S.-Q. Wang and Q.-S. Wu**, Poincaré duality for smooth Poisson algebras and BV structure on Poisson cohomology, J. Algebra 649 (2024), 169–211.

Suppose $\mathrm{Ext}^d(A, A^e) = A\pi \cong A^{\mu_A}$, such that, for any $a \in A$

$$\pi a = \mu_A(a)\pi.$$

$\pi$ is called a $\mu_A$-**twisted volume** of $A$.

$$\big(\mu_A(a) - a\big)\pi = \pi a - a\pi = [\pi, a] \ (\forall\, a \in A)$$

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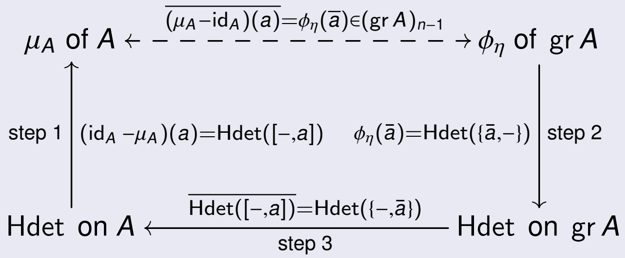
- $[-, a] : M \to M, x \mapsto xa - ax$ for any ${}_A M_A$.
- $\delta_a := [-, a] : A \to A$, which is a derivation on $A$.
- $H_{\bar{a}} = \{\bar{a}, -\} : \mathrm{gr}\, A \to \mathrm{gr}\, A$ is the Hamiltonian derivation.

## Main ideas

Using **homological determinant** as a bridge to give a connection between Nakayama auto. $\mu_A$ of $A$ and modular deri. $\phi_\eta$ of gr $A$.




**Q.-S. Wu, R.-P. Zhu**, Nakayama automorphisms and modular derivations in filtered quantizations, J. Algebra 572 (2021), 381-421.

Let $H$ be a Hopf algebra acting on an algebra $A$.

**$H \curvearrowright A \Rightarrow H \curvearrowright \mathrm{C}^*(A, A^e) \Rightarrow H \curvearrowright \mathrm{H}^*(A, A^e)$**

There is a left $H$-module structure on the Hochschild cochain

$$\cdots \longrightarrow C^{n-1}(A, A^e) \longrightarrow C^n(A, A^e) \longrightarrow C^{n+1}(A, A^e) \longrightarrow \cdots$$

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For any $f \in C^n(A, A^e)$, any $h \in H$ and $a_1, \ldots, a_n \in A$,

$$(h \rightharpoonup f)(a_1, \ldots, a_n) = \sum_{(h)} (S^2 h_{n+2} \otimes h_1) \cdot f(Sh_{n+1} \cdot a_1, \ldots, Sh_2 \cdot a_n)$$

$A^e$ is viewed as a left $H \otimes H$-module: $(g \otimes h)(a \otimes b) = (g \cdot a \otimes h \cdot b)$.

General action:

$$(h \rightharpoonup f)(a_1, \ldots, a_n) = \sum_{(h)} (S^2 h_{n+2} \otimes h_1) \cdot f(Sh_{n+1} \cdot a_1, \ldots, Sh_2 \cdot a_n)$$

Group action: $(g \rightharpoonup f)(a_1, \ldots, a_n) = (g \otimes g) \cdot f(g^{-1} \cdot a_1, \ldots, g^{-1} \cdot a_n)$

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Li (derivation) action:

$$(\delta \rightharpoonup f)(a_1, \ldots, a_n) = (\delta \otimes 1 + 1 \otimes \delta) \cdot f(a_1, \ldots, a_n) + \sum_{i=1}^{n} f(a_1, \ldots, -\delta(a_i), \ldots, a_n)$$

Suppose $\mu_A$-**twisted volume** of $A$. Then $\pi a = \mu_A(a)\pi$, $\forall\, a \in A$.

$$(\mathrm{Ext}^d(A, A^e) = A\pi \cong A^{\mu_A})$$

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$$(\mathrm{Ext}^d(A, A^e) = A\pi \cong A^{\mu_A})$$

For any $h \in H$, there is a unique $a \in A$ such that

$$h \rightharpoonup \pi = a\,\pi \in A\pi = \mathrm{Ext}^d(A, A^e).$$

Let $\phi : H \to A$ be the map such that

$$h \rightharpoonup \pi = \phi(h)\pi \ (\in A\pi).$$

# $\phi$ is convolution invertible

By the $H$-action on $\mathrm{Ext}^d(A, A^e)$, $\phi(gh)\pi = g \rightharpoonup (\phi(h)\pi)$, and

$$\phi(gh) = \sum_{(g)} \big(g_1 \rightharpoonup \phi(h)\big)\phi(g_2).$$

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In fact, $\phi \in \mathrm{Hom}_k(H, A)$ is convolution invertible, with the inverse

$$\phi^{-1}(h) = \sum h_2 \rightharpoonup \phi(S^{-1}h_1).$$

## Definition

The **homological determinant** $\mathrm{Hdet} = \mathrm{Hdet}_\pi$ of the Hopf action $H$ on $A$ is defined to be $\phi^{-1} \in \mathrm{Hom}_k(H, A)$, that is,

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If $A$ is connected graded skew CY, Hdet coincides with the definition by Jørgensen-Zhang; Kirkman-Kuzmanovich-Zhang. Hdet agrees with the definition by Meur (2019).

**P. Le Meur**, Patrick Smash products of Calabi-Yau algebras by Hopf algebras, J. Noncommut. Geom. 13 (2019), 887–961.

**Q.-S. Wu, R.-P. Zhu**, Nakayama automorphisms and modular derivations in filtered quantizations, J. Algebra 572 (2021), 381-421.

# Step 1: $(\mathrm{id}_A - \mu_A)(a) = \mathrm{Hdet}_\pi([-, a])$

Given ${}_A M_A$, consider the **commutator action** $a \in A$ on $M$

$$[-, a] : M \to M, x \mapsto xa - ax \quad ([-, a] \curvearrowright x = xa - ax)$$

$$\big(\mu_A(a) - a\big)\pi = \pi a - a\pi = [\pi, a] \ (\forall\, a \in A)$$

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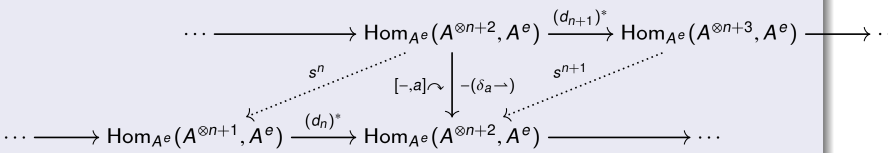
$$(\pi \in \mathrm{Ext}^d(A, A^e) = A\pi \cong A^{\mu_A})$$

Consider $\delta_a := [-, a] : A \to A$, which is a derivation on $A$.
Let $H = \mathcal{U}(k\delta_a) = k[\delta_a]$, which acts on $A$ by $\delta_a \rightharpoonup x = [x, a]$.

# Step 1: $(\mathrm{id}_A - \mu_A)(a) = \mathrm{Hdet}_\pi([-, a])$

## Key fact 1

$$[-, a] \curvearrowright \mathrm{H}^*(A, A^e) \ \text{“} = \text{”} \ \delta_a \rightharpoonup \mathrm{H}^*(A, A^e).$$




$$s^n(f)(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i f(x_0 \otimes \cdots \otimes x_i \otimes a \otimes x_{i+1} \otimes \cdots \otimes x_n)$$

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$$\begin{aligned}(\mu_A(a) - a)\pi &= [\pi, a] \overset{\text{Key fact 1}}{=} \delta_a \rightharpoonup \pi,\\ \mathrm{Hdet}_\pi(\delta_a)\pi &= \phi^{-1}(\delta_a)\pi = (1_H \rightharpoonup \phi(S^{-1}\delta_a) + \delta_a \rightharpoonup \phi(1_H))\pi\\ &= -(\delta_a \rightharpoonup \pi)\end{aligned}$$

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&= -(\delta_a \rightharpoonup \pi)
\end{aligned}$$

## Proposition 4.1

$(\mathrm{id}_A - \mu_A)(a) = \mathrm{Hdet}_\pi(\delta_a)$.

# Step 2: $\phi_\eta \longleftrightarrow \mathrm{Hdet}_{\eta^*}$

## Key fact 2

Let $R(:= \mathrm{gr}\, A)$ be a $d$-dim affine smooth commutative algebra.

$$\begin{array}{ccc} \mathrm{H}_{p+q}(R,R) \otimes \mathrm{H}^q(R,R) & \xrightarrow{- \cap -} & \mathrm{H}_p(R,R) \, . \\ HKR \Big\downarrow \wr & \circlearrowright & HKR \Big\downarrow \wr \\ \Omega^{p+q}(R) \otimes \Omega^q(R)^* & \xrightarrow{\iota_-(-)} & \Omega^p(R) \end{array}$$

where $\iota_-(-)$ is induced by the **contraction map**.

# Step 2: $\phi_\eta \longleftrightarrow \mathrm{Hdet}_{\eta^*}$

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where $\iota_{-}(-)$ is induced by the **contraction map**.

If $H$ is cocommutative and $H \curvearrowright R$, then the morphisms above are *H*-**morphisms**.

$\iota_-(-) : \Omega^{p+q}(R) \otimes \mathrm{Der}^q(R) \to \Omega^p(R)$ is the map $\omega \otimes F \mapsto \iota_F(\omega)$ induced by the **contraction map**, which is an $H$-**morphism**.

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If $\omega = a_0\, \mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_{p+q} \in \Omega^{p+q}(R)$,

$$\begin{aligned}&\iota_F(\omega)\\ =&\sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma) a_0 F(a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(p)})\, \mathrm{d}a_{\sigma(p+1)} \wedge \cdots \wedge \mathrm{d}a_{\sigma(p+q)}.\end{aligned}$$

The $H$-module structure of $\Omega^n(R)$ is given by

$$\begin{aligned}& h \rightharpoonup (a_0\, \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n)\\ :=& \sum (h_1 \rightharpoonup a_0)\, \mathrm{d}(h_2 \rightharpoonup a_1) \wedge \cdots \wedge \mathrm{d}(h_{n+1} \rightharpoonup a_n).\end{aligned}$$

The **cap product** $\cap : \mathrm{H}_{p+q}(A, M) \otimes \mathrm{H}^q(A, N) \longrightarrow \mathrm{H}_p(A, M \otimes_A N)$ is the map defined on the level of Hochschild complexes by

$$\mathrm{C}_{p+q}(A, M) \otimes \mathrm{C}^q(A, N) \longrightarrow \mathrm{C}_p(A, M \otimes_A N)$$

$$(m \otimes a_1 \otimes \cdots \otimes a_{p+q}) \cap f := (m \otimes f(a_1, \ldots, a_q)) \otimes a_{q+1} \otimes \cdots \otimes a_{p+q}.$$

The cap product $\cap$ is also an $H$**-module morphism**:

$$\begin{aligned} & h \rightharpoonup \big((m \otimes a_1 \otimes \cdots \otimes a_{p+q}) \cap f\big) \\ = & \sum_{(h)} (h_1 \rightharpoonup \big(m \otimes a_1 \otimes \cdots \otimes a_{p+q})\big) \cap (h_2 \rightharpoonup f). \end{aligned}$$

Recall the **HKR theorem**. If $R$ is smooth, $H$ is cocommutative, then

$$\Omega^n(R) \overset{\varepsilon_n}{\cong} \mathrm{H}_n(R, R)$$

is an ***H*-module isomorphism**, where $\varepsilon_n$ given by

$$\varepsilon_n(a_0 da_1 \wedge \cdots \wedge da_n) := \overline{\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)})}$$

where $\overline{z}$ denotes the image of $z \in Z_n(\mathrm{C}_\bullet(R, R))$ in $\mathrm{H}_n(R, R)$.

Suppose $R$ is smooth. Then

$$\mathrm{H}^n(R,R) \overset{\varrho^n}{\cong} \mathrm{Der}^n(R) \text{ is an } H\textbf{-module isomorphism}, \text{ where}$$

$$\bar{f} \mapsto \varrho^n(\bar{f}) : b_1 \wedge \cdots \wedge b_n \mapsto \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$$

for any $f \in Z(\mathrm{C}^n(R,R))$ and $b_1, \cdots, b_n \in R$.

## Homo. determinants of Hopf actions on comm. CYs

$$\mathrm{Ext}^d_{R^e}(R, R^e) \cong \mathrm{Ext}^d_{R^e}(R, R) \cong \mathrm{Der}^d(R) \cong \mathrm{Hom}_R(\Omega^d(R), R) = R\eta^*$$

are *H*-**module isomorphisms**.

### Homo. determinants of Hopf actions on comm. CYs

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are *H*-**module isomorphisms**.

The generator of $\mathrm{H}_d(R, R) \cong \Omega^d(R)$ is still denoted by $\eta$, and the generator of $\mathrm{H}^d(R, R) \cong \mathrm{Hom}_R(\Omega^d(R), R)$ is still denoted $\eta^*$.

The homological determinant can be computed by using the generator $\eta^*$ of $\mathrm{H}^d(R, R) \cong \mathrm{Hom}_R(\Omega^d(R), R) = R\eta^*$: $\forall\, h \in H$,

$$h \rightharpoonup \eta^* = \phi(h)\eta^* = \sum_{(h)} (h_1 \rightharpoonup \mathrm{Hdet}_{\eta^*}(Sh_2))\eta^*$$

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## Proposition 4.2

*Let $R$ be a smooth commutative algebra with trivial canonical bundle $\Omega^d(R) = R\eta$. Then, the* **homological determinant** *of a cocommutative Hopf algebra $H$ acting on $R$ is given by*

$$\mathrm{Hdet}_{\eta^*}(h) = \eta^*(h \rightharpoonup \eta),\ \forall\, h \in H.$$

By using the **Key fact 2** that

$$\begin{array}{ccc} \mathrm{H}_d(R,R) \otimes \mathrm{H}^d(R,R) & \xrightarrow{- \cap -} & \mathrm{H}_0(R,R) \\ {\scriptstyle HKR}\downarrow \wr & \circlearrowleft & {\scriptstyle HKR}\downarrow = \\ \Omega^d(R) \otimes \Omega^d(R)^* & \xrightarrow{\iota_-(-)} & \Omega^0(R) \end{array}$$

is a commutative diagram of $H$-morphisms,

$$(h \rightharpoonup \eta) \cap \eta^* = \iota_{\eta^*}(h \rightharpoonup \eta) = \eta^*(h \rightharpoonup \eta) = \mathrm{Hdet}_{\eta^*}(h)$$

**Lemma.** For any $\delta \in \mathrm{Der}(R)$, $a \in R$ and $I = \{i_1 < i_2 < \cdots < i_d\} \in S$,

$$\delta(a)\,\mathrm{d}x_I = \sum_{j=1}^{d} (-1)^{j-1} \delta(x_{i_j})\,\mathrm{d}a \wedge \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \widehat{\mathrm{d}x_{i_j}} \cdots \wedge \mathrm{d}x_{i_d}.$$

**Lemma.** For any $\delta \in \mathrm{Der}(R)$, $a \in R$ and $I = \{i_1 < i_2 < \cdots < i_d\} \in S$,

$$\delta(a)\, dx_I = \sum_{j=1}^{d} (-1)^{j-1} \delta(x_{i_j})\, da \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \widehat{dx_{i_j}} \cdots \wedge dx_{i_d}.$$

The proof follows by applying the contraction map $\iota_\delta : \Omega^{d+1}(R) \to \Omega^d(R)$ to $0 = da \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_d}$.

$$\begin{aligned}
&\mathrm{Hdet}_{\eta^*}(\delta) = \eta^*(\delta \rightharpoonup \eta) \\
=&\Big( \sum_I (\delta a_I)\, dx_{i_1} \wedge \cdots \wedge dx_{i_d} + \sum_{I,s} a_I\, dx_{i_1} \wedge \cdots \wedge d\delta(x_{i_s}) \wedge \cdots \wedge dx_{i_d} \Big) \cap \eta^* \\
=&\eta^*\Big( \sum_I (\delta a_I)\, dx_{i_1} \wedge \cdots \wedge dx_{i_d} + \sum_{I,s} a_I\, dx_{i_1} \wedge \cdots \wedge d\delta(x_{i_s}) \wedge \cdots \wedge dx_{i_d} \Big) \\
=&\eta^*(L_\delta(\eta)) = \frac{L_\delta(\eta)}{\eta} = \mathrm{div}_\eta(\delta).
\end{aligned}$$

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&\mathrm{Hdet}_{\eta^*}(\delta) = \eta^*(\delta \rightharpoonup \eta) \\
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=&\eta^* \Big( \sum_I (\delta a_I)\, dx_{i_1} \wedge \cdots \wedge dx_{i_d} + \sum_{I,s} a_I\, dx_{i_1} \wedge \cdots \wedge d\delta(x_{i_s}) \wedge \cdots \wedge dx_{i_d} \Big) \\
=&\eta^*(L_\delta(\eta)) = \frac{L_\delta(\eta)}{\eta} = \mathrm{div}_\eta(\delta).
\end{aligned}$$

## Proposition 4.3

$\phi_\eta(r) = \mathrm{Hdet}_{\eta^*}(\{r, -\})$, *for any* $r \in R$ $\Big( = L_{H_r}(\eta)/\eta \Big)$.

# Step 3: $\overline{\mathrm{Hdet}_\pi(\delta_a)} = \mathrm{Hdet}_{\eta^*}(\{-, \bar{a}\})$

Let $Q_\bullet \to {}_{A^e}A \to 0$ be a f.g. filt-projective resolution, such that $\operatorname{gr} Q_\bullet \to {}_{(\operatorname{gr} A^e)} \operatorname{gr} A \to 0$ is a f.g. graded-projective resolution. Note that $\operatorname{gr} A^e \cong (\operatorname{gr} A)^e$.

Let $\mathrm{B}_\bullet(A) \to A \to 0$ be the bar resolution of $A$, which is filtered.

## Step 3: $\overline{\mathrm{Hdet}_\pi(\delta_a)} = \mathrm{Hdet}_{\eta^*}(\{-, \bar{a}\})$

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Let $\mathrm{B}_\bullet(A) \to A \to 0$ be the bar resolution of $A$, which is filtered.

$$Q_\bullet \overset{\text{homo. equiv.}}{\longrightarrow} \mathrm{B}_\bullet(A) \qquad \text{"} \Rightarrow \text{"}$$

$$\operatorname{gr} Q_\bullet \overset{\text{homo. equiv.}}{\longrightarrow} \operatorname{gr} \mathrm{B}_\bullet(A) \cong \mathrm{B}_\bullet(\operatorname{gr} A).$$

$$\begin{array}{ccc} \mathrm{HOM}_{A^e}(Q_\bullet, A^e) & \xrightarrow{\text{homo. equiv.}} & \mathrm{HOM}_{A^e}(B_\bullet(A), A^e) \\ \downarrow = & & \downarrow \\ \mathrm{Hom}_{A^e}(Q_\bullet, A^e) & \xrightarrow{\text{homo. equiv.}} & \mathrm{Hom}_{A^e}(B_\bullet(A), A^e), \end{array}$$

the inclusion $\mathrm{HOM}_{A^e}(\mathrm{B}_\bullet(A), A^e) \subseteq \mathrm{Hom}_{A^e}(\mathrm{B}_\bullet(A), A^e)$ is also a homotopy equivalence.

$$\begin{array}{ccc} \mathrm{HOM}_{A^e}(Q_\bullet, A^e) & \xrightarrow{\text{homo. equiv.}} & \mathrm{HOM}_{A^e}(B_\bullet(A), A^e) \\ \downarrow = & & \downarrow \\ \mathrm{Hom}_{A^e}(Q_\bullet, A^e) & \xrightarrow{\text{homo. equiv.}} & \mathrm{Hom}_{A^e}(B_\bullet(A), A^e), \end{array}$$

the inclusion $\mathrm{HOM}_{A^e}(\mathrm{B}_\bullet(A), A^e) \subseteq \mathrm{Hom}_{A^e}(\mathrm{B}_\bullet(A), A^e)$ is also a homotopy equivalence.

$$\mathrm{Hom}_{\mathrm{gr}\, A^e}(\mathrm{gr}\, Q_\bullet, \mathrm{gr}\, A^e) \xrightarrow{\text{homo. equiv.}} \underline{\mathrm{Hom}}_{\mathrm{gr}\, A^e}(\mathrm{B}_\bullet(\mathrm{gr}\, A), \mathrm{gr}\, A^e).$$

The spectral sequence of the filtered complex $\mathrm{HOM}_{A^e}(\mathrm{B}_\bullet(A), A^e)$ yields that

• $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq d$, and

• $\mathrm{Ext}^d_{A^e}(A, A^e) = \mathrm{H}^d(\mathrm{HOM}_{A^e}(\mathrm{B}_\bullet(A), A^e))$ has a bounded below filtration such that $\mathrm{gr}\,\mathrm{Ext}^d_{A^e}(A, A^e) \cong \underline{\mathrm{Ext}}^d_{(\mathrm{gr}\,A)^e}(\mathrm{gr}\,A, (\mathrm{gr}\,A)^e)$

$$= \mathrm{H}^d(\mathrm{Hom}_{(\mathrm{gr}\,A)^e}(B_\bullet(\mathrm{gr}\,A), (\mathrm{gr}\,A)^e))$$

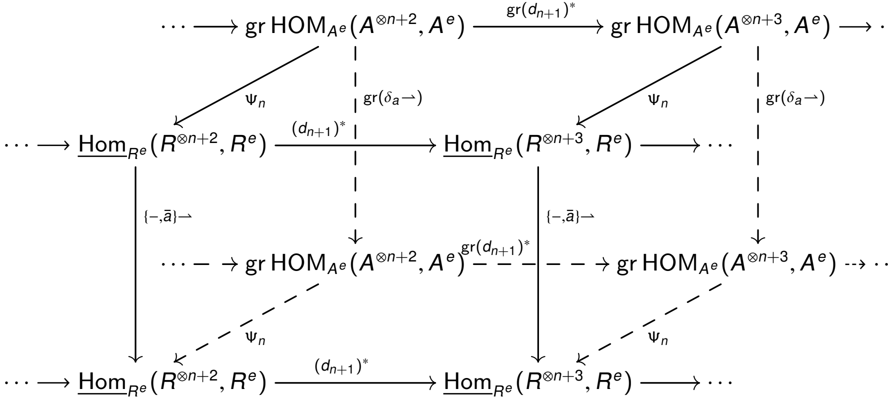
as right $(\mathrm{gr}\,A)^e$-module,

# Step 3: $\overline{\delta_a \rightharpoonup \pi} = \{-, \bar{a}\} \rightharpoonup \overline{\pi}$

Let $R := \mathrm{gr}\, A$. The following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{gr}\, \mathrm{HOM}_{A^e}(A^{\otimes n+2}, A^e) & \xrightarrow{\mathrm{gr}(\delta_a \rightharpoonup)} & \mathrm{gr}\, \mathrm{HOM}_{A^e}(A^{\otimes n+2}, A^e) \\
\cong \downarrow \Psi_n & & \cong \downarrow \Psi_n \\
\underline{\mathrm{Hom}}_{(\mathrm{gr}\, A)^e}((\mathrm{gr}\, A)^{\otimes n+2}, (\mathrm{gr}\, A)^e) & \xrightarrow{\{-,\bar{a}\}\rightharpoonup} & \underline{\mathrm{Hom}}_{(\mathrm{gr}\, A)^e}((\mathrm{gr}\, A)^{\otimes n+2}, (\mathrm{gr}\, A)^e).
\end{array}$$

$\pi$ is a twisted volume "$\Rightarrow$" $\eta^* := \overline{\pi}$ is a dual basis of $\eta$.


$\mathrm{gr}\,\mathrm{HOM}_{A^e}(A^{\otimes n+2}, A^e)$
$\mathrm{gr}(d_{n+1})^*$
$\mathrm{gr}\,\mathrm{HOM}_{A^e}(A^{\otimes n+3}, A^e)$
$\Psi_n$
$\mathrm{gr}(\delta_a \rightharpoonup)$
$\underline{\mathrm{Hom}}_{R^e}(R^{\otimes n+2}, R^e)$
$(d_{n+1})^*$
$\underline{\mathrm{Hom}}_{R^e}(R^{\otimes n+3}, R^e)$
$\{-,\bar{a}\}\rightharpoonup$

## Proposition 4.4

- $\overline{\delta_a \rightharpoonup \pi} = \{-, \bar{a}\} \rightharpoonup \eta^*$
- $\overline{\mathrm{Hdet}_\pi(\delta_a)} = \mathrm{Hdet}_{\eta^*}(\{-, \bar{a}\})$

$$\overline{(\mathrm{id}_A - \mu_A)(a)} \overset{step1}{=\!=} \overline{\mathrm{Hdet}_\pi(\delta_a)}$$
$$\overset{step3}{=\!=} \mathrm{Hdet}_{\eta^*}(\{-, \bar{a}\})$$
$$\overset{step2}{=\!=} -\phi_\eta(\bar{a}).$$

$$\begin{aligned}\overline{(\mathrm{id}_A - \mu_A)(a)} &\overset{step1}{=\!=} \overline{\mathrm{Hdet}_\pi(\delta_a)} \\ &\overset{step3}{=\!=} \mathrm{Hdet}_{\eta^*}(\{-, \bar{a}\}) \\ &\overset{step2}{=\!=} -\phi_\eta(\bar{a}).\end{aligned}$$

**Theorem 4.5**

$\overline{(\mu_A - \mathrm{id}_A)(a)} = \phi_\eta(\bar{a}) \in (\mathrm{gr}\, A)_{n-\ell}$ *for any* $a \in F_n A$.

- $(\mu_A - \mathrm{id}_A)|_{F_{\ell-1}A} = 0$.
- *for any* $a \in F_\ell A$, $\mu_A(a) - a = \phi_\eta(\bar{a}) \in F_0 A$.

## Corollary 4.6 (Calabi-Yau $\longleftrightarrow$ unimodular Poisson)

*Suppose that $A$ is generated by $F_1 A$ as $k$-algebra.*

- *If the poisson structure on* $\mathrm{gr}\, A$ *is* **unimodular** *(that is, $\phi_\eta = 0$ for some volume form $\eta$), then $A$ is* **Calabi-Yau**.
- *If* $\mathrm{U}(A) \subset F_0 A$ *($\Leftarrow$ $\mathrm{gr}\, A$ is a domain), then* $\mathrm{gr}\, A$ *is* **unimodular** *if and only if $A$ is* **Calabi-Yau**.

# Ring of differential operators

Let $R$ be a $d$-dim affine smooth domain over a field $k$ with $char(k) = 0$.

The **ring of differential operators** $\mathcal{D}(R) := \bigcup_p \mathcal{D}(R)_p$ of $R$ is defined inductively by Grothendieck as $\mathcal{D}(R)_{-1} = 0$ and

$$\mathcal{D}(R)_p := \{f \in \mathrm{End}_k(R) \mid fr - rf \in \mathcal{D}(R)_{p-1} \text{ for all } r \in R\}.$$

Obviously, $\mathcal{D}(R) = \bigcup_p \mathcal{D}(R)_p$ is a filtered $k$-algebra.

Weyl algebra $A_d(\mathbb{C}) = \mathcal{D}(\mathbb{C}[x_1, \ldots, x_d])$.

$\operatorname{gr} \mathcal{D}(R) \cong$ the symmetric $R$-algebra $Sym_R(\operatorname{Der}(R))$ as graded $R$-algebra.

**Proposition 5.1**

*The poisson structure on $\operatorname{gr} \mathcal{D}(R)$ is unimodular.*

**Theorem 5.2**

*The ring of differential operators $\mathcal{D}(R)$ is a 2d-dim CY algebra.*

# Thank you for your attention!