On Non-simple Blowups for Quantized Liouville Equation

joint work with Lei Zhang and Teresa D'Aprile

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Nonlocal Problems in Mathematical Physics, Analysis and Geometry

The singular Liouville equation

In this talk I will talk about the following simple equation defined in two dimensional spaces:

$$\Delta v(x) + h(x)e^{v(x)} = 4\pi\alpha\delta_0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2$$

where h is a positive smooth function and B_1 is the unit ball, δ_0 is a Dirac mass placed at the origin and $\alpha > -1$. Since

$$\Delta(\frac{1}{2\pi}\log|x|) = \delta_0,$$

Setting $u(x) = v(x) - 2\alpha\log|x|$ we have

$$\Delta u + |x|^{2\alpha}h(x)e^{u(x)} = 0.$$

Geometric background

Nirenberg problem with conic singularity: which smooth functions K on $\mathbb{S}^2$ are realized as the Gauss curvature of a metric $g = e^{2u} g_0$ on $\mathbb{S}^2$ pointwise conformal to the standard round metric g_0 of $\mathbb{S}^2 \subset \mathbb{R}^3$?

$$\Delta u + K(x) e^{2u} = 1 \tag{1}$$

If in a neighborhood of one point, the metric has a conic singularity of order α:

$$g = e^{2u} |z|^{2\alpha} |dz|^2,$$

The corresponding PDE to study is

$$\Delta u + K(x) |x|^{2\alpha} e^{u} = 0.$$

Physical Background

Mean Field Equation:

$$\Delta_g u + \rho(\frac{h(x)e^u}{\int_M h(x)e^u} - \frac{1}{|M|}) = 4\pi \sum_j (\delta_{p_j} - \frac{1}{|M|}), \text{ on } (M, g)$$

If the singular source is quantized, i.e. $\alpha_j \in \mathbf{N}$, the Liouville equation has close ties with Toda System, Algebraic geometry, integrable system, number theory and complex Monge-Ampere equations.
Chen-Lin, Jost-Wang, Lin-Wei-Ye, Malchiodi, Bartolucci-Tarantello, ...

Chipot-Shafrir-Wolansky 1997

$$\Delta_g u_i + \sum_j a_{ij}(\frac{h(x)e^{u_j}}{\int_M h(x)e^{u_j}} - \frac{1}{|M|}) = 4\pi \sum_j \sum_{ij} (\delta_{p_{ij}} - \frac{1}{|M|}), \text{ on } (M, g)$$

Case 1. No singularity ($\alpha = 0$)

$$\Delta u_k + h_k(x)e^{u_k} = 0 \quad \text{in } B_1$$

$$u_k(0) = \max_{B_1} u_k(x) \to +\infty$$

$$\int h_k(x)e^{u_k} < +\infty$$

$$0 < C_1 \leq h(x) \leq C_2 < +\infty$$

Theorem: All bubbles are simple

- Brezis-Merle (CPDE91)
- Brezis-Li-Shafrir (IUMJ93)
- Li-Shafrir (IUMJ94)
- Li (CMP1995)

classification of global solutions

Theorem

(Chen-Li (Duke94)) Let u be a solution of

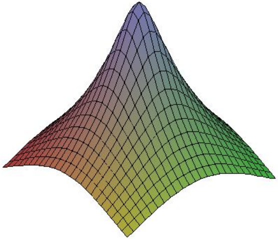
$$\Delta u + e^u = 0, \quad in \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < \infty,$$

then

$$u(x) = U_{\lambda,x_0} = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8}|x - x_0|^2)^2}$$

for some $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^2$. (Liouville 1836)

$\int_{\mathbb{R}^2} e^u = 8\pi$. If $v(y) = u(\delta y) + 2\log\delta$, then $\int_{\mathbb{R}^2} e^v = \int_{\mathbb{R}^2} e^u$.



local blowup for regular equation

Let u_k be a sequence of bubbling solutions of

$$\Delta u_k + he^{u_k} = 0, \quad \text{in} \quad B_1,$$

where h is a positive smooth function. If

1. $$\max_x u_k(x) = u_k(0) \to \infty, \quad \text{and} \quad \max_{K \subset\subset B_1 \setminus \{0\}} u_k \le C(K)$$

2. $$\int_{B_1} he^{u_k} \le C,$$

3. $$|u_k(x) - u_k(y)| \le C, \quad \forall x, y \in \partial B_1,$$

Theorem

(Y.Y.Li, (CMP95)) Suppose $\lambda_k = u_k(0) = \max u_k \to \infty$, *then*

$$u_k(x) - \log \frac{e^{\lambda_k}}{(1 + \frac{e^{\lambda_k} h(0)}{8}|x|^2)^2} = O(1), \quad \forall x \in B_1.$$

Simple-vs-Non-simple blow-up

A blow-up is simple if after suitable rescaling

$$|u_k - U_{\lambda_k, p_k}| \leq C \text{ in } B_1$$

Equivalently

$$u_k + 2\log|x| \leq C \text{ in } B_1$$

Equivalently u satisfies spherical Harnack inequality around 0, which implies that, after scaling, the sequence u_k behaves as a single bubble around the maximum point.

Non-simple Blow-ups

$$\Delta u_k + h(x)|x|^{2\alpha} e^{u_k} = 0, \quad \text{in} \quad B_1.$$

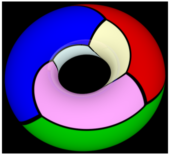
A blow-up is non-simple if after suitable rescaling

$$|u_k - U_{\lambda_k, p_k}| >> C \text{ in } B_1$$

Equivalently

$$\max_{B_1}(u_k + 2(1+\alpha)\log|x|) \to +\infty$$

Applications: Uniform Estimate

If we consider a mean field equation on a surface, say 

$$\Delta_g u + \rho\left(\frac{he^u}{\int_M he^u} - 1\right) = 0. \quad vol(M) = 1.$$

Since all bubbles are simple, the uniform estimate implies

1. Around each blowup point, there is only one bubble profile: $he_k^u \rightharpoonup 8\pi\delta_p$
2. The height of bubbles are roughly the same.
3. The energy $(\int_M he^{u_k})$ is concentrated around a few blowup points.
4. Further refined estimates are possible

- if $8\pi N < \rho < 8\pi(N+1)$ we have $|u| < C$
-
$$T_\rho = -\rho \Delta_g^{-1}\left(\frac{he^u}{\int_M he^u} - 1\right)$$
-
$$d_\rho := deg(I - T_\rho, B_R, 0)$$

  is well defined for $\rho \neq 8N\pi$. (YY Li (2000))

Theorem

(Chen-Lin CPAM02, CPAM03)

$$d_\rho = \begin{cases} 1 & \rho < 8\pi, \\ \frac{(-\chi_M+1)\dots(-\chi_M+N)}{N!} & 8N\pi < \rho < 8(N+1)\pi. \end{cases}$$

$\chi(M) = 2 - 2g_e$, the g_e is the genus of the manifold, which is the number of handles.

Case 2. Non-quantized singularity ($\alpha \notin \mathbb{N}$)

$$\Delta u + h(x)|x|^{2\alpha} e^u = 0.$$

$$\int h(x)|x|^{2\alpha} e^u < +\infty$$

$$\alpha \notin \mathbb{N}$$

Theorem: All bubbles are simple

Classification Theorem

Theorem

(Prajapat-Tarantello 01) If $\alpha > -1$ *is not an integer, all solutions to*

$$\Delta u + |x|^{2\alpha} e^u = 0, \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2\alpha} e^u < \infty,$$

are radially symmetric and can be written as

$$u(x) = U_\lambda = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8(1+\alpha)^2} |x|^{2+2\alpha})^2}$$

for some $\lambda \in \mathbb{R}$. *The total integration is*

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^u = 8\pi(1 + \alpha).$$

Non-quantized singularity

Theorem

(Bartolucci-Chen-Lin-Tarantello (CPDE 04)) Let u_k be blowup solutions to

$$\Delta u_k + |x|^{2\alpha} h e^{u_k} = 0, \quad B_1$$

with $\alpha > -1$ and bounded oscillation on ∂B_1. Suppose 0 *is the only blowup point in B_1, then*

$$h e^{u_k} \rightharpoonup 8\pi(1+\alpha)\delta_0$$

and if α is not a positive integer

$$u_k(x) - \log \frac{e^{u_k(0)}}{(1 + \frac{h(0)}{8(1+\alpha)^2} e^{u_k(0)} |x|^{2\alpha+2})^2} = O(1) \quad B_1.$$

Chen-Lin (CPAM 07): Topological degree when $\alpha \notin N$.

Case 3. Quantized singularity ($\alpha \in \mathbb{N}$)

$$\Delta u + h(x)|x|^{2N} e^u = 0.$$

$$\int h(x)|x|^{2N} e^u < +\infty$$

$$\alpha = N \in \mathbb{N}$$

Classification Theorem

Theorem

(Prajapat-Tarantello 01) All solutions of

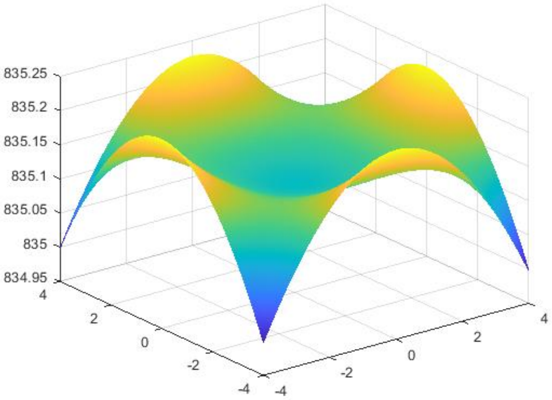
$$\Delta u + |x|^{2N} e^u = 0, \quad in \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2N} e^u < \infty,$$

are of the form

$$u(z) = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8(1+N)^2} |z^{N+1} - \xi|^2)^2}$$

for some $\xi \in \mathbb{C}$. $\int_{\mathbb{R}^2} |x|^{2N} e^u = 8\pi(1 + N)$.

Non-simple-blow-ups



If we choose $\xi_k \to 0$ and $\lambda_k \to \infty$ we can see non-simple blowup solutions.

Quantized singularity, Non-simple blowup

Let u_k be a sequence of solutions to

$$\Delta u_k + |x|^{2N} h(x) e^{u_k} = 0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2,$$

where $h > 0$ is smooth. Suppose 0 is the only blowup point and N is a positive integer, u_k has bounded oscillation on ∂B_1 and $\int_{B_1} |x|^{2N} h e^{u_k} < C$.

Theorem

(Kuo-Lin (JDG 16), Bartolucci-Tarantello (CPDE 18)) For $N \in \mathbb{N}$, if u_k has a non-simple blowup point at 0:

$$\max_{x \in B_1} u_k(x) + 2(1+N) \log |x| \to \infty.$$

u_k has exactly $N+1$ local maximum points evenly distributed around 0.

Quantized singularity, Non-simple blowup

Let u_k be a sequence of solutions to

$$\Delta u_k + |x|^{2N} h(x) e^{u_k} = 0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2,$$

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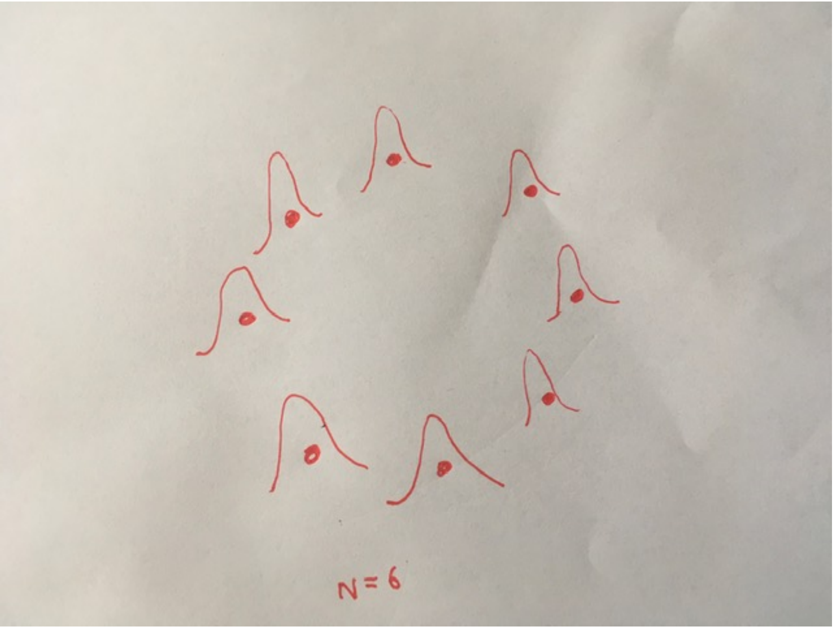
Question: how to analyze non-simple blow-ups?


N=6

Vanishing Theorems

Theorem

(Wei-Zhang, (Adv Math.2021,PLMS2022, JEMS 2024)) Let u_k be non-simple blowup solutions to

$$\Delta u_k + |x|^{2N} h_k(x) e^{u_k} = 0, \quad in \quad B_1 \subset \mathbb{R}^2,$$

under the usual assumptions. Then along a sub-sequence

$$\lim_{k\to\infty} \nabla h_k(0) = 0.$$

$$\lim_{k\to\infty} \Delta h_k(0) = 0.$$

More Vanishing Theorems when $N \geq 1$

Theorem

(D'Aprile-Wei-Zhang (2024-09)) Let u_k be non-simple blowup solutions

$$\Delta u_k + h_k |x|^{2N} e_k^u = 0 \ in \ B_1$$

Then

$$\nabla h_k(0) = o(1)$$

$$\partial^2_{xx} h_k(0) = o(1), \partial^2_{xy} h_k(0) = \partial^2_{yx} h_k(0) = o(1), \partial^2_{yy} h_k(0) = o(1)$$

More Vanishing Theorems when $N \geq 1$

Theorem

(D'Aprile-Wei-Zhang (2024-09)) Let u_k *be non-simple blowup solutions*

$$\Delta u_k + h_k |x|^{2N} e_k^u = 0 \text{ in } B_1$$

Then

$$\nabla h_k(0) = o(1)$$

$$\partial_{xx}^2 h_k(0) = o(1), \partial_{xy}^2 h_k(0) = \partial_{yx}^2 h_k(0) = o(1), \partial_{yy}^2 h_k(0) = o(1)$$

Higher order vanishing

$$\nabla^\alpha h_k(0) = o(1), |\alpha| \leq 2^M + 1, \text{ if } N \geq 2^{M+1} - 2$$

Comparison with simple blowups

Simple blow-ups

$$\Delta u_k + h(x)|x|^{2\alpha} e^{u_k} = 0, \ \alpha = 0, \ \notin N$$

First order vanishing

$$\nabla h(0) \to 0$$

Comparison with simple blowups

Simple blow-ups

$$\Delta u_k + h(x)|x|^{2\alpha} e^{u_k} = 0, \ \alpha = 0, \ \notin N$$

First order vanishing

$$\nabla h(0) \to 0$$

Non-simple blowups

$$\Delta u_k + h(x)|x|^{2N} e^{u_k} = 0$$

Second order vanishing

$$\nabla^\alpha h(0) \to 0, 1 \le |\alpha| \le [\frac{N+1}{2}] + 1$$

Comparison with simple blowups

Simple blow-ups

$$\Delta u_k + h(x)|x|^{2\alpha} e^{u_k} = 0, \; \alpha = 0, \; \notin N$$

First order vanishing

$$\nabla h(0) \to 0$$

Non-simple blowups

$$\Delta u_k + h(x)|x|^{2N} e^{u_k} = 0$$

Second order vanishing

$$\nabla^{\alpha} h(0) \to 0, 1 \leq |\alpha| \leq [\frac{N+1}{2}] + 1$$

This is optimal when $N = 1$.

Existence of non-simple blow-up

Theorem (D'Aprile, Wei, Zhang, CVPDE 2023)

Consider the following

$$-\Delta u = \lambda h(x) e^u - 4\pi\delta_0 \ \text{in}\ B_1, \quad u = 0 \ \text{ on }\ \partial B_1.$$

Assume that

$$h(x) = 1 + 2(x_1^4 + x_2^4) + 4x_1^2 x_2^2 + \\ D_0(x_1^6 - x_2^6) + D_1(x_1^5 x_2 + x_1 x_2^5) + D_2(x_1^4 x_2^2 - x_1^2 x_2^4) + D_3 x_1^3 x_2^3 + O(|x|^7)$$

Let $\xi \in \mathbb{R}^2$, $\xi \neq 0$, be a zero for the following vector field which is stable under uniform perturbations

$$(\xi_1, \xi_2) \longmapsto \begin{pmatrix} 3D_0\xi_1^2 + D_1\xi_1\xi_2 + \frac{3D_0 - D_2}{4}\xi_2^2 + \frac{15D_0 - D_2}{4} \\ \frac{D_1}{2}\xi_1^2 + \frac{3D_0 - D_2}{2}\xi_1\xi_2 + 3\frac{2D_1 + D_3}{8}\xi_2^2 + \frac{10D_1 + 3D_3}{8} \end{pmatrix}.$$

Then non-simple blow-up exists.

Non-existence theorems

Surprisingly in many general situations, non-simple blow-up does not happen. D'Aprile-Wei studied the following classical Liouville equation

$$\Delta u + \lambda e^u = \sum_{i=1}^{M} 4\pi\gamma_i\delta_{p_i} \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \tag{2}$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is an open and bounded subset of $\mathbb{R}^2$, $p_1, ..., p_M \in \Omega$, $\partial\Omega$ is smooth, $\lambda > 0$ and $\gamma_i > -1$.

Nonsimple blow-ups?

D'Aprile-Wei (JFA2020): Let $\Omega = B_1$ and $p_1 = 0$.

$$\Delta u + \lambda e^u = 4\pi N_\lambda \delta_0 \quad \text{in} \quad B_1, \qquad (3)$$
$$u = 0 \quad \text{on} \quad \partial B_1,$$

Then non-simple blow-ups exist if

$$N_\lambda - N \sim C\lambda \log^2 \lambda$$

Nonsimple blow-ups?

D'Aprile-Wei (JFA2020): Let $\Omega = B_1$ and $p_1 = 0$.

$$\Delta u + \lambda e^u = 4\pi N_\lambda \delta_0 \quad \text{in} \quad B_1, \qquad (3)$$
$$u = 0 \quad \text{on} \quad \partial B_1,$$

Then non-simple blow-ups exist if

$$N_\lambda - N \sim C\lambda \log^2 \lambda$$

Conjecture: When $\Omega = B_1, N_\lambda = N$, there are no non-simple blow-up phenomena (3).

Theorem

(D'Aprile-Wei-Zhang-22) Let u_k be a sequence of blowup solutions of

$$\Delta u + \lambda e^u = \sum_{i=1}^{M} 4\pi\gamma_i \delta_{p_i} \quad in \quad \Omega \subset \mathbb{R}^2, \tag{4}$$
$$u = 0 \quad on \quad \partial\Omega,$$

with parameter λ_k that satisfies $\int_\Omega \lambda_k e^{u_k} < C$. Then u_k is simple around any blowup point in Ω.

Theorem

(D'Aprile-Wei-Zhang-22) Let u_k be a sequence of blowup solutions of

$$\Delta u + \lambda e^u = \sum_{i=1}^{M} 4\pi\gamma_i\delta_{p_i} \quad in \quad \Omega \subset \mathbb{R}^2, \qquad (4)$$
$$u = 0 \quad on \quad \partial\Omega,$$

with parameter λ_k that satisfies $\int_\Omega \lambda_k e^{u_k} < C$. Then u_k is simple around any blowup point in Ω.

Corollary: When $\Omega = B_1, \gamma_i = N$, there are no non-simple blow-up phenomena (2).

$$\Delta u + \lambda e^u = 4\pi N\delta_0 \quad \text{in} \quad B_1, \qquad (5)$$
$$u = 0 \quad \text{on} \quad \partial B_1,$$

Non-simple blow-ups are lonely

$$\Delta u_k + e^{u_k} = \sum_{i=1}^{M} 4\pi\gamma_i \delta_{p_i} \quad \text{in} \quad \Omega \tag{6}$$

$$\int_\Omega e^{u_k} \le C \tag{7}$$

and

$$|u_k(x) - u_k(y)| \le C, \quad \forall x, y \in \partial\Omega. \tag{8}$$

Theorem

(D'Aprile-Wei-Zhang-22) Let u_k be a sequence of blowup solutions of (6) such that (7) and (8) hold. If there are at least two blowup points in Ω, each blowup point is simple.

Summary

$$\Delta u_k + h_k |x|^{2N} e^{u_k} = 0$$

- Vanishing Theorems

$$\nabla h_k(0) = 0, \Delta h_k(0) = 0$$

$$\nabla h_k = 0, D^\alpha h_k = 0, |\alpha| \geq 2$$

- 2. No-simple blow-ups does not exist when Dirichlet BC imposed

$$\Delta u + \lambda e^u = 4\pi \sum_{i=1}^{M} \gamma_i \delta_{p_i} \text{ in } \Omega; \; u = 0 \text{ on } \partial\Omega$$

- 3. Non-simple blow-ups are lonely
  If there are two blow-ups then non-simple blow-ups do not exist.

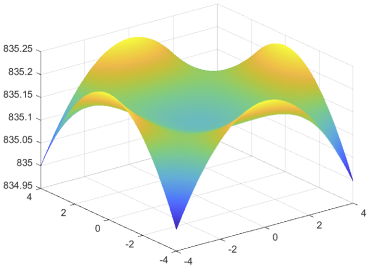
Local maximum points

Let p_0^k, ..., p_N^k be the $N+1$ local maximums of u_k

$$\Delta u_k + |x|^{2N} h_k(x) e^{u_k} = 0, \quad \text{in} \quad B_1.$$

Let

$$\delta_k = |p_0^k|, \quad \mu_k = u_k(p_0^k) + 2(1+N)\log \delta_k.$$



Trivial Observations

- The study of blowup solutions looks like that of a single Liouville equation near each local maximum point.
- The relations between these local maximums plays a crucial role.
- The blowup solutions look almost like a harmonic function away from the $N+1$ local maximums.

Difficulty and Problem

- The main difficulty: a priori we don't have any relation between

$$\delta_k \quad \text{(the distance between small bubbles)}$$

  and

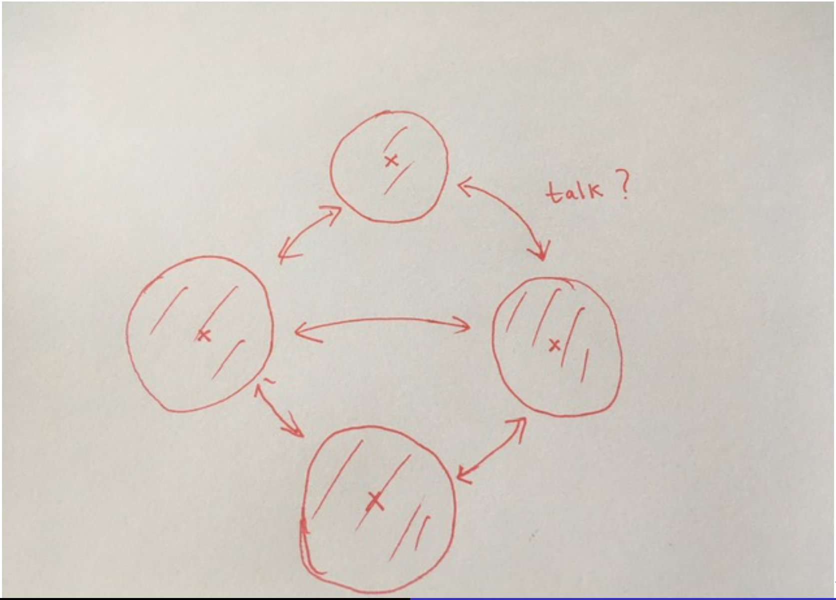
$$\mu_k = u_k(p_0^k) + 2(1+N)\log\delta_k.\text{(the height of bubbles)}$$

  In fact it should be no relation at all, from the ground state solution:

$$U(z) = \log \frac{e^{\lambda}}{(1 + \frac{e^{\lambda}}{8(1+N)^2}|z^{N+1} - \xi|^2)^2}$$

- The main problem: how do different bubbles talk to each other?

Nonlocal Interactions!



key ideas of the proof

A lot of local Pohozaev identities:

- A Pohozaev identity for $\Delta u_k + h_k e^{u_k} = 0$ on $B_\sigma(x_j)$ is

$$\int_{B_\sigma} (\nabla h_k \cdot x) e^{u_k} = \int_{\partial B_\sigma} \left(\frac{\sigma}{2} (|\partial_\nu u_k|^2 - |\partial_\tau u_k|^2) + \sigma h_k e^{u_k} + 2\partial_\nu u_k \right) dS.$$

- $$\delta_k \nabla (\log h_k)(\delta_k Q_l^k) + 2N \frac{Q_l^k}{|Q_l^k|^2} + \nabla \phi_{l,k}(Q_l^k) = O(\mu_k e^{-\mu_k}).$$

- $$\nabla \phi_l^k(Q_l^k) = -4 \sum_{m \neq l} \frac{Q_l^k - Q_m^k}{|Q_l^k - Q_m^k|^2} + O(\delta_k^2) + O(\mu_k e^{-\mu_k}).$$

- Denoting $Q_l^k = e^{i\frac{2\pi l}{N+1}}(1 + m_l^k)$ and use this in the long computation of each Pohozaev identity, we have
- $$\begin{pmatrix} m_1^k \\ m_2^k \\ \vdots \\ m_N^k \end{pmatrix} = A^{-1}\delta_k \bar{\nabla}(\log h_k)(0) \begin{pmatrix} e^{i\beta_1} \\ e^{i\beta_2} \\ \vdots \\ e^{i\beta_N} \end{pmatrix} + O(\delta_k^2) + O(\mu_k e^{-\mu_k})$$

  where $\beta_l = 2\pi l/(N+1), l = 0, ..., N$.

$$A = \begin{pmatrix} D & -d_1 & \dots & -d_{N-1} \\ -d_1 & D & \dots & -d_{N-2} \\ \vdots & \vdots & \dots & \vdots \\ -d_{N-1} & -d_{N-2} & \dots & D \end{pmatrix}$$

where

$$d_i = \frac{1}{\sin^2(\frac{i\pi}{N+1})}, \quad i = 1, ..., N, \quad D = d_1 + ... + d_N.$$

Main Idea for the proof of non-existence of nonsimple blowups

Theorem (D'Aprile-Wei-Zhang): All blow-ups for the following problem is simple:

$$-\Delta u = \lambda e^u - 4\pi \sum_{i=1}^{M} \gamma_i \delta_{p_i} \text{ in } \Omega$$

$$\lambda \int_\Omega e^u < C$$

$$u = 0 \text{ on } \partial\Omega$$

We found that when non-simple blowup happens, the oscillation on the boundary has to be very special. This is the main reason that we can prove the conjecture in a very general setting. Basically, as long as we know the behavior of the blowup solutions on the boundary and it is different from that of a non-simple blowup global solutions, we can capture this difference and say that non-simple blowup cannot happen.

D'Aprile-Wei conjecture. Key Theorem

Basic set-up: Let u_k be a sequence of solutions of the following equation that blows up at 0:

$$\Delta u_k + |x|^{2N} e^{u_k} = 0, \quad \text{in} \quad B_1 \tag{9}$$

Suppose the oscillation of u_k on the boundary of B_1 is finite:

$$|u_k(x) - u_k(y)| \leq C, \quad \forall x, y \in \partial B_1 \tag{10}$$

for some $C > 0$ independent of k, and there is a uniform bound on the integration of $|x|^{2N} e^{u_k}$:

$$\int_{B_1} |x|^{2N} e^{u_k} < C. \tag{11}$$

D'Aprile-Wei-Conjecture, key theorem

Set

$$\Phi_k(x) = u_k(x) - \frac{1}{2\pi}\int_{\partial B_1} u_k, \quad x \in B_1,$$

and let Φ be the limit of Φ_k over any fixed compact subset of B_1. Then our assumption of Φ_k is

$$\text{Either } \Phi \neq 0 \quad \text{or} \quad \Phi_k \equiv 0. \tag{12}$$

Theorem

Let 0 be the only blowup point of u_k in B_1, which has a uniformly bounded integration. Suppose (12) holds. Then u_k is a simple blowup sequence:

$$u_k(x) + 2(1+N)\log|x| \leq C$$

for some $C > 0$.

key-ideas

Let v_k be the scaled u_k with $p_0^k = e_1$:

$$v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1)\log\delta_k, \quad |y| < \delta_k^{-1}.$$

Other local maximums are very close to $e^{\frac{2i\pi l}{N+1}}$ for $l = 1, ..., N$. Let

$$V_k(x) = \log \frac{e^{\bar{\mu}_k}}{(1 + \frac{e^{\bar{\mu}_k}}{8(1+N)^2}|y^{N+1} - e_1|^2)^2}.$$

that agrees with v_k at e_1 as a common local maximum. Now we use the following expansion of V_k for $|y| = L_k$ ($L_k = \delta_k^{-1}$)

$$\begin{aligned} V_k(y) &= -\bar{\mu}_k + 2\log(8(N+1)^2) - 4(N+1)\log L_k + \frac{2}{L_k^{2N+2}} \\ &+ \frac{4\cos((N+1)\theta)}{L_k^{N+1}} + \frac{4}{L_k^{2N+2}}\cos((2N+2)\theta) \\ &+ O(L_k^{-3N-3}) + O(e^{-\bar{\mu}_k} L_k^{-2N-2}). \end{aligned}$$

key ideas

The oscillating part of V_k is mainly

$$4\cos((N+1)\theta)\delta_k^{N+1} + 4\delta_k^{2N+2}\cos((2N+2)\theta).$$

based on this we set $\phi_{v,k}(\delta_k \cdot)$ to be the harmonic function that is equal to 0 at 0 and represents the oscillation of V_k $\partial\Omega_k$:

$$\phi_{v,k}(\delta_k y) = 4\delta_k^{2N+2} r^{N+1}\cos((N+1)\theta) + 4\delta_k^{4N+4} r^{2N+2}\cos((2N+2)\theta) + \ldots.$$

Recall that the oscillation of v_k is $\Phi_k(\delta_k \cdot)$. Thus if we set

$$\phi_{0,k}(y) = \Phi_k(\delta_k y) - \phi_{v,k}(\delta_k y)$$

and $v_{0,k} = v_k - \phi_{0,k}$, then $v_{0,k} - V_k$ is a constant on the boundary, but the equation of $v_{0,k}$ is

$$\Delta v_{0,k} + h_{0,k}|y|^{2N} e^{v_{0,k}} = 0, \quad \text{in} \quad \Omega_k$$

where $h_{0,k} = e^{\phi_{0,k}}$.

key-ideas

Because of the difference on the oscillations, we can prove that $\nabla h_{0,k}(e^{\frac{2\pi is}{N+1}})$ is different from zero to some extent (based on the Fourier expansions of these harmonic functions):

Lemma

There exist an integer $L > 0$,$\delta_k^* \in (\delta_k^L, \delta_k)$ *an integer* $0 \le s \le N$ *such that*

$$\nabla h_{0,k}(e^{\frac{2\pi is}{N+1}})/\delta_k^* \neq 0.$$

This lemma eventually leads to a contradiction: If non-simple blowup does exist, and v_k is so close to a global solution V_k, there is no way for $v_{0,k}$ to have a coefficient function different from 1. This part of the proof is similar to that of the vanishing theorems: two functions are extremely close near one local maximum, using Harnack this closeness can be passed to the neighborhood of other local maximums, then Pohozaev identities say this is not possible.

Application to Mean Field Equation

Corollary

(Wei-Zhang-PLMS22) Let u be a solution of

$$\Delta_g u + \rho(\frac{he^u}{\int_M he^u} - 1) = 4\pi \sum_{j=1}^{d} \alpha_j(\delta_{p_j} - 1).$$

If all $\alpha_j \in \mathbb{N}$ *and*

$$\Delta(\log h)(p_j) - 2K(p_j) \notin 4\pi\mathbb{N}, \quad j = 1, .., d.$$

Then any blowup solutions u^k *satisfy a spherical Harnack inequality around any blowup point.*

Applications to Classifications

$$\Delta u + \lambda e^u = 4\pi\alpha\delta_0, \quad \text{in } B_1(0), \quad u = 0, \quad \text{on } \partial B_1(0), \tag{13}$$

Complete classifications of solutions (Li-Wei 2024):

(1) For any $\alpha \in (0, 1]$, there exists a small $\lambda_\alpha > 0$ such that for $\lambda \in (0, \lambda_\alpha)$, the equation has exactly **three** solutions (up to rotation).

(2) For any $\alpha \in (1, 2)$, there exists a small $\lambda_\alpha > 0$ such that for $\lambda \in (0, \lambda_\alpha)$, the equation (13) has exactly **four** solutions (up to rotation):

Extensions

Non-simple blowups for 4−th order Liouville:

$$\Delta^2 u = |x|^{4N} e^{4u} \quad B_1(0) \subset \mathbf{R}^4$$

Theorem (Hyder-Wei-Yang Let u_k be a sequence of blow-up solutions to

$$\begin{cases} \Delta^2 u_k(x) = |x|^{4N} h_k(x) e^{u_k} \quad \text{in} \quad B_1, \\ u_k = \Delta u_k \equiv 0 \quad \text{on} \quad \partial B_1, \\ \int_{B_1} |x|^{4N} e^{u_k} dx < +\infty, \\ u_k \leq C_K \quad \text{for} \quad x \in K \subset B_1 \setminus \{0\}, \end{cases}$$

where $N = 1, 2, 3$ and

$$\begin{cases} \|h_k\|_{C^3(B_1)} \leq C, \quad \frac{1}{C} \leq h_k(x) \leq C, \quad x \in \bar{B}_1, \\ \int_{B_1} h_k e^{u_k} \leq C, \\ |u_k(x) - u_k(y)| \leq C, \quad \forall x, y \in \partial B_1. \end{cases}$$

Then u_k is a simple blow-up solution.

Among the others we prove the following points:

1. We assume that the sequence of points $p_{k,0}$ represents the local maximum point of u_k.

$$\hat{u}_k(y) = u_k(\theta_k y) + 4(1+N)\log|\theta_k|, \quad \theta_k = |p_{k,0}|,$$

then $\hat{u}_k(y)$ verifies the following equation

$$\Delta^2 \hat{u}_k(y) = |y|^{4N}\hat{h}_k(y)e^{\hat{u}_k} \quad \text{in} \quad B_{1/\theta_k}(0),$$

where $\hat{h}_k(y) = h(\theta_k y)$. The $N+1$ blow-up points $e_0, \cdots, e_N$ satisfy the following:

$$N\frac{e_\ell}{|e_\ell|^2} = 2\sum_{m\neq\ell}\frac{e_\ell - e_m}{|e_\ell - e_m|^2}, \quad \ell = 0, \cdots, N.$$

When $N = 1, 2$, we can show that $e_0, \cdots, e_N$ constitutes the **vertex point of a regular polygon**. When $N = 3$ we can prove that $e_0, \cdots, e_3$ constitutes the **vertex point of a regular polygon or Tetrahedron.**

2. Suppose that the non simple blow-up happens. If the blow-up points constitute the vertex point of a regular polygon. Then

$$\nabla(\log h_k)(0) = O(\theta_k) + O(\theta_k^{-2}e^{-\mu_k/2}),$$

where $\mu_k = u_k(p_{k,0}) + 4(N+1)\log\theta_k$. If $N = 3$ and the blow-up

THANKS FOR YOUR ATTENTION