Dual formulation of the multi-state Schrödinger Poisson Equation

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Nonlocal Shrodinger equations

$$(-\Delta + V)\,\phi + \lambda\phi = 0 \tag{1}$$

on $\mathbb{R}^3$, where $V = K * |\phi|^2$ is some symmetric kernel.

Nonlocal Shrodinger equations

The kernel is induced by electrostatic/gravitational potential $K(x) \approx \pm |x|^{-1}$:

$$-\Delta\phi(x) \pm \left(\int \frac{|\phi(y)|^2}{|x-y|} dy \right) \phi(x) + \phi(x) = 0$$

Nonlocal Shrodinger equations

The "gravitational" case $K(x) \approx -|x|^{-1}$

$$-\Delta\phi(x) - \left(\int \frac{|\phi(y)|^2}{|x-y|} dy \right) \phi(x) + \phi(x) = 0$$

was originally proposed by Ph. Choquard, as an approximation to Hartree–Fock theory for a one component plasma. Equation of similar types also appear to be a prototype of the so-called nonlocal problems, which arise in many situations and as a model of self-gravitating matter.

A generalized version in $\mathbb{R}^n$ takes the form

$$-\Delta\phi - \left(I_\alpha * |\phi|^p\right)|\phi|^{p-2}\phi + \phi = 0 \tag{1}$$

where

$$I_\alpha = A(\alpha)|x|^{\alpha-n}; \quad A(\alpha) := \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2}\Gamma\left(\alpha/2)\right)} \tag{2}$$

is the Rietz potential, $\alpha \in (0, n)$, $p \in (1, \infty)$ was considered by many authors in the last decades, using its variational structure as a critical point of the functional

$$E_{p,\alpha}(\phi) = \frac{1}{2}\int_{\mathbb{R}^n}\left(|\nabla u|^2 + |\phi|^2 - \frac{1}{2p}\left(I_\alpha * |\phi|^p\right)|\phi|^p\right) \tag{3}$$

on an appropriate space.

In particular, existence of solutions the case $p = 2$ (and for more general singular interaction kernels) was studied by

- E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's non-linear equation* ,Studies in Appl. Math.57(2)93-105, 1976/77
- P.-L. Lions: *The Choquard equation and related questions*, Nonlinear Anal., 4 (6) pp. 1063-1072, 1980
- G.P. Menzala: *On regular solutions of a nonlinear equation of Choquard's type*, Proc. Roy. Soc. Edinburgh Sect. A, 86 (3-4) pp. 291-301, 1980

For existence, regularity and asymptotic behavior of solutions in the general case see V. Moroz and J. Van Schaftingen

- *Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent*, Commun. Contemp. Math. 17 , no. 5, 2015
- *A guide to the Choquard equation*, Vitaly Moroz & Jean Van Schaftingen. J. Fixed Point Theory Appl. 19 , no. 1, 773-813, 2017 and references therein.

The non-linear Schrödinger equation associated with $E_{p,\alpha}$ takes the form

$$-i\partial_t\psi - \Delta\psi - a(I_\alpha * |\psi|^p)|\psi|^{p-2}\psi = 0 \ . \tag{4}$$

The number $a \in \mathbb{R}$ is the strength of interaction. The case $a > 0$ corresponds to the *attractive, gravitation-like* dynamics, and is related to Choquard's equation. The case $a < 0$ is the repulsive, electrostatic case and is related to the Hartree system. In this lecture we deal with the attractive case.

Considering an eigenmode $\psi = e^{-i\lambda t}\phi$ we get that ϕ satisfy the non-linear eigenvalue problems

$$-\Delta\phi - a\left(I_\alpha * |\phi|^p\right)|\phi|^{p-2}\phi - \lambda\phi = 0 \qquad (5)$$

which can be reduced to $a = \lambda = 1$ by a proper scaling. However, the solutions of the time dependent nonlinear equation preserve the $\mathbb{L}^2$ norm in time, so it is natural to look for stationary solutions under a prescribed $\mathbb{L}^2$ norm (say, $\|\phi\|_2 = 1$). It is not difficult to see that, in general, one can find a scaling $\phi \mapsto \phi_\beta(x) = \beta^{-n/2}\phi(x/\beta)$ which preserves the $\mathbb{L}^2$ norm and transform the strength of interaction in (5) into $a = 1$, making this parameter mathematically insignificant. There is, however, an exceptional case $\alpha = n(p-1) - 2$. In that case the first two terms in (5) are scaled identically under $\mathbb{L}^2$ preserving scaling, so the size of the interaction coefficient a is mathematically significant in that case.

In the case $p = 2$ and in the presence of a prescribed, scalar potential W, the $\mathbb{L}^2-$ constraint version of (5) takes the form

$$-\Delta\phi + W\phi - a\left(\int_{\mathbb{R}^n} \frac{|\phi(y)|^2}{|x-y|^{n-\alpha}} dy\right)\phi - \lambda\phi = 0, \quad \|\phi\|_2 = 1 \,. \qquad (6)$$

A solution is given by a minimizer of the functional

$$E_a^W(\phi) := \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla\phi|^2 + W|\phi|^2\right) dx - \frac{a}{4}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^{n-\alpha}} dxdy \quad (7)$$

restricted to the $\mathbb{L}^2$ unit sphere $\|\phi\|_2 = 1$. W a prescribed function satisfying

$$\lim_{|x|\to\infty} W(x) = \infty$$

Exceptional case

D. Yinbin, L. Lu and S. Wei studied the equation in the exceptional case $\alpha = n - 2$, for $n \geq 3$, $a > 0$.

$$-\Delta\phi + W\phi - a\left(\int_{\mathbb{R}^n} \frac{|\phi(y)|^2}{|x-y|^2}dy\right)\phi - \lambda\phi = 0, \quad \|\phi\|_2 = 1 \ .$$

Constraint minimizers of mass critical Hartree functionals: Existence and mass concentrations, J. Mathematical physics, 56, 2015

In particular, they showed the existence of a critical strength $\bar{a}_c > 0$, depending on n but independent of W, such that E_a^W is bounded from below on the sphere $\|\phi\|_2 = 1$ iff $a \leq \bar{a}_c$. Moreover, a minimizer of E_a^W exists if $a < \bar{a}_c$, and is a solution. It was also shown that $a_c = \|\bar{\phi}\|_2$, where $\bar{\phi}$ is the unique, positive unconstrained solution of the equation of

$$-\Delta\bar{\phi}(x) - \left(\int_{\mathbb{R}^n} \frac{|\bar{\phi}(y)|^2}{|x-y|^2}dy\right)\bar{\phi}(x) + \bar{\phi}(x) = 0 \ . \qquad (8)$$

Multi state system

The first object is to extend the $\mathbb{L}^2$-constraint Choquard equation (6) into a $k-$ state system

$$-\Delta\phi_j + (W - aV)\,\phi_j - \lambda_j\phi_j = 0 \quad \|\phi_j\|_2 = 1, \quad ; \quad j = 1\ldots k$$

where

$$V(x) = I_\alpha * \left(\sum_{i=1}^{k}\beta_i \int_{\mathbb{R}^n}\frac{\rho_i(y)}{|x-y|^{n-\alpha}}dy\right)$$

$$\rho_i := |\phi_i|^2$$

where $(\phi_1, \ldots \phi_k)$ constitutes an orthonormal $k-$sequence in $\mathbb{L}^2(\mathbb{R}^n)$ and

$$\beta_j > 0, \quad \sum_1^k \beta_j = 1 \tag{9}$$

are the *probabilities of occupation* of the states $j = 1\ldots k$.

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and the exceptional case $\alpha = n - 2$

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are the *probabilities of occupation* of the states $j = 1 \dots k$.

Theorem

If $\{\phi_1, \dots, \phi_k\}$ *is a minimizer of*

$$\mathcal{E}^{(\alpha)}_{\beta,a}(\vec{\psi}) := \frac{1}{2}\sum_{j=1}^{k} \beta_j \left[\langle \nabla\psi_j, \nabla\psi_j \rangle + \langle W\psi_j, \psi_j \rangle - \frac{a}{2}\sum_{i=1}^{k} \beta_i \left\langle |\psi_j|^2, I_\alpha * |\psi_i|^2 \right\rangle \right]$$

on all $k-$ *orthonormal frames* $\vec{\psi} = (\psi_1, \dots \psi_k)$*, then it is a solution of the* $k-$ *state Choquard system, while*

$$\lambda_1 < \lambda_2 \leq \dots, \leq \lambda_k$$

are successive eigenvalues starting form the minimal level λ_1 *and* ϕ_j *the corresponding eigenstates.*

Dual formulation

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For the case of single state $k = 1$, the dual formulation of E_a^W for $\alpha = 2$ on the constraint $\mathbb{L}^2$ sphere takes the form of the functional $V \mapsto \mathcal{H}_a^{W,\alpha}(V)$

$$\mathcal{H}_a^{W,2}(V) = \frac{a}{2} \int_{\mathbb{R}^n} |\nabla V|^2 + \lambda_1(V)$$

over the *unconstrained* Beppo-Levi space $V \in \dot{\mathbb{H}}_1(\mathbb{R}^n)$. Here the functional $\lambda_1 = \lambda_1(V)$ is the leading (minimal) eigenvalue of the Schrdinger operator

$$-\Delta + W - aV \quad \text{on} \quad \mathbb{R}^n \ .$$

The extension of this dual formulation to the $k-$system for $\alpha \in (0,2]$ In case $\alpha = 2$ it takes the form

$$\mathcal{H}^{W,2}_{\beta,a}(V) = \frac{a}{2}\int_{\mathbb{R}^n} |\nabla V|^2 + \sum_{j=1}^{k} \beta_j \lambda_j(V)$$

where $\lambda_1(V) < \lambda_2(V) \leq \ldots \lambda_k(V)$ are the leading k eigenvalues of the Schrdinger operator, while $\beta_1 > \beta_2 > \ldots \beta_k > 0$.

If $\alpha \neq 2$?

Let us recall some definitions and theorems we use later
For $V_1, V_2 \in C_0^\infty(\mathbb{R}^n)$ and $\alpha \in (0,2)$, consider the quadratic form

$$\langle V_1, V_2 \rangle_{\alpha/2} := A(-\alpha) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(V_1(x) - V_1(y))(V_2(x) - V_2(y))}{|x-y|^{n+\alpha}} dxdy$$

where the constant $A(-\alpha)$ is defined as in (2). If $\alpha = 2$

$$\langle V_1, V_2 \rangle_{(1)} := \int_{\mathbb{R}^n} \nabla V_1 \cdot \nabla V_2 dx \ .$$

The closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\alpha/2}$ is denoted by $\dot{\mathbb{H}}_{\alpha/2}$. We denote the associated norm by $|||\cdot|||_{\alpha/2}$. [1] Recall that $\dot{\mathbb{H}}_{\alpha/2}$ is a Hilbert space so, in particular, is weakly locally compact.

[1] Note that $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$ does not contain $\mathbb{L}^2(\mathbb{R}^n)$. In case $\alpha = 2$ it is sometimes called Beppo-Levi space.

$\alpha = 2$:

$$\mathcal{H}_{\beta,a}^{W,2}(V) = \frac{a}{2}\int_{\mathbb{R}^n} |\nabla V|^2 + \sum_{j=1}^{k} \beta_j \lambda_j(V)$$

where $\lambda_1(V) < \lambda_2(V) \leq \ldots \lambda_k(V)$ are the leading k eigenvalues of the Schrdinger operator, while $\beta_1 > \beta_2 > \ldots \beta_k > 0$.

$0 < \alpha < 2$:

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V) = \frac{a}{2}\langle V, V\rangle_{\alpha/2} + \sum_{j=1}^{k}\beta_j\lambda_j(V)$$

where $\lambda_1(V) < \lambda_2(V) \leq \ldots \lambda_k(V)$ are the leading k eigenvalues of the Schrdinger operator, while $\beta_1 > \beta_2 > \ldots \beta_k > 0$.

Duality for a single component $k = 1$

$$\frac{1}{2}\langle I_\alpha * \rho, \rho\rangle := \sup_{V \in C_0^\infty(\mathbb{R}^n)} \langle \rho, V\rangle - \frac{1}{2}\langle V, V\rangle_{\alpha/2} \ . \tag{10}$$

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$$\mathbf{H}^{(\alpha)}(\phi, V) = \langle (-\Delta + W)\phi, \phi \rangle + a\left[\left\langle V, V \right\rangle_{\alpha/2} - \left\langle V, |\phi|^2 \right\rangle \right] \ . \tag{11}$$

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$$\inf_{V\in C_0^\infty(\mathbb{R}^n)} \mathbf{H}^{(\alpha)}(\phi, V) = 2\mathcal{E}_a^{(\alpha)}(\vec{\phi}) \equiv$$

$$\frac{1}{2}\left[\langle\nabla\phi, \nabla\phi\rangle + \langle W\phi, \phi\rangle - \frac{a}{2}\left\langle|\phi^2, I_\alpha * |\phi|^2\right\rangle\right] \ .$$

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$$\inf_{V\in C_0^\infty(\mathbb{R}^n)} \mathbf{H}^{(\alpha)}(\phi, V) = 2\mathcal{E}_a^{(\alpha)}(\vec{\phi}) \equiv$$

$$\frac{1}{2}\left[\langle\nabla\phi, \nabla\phi\rangle + \langle W\phi, \phi\rangle - \frac{a}{2}\left\langle|\phi^2, I_\alpha * |\phi|^2\right\rangle\right] \ .$$

Let $\mathcal{H}_a^{W,\alpha}(V) = \inf_{\|\phi\|=1} \mathbf{H}^{(\alpha)}(\phi, V)$.

$$\mathcal{H}_{,a}^{W,\alpha}(V) = \frac{a}{2}\langle V, V\rangle_{\alpha/2} + \inf_{\|\phi\|_2=1}\langle(-\Delta + W - aV)\phi, \phi\rangle \ .$$

Multi-level case

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V) = \frac{a}{2}\langle V, V\rangle_{\alpha/2} + \inf_{\vec{\phi}\in\otimes^k\mathbb{H}^1} \sum_{j=1}^{k} \beta_j \langle(-\Delta + W - aV)\phi_j, \phi_j\rangle \ .$$

$$V \in \dot{\mathbb{H}}_{\alpha/2} \mapsto G_{\beta,a}(V) := \inf_{\vec{\phi}\in\otimes^k\mathbb{H}^1} \sum_{j=1}^{k} \beta_j \langle(-\Delta + W - aV)\phi_j, \phi_j\rangle \ . \quad (12)$$

Lemma

[Extended Rayleigh-Ritz] If $\beta_1 \geq \beta_2 \ldots \geq \beta_k$ *then*

$$G_{\beta,a}(V) = \sum_{j=1}^{k} \beta_j \lambda_j(V)$$

where $\lambda_j(V)$ *are the* k *lowest eigenvalues of the operator* $-\Delta + W - aV$ *arranged by increasing order. In particular,* $G_{\beta,a}$ *is concave on* $\mathbb{H}_{\alpha/2}$*.*
If $2 < n < 4 + \alpha$, $0 < \alpha \leq 2$ *then* $G_{\beta,a}$ *is weakly continuous on* $\mathbb{H}_{\alpha/2}$*.*
Moreover, the minimum in (12) is obtained at the eigenfunction $\bar{\phi}_j$ *of* $-\Delta + W - aV$ *corresponding to* λ_j*.*

The Euler-Lagrange equation

$$\underline{\partial}_V \frac{1}{2} \langle V, V \rangle_{\alpha/2} = (-\Delta)^{\alpha/2} V$$

$$\overline{\partial}_V G_{\beta,a} = -a \sum_{j=1}^{k} \beta_j |\bar{\phi}_j|^2$$

$$(-\Delta)^{\alpha/2} V - \sum_{j=1}^{k} \beta_j |\bar{\phi}_j|^2 = 0 \quad \Longleftrightarrow V = I_\alpha * (\sum_{j=1}^{k} \beta_j |\bar{\phi}_j|^2)$$

Partial Proof

Let $\bar{\phi}_j$ be the normalized eigenvalues of $-\Delta + W - aV$ corresponding to $\lambda_j(V)$. Fix some $m \geq j$ and let $\mathbb{H}_m = Sp(\bar{\phi}_1, \ldots \bar{\phi}_m)$. Let us restrict the supremum (12) to $\mathbb{H}_m^k := \{\vec{\phi} := (\phi_1, \ldots \phi_k), \phi_j \in \mathbb{H}_m\} \subset \mathbb{H}^k$.
Then

$$\phi_j = \sum_{i=1}^m \left\langle \phi_j, \bar{\phi}_i \right\rangle \bar{\phi}_i, \quad (-\Delta + W - aV)\phi_j = \sum_{i=1}^m \lambda_i \left\langle \phi_j, \bar{\phi}_i \right\rangle \bar{\phi}_i \quad .$$

Define $\beta_{k+1} = \ldots = \beta_m = 0$. Then we can write, for any $\vec{\phi} \in \mathbb{H}_m^k$

$$\sum_{j=1}^k \beta_j \left\langle (-\Delta + W - aV)\phi_j, \phi_j \right\rangle = \sum_{i=1}^m \sum_{j=1}^m \beta_j \lambda_i | \left\langle \phi_j, \bar{\phi}_i \right\rangle |^2 \, . \tag{13}$$

Denote now $\gamma_{i,j} := | \left\langle \phi_j, \bar{\phi}_i \right\rangle |^2$. Then $\{\gamma_{i,j}\}$ is $m \times m$, bi-stochastic matrix, i.e $\sum_{i=1}^m \gamma_{i,j} = \sum_{j=1}^m \gamma_{i,j} = 1$ for all $i, j = 1 \ldots m$. Consider now the infimum of $\sum_{i=1}^m \sum_{j=1}^m \tilde{\gamma}_{i,j} \lambda_i \beta_j$ over all bi-stochastic martices $\{\tilde{\gamma}_{i,j}\}$. By Krain-Milman theorem, the minimum is obtained on an extreme point in the convex set of bi- stochastic matrices. By Birkhoff theorem, the extreme points are permutations so, from(13)

$$\forall \vec{\phi} \in \mathbb{H}^k_m, \quad \sum_{j=1}^{k} \beta_j \langle(-\Delta + W - aV)\phi_j, \phi_j\rangle \geq \sum_{j=1}^{m} \beta_{\pi(j)}\lambda_j$$

for some permutation $\pi : \{1, \dots m\} \mapsto \{1, \dots m\}$. Now, recall that β_j are assumed to be in descending order while λ_j are in ascending order by definition. By the discrete rearangment theorem of Hardy, Littelwood and Polya [**?**] we obtain that the maximum on the right above is obrained at the identity permutation $\pi(i) = i$, that is, at the identity matrix $\tilde{\gamma}_{i,j} := \langle \phi_j, \bar{\phi}_i \rangle = \delta_{i,j}$. This implies thet the eigenbasis $\bar{\phi}_1, \dots \bar{\phi}_k$ of the k leading eigenvalues is the minimizer of (12) on $\mathbb{H}^k_m$ for any $m \geq k$.
In particular, the minimizer of (12) in $\mathbb{H}^k_m$ is independent of m, as long as $m \geq k$. Suppose there exists some $\vec{\psi} \in \mathbb{H}^k$ which is not contained in and finite dimensional subspace generated by eigenstates, for which (12) is strictly smaller than its value on the first $k-$ leading eigenspace. Since the eigenstates of the Schrdinger operator under assumption generate the whole space we can find, for a sufficiently large m, an orthonormal base in $\mathbb{H}^k_m$ for on which the left side of (12) is strictly larger than $\sum_{j=1}^{k} \beta_j \lambda_j(V)$, and we get a contradiction for this value of m.

Theorem (Main)

a: *The functional* $V \mapsto \frac{a}{2}\langle V, V\rangle_{\alpha/2} + G_{\beta,a}(V)$ *is bounded from below on* $\dot{\mathbb{H}}_{\alpha/2}$ *for any* $a > 0$ *if* $3 \le n < 2+\alpha$.

b: *In the critical cases* $n = 3, \alpha = 1$ *or* $n = 4, \alpha = 2$ *there exists* $a = a_c^{(n)}(\vec{\beta}) > 0$ *independent of* W *for which the functional is bounded from below if* $a < a_c^{(n)}(\vec{\beta})$ *and unbounded if* $a > a_c^{(n)}(\vec{\beta})$.

c: *Moreover, in the cases* $n = 3$ *and* $n = 4$, $a < a_c^{(n)}(\vec{\beta})$ *the functional is coersive on* $\dot{\mathbb{H}}_{\alpha/2}(\mathbb{R}^n)$, *namely*

$$\lim_{|||V|||_{\alpha/2}\to\infty} \frac{1}{2}\langle V, V\rangle_{\alpha/2} + G_{\beta,a}(V) = \infty \tag{14}$$

and a minimizer $\bar{V}$ *exists.*

d: *If case [c] holds then there exists a solution of the Choquard system, induced by the minimizing potential* $\bar{V}$.

Proof of Main Theorem

Theorem (Lieb-Thirring)

For the Schrödinger operator $-\Delta - V$ on $\mathbb{R}^n$ with a real valued potential V the numbers $\mu_1(V) \leq \mu_2(V) \leq \cdots \leq 0$ denote the (not necessarily finite) sequence of its negative eigenvalues. Then, for $n \geq 3$ and $\gamma \geq 0$

$$\sum_{j;\mu_j(V)<0} |\mu_j(V)|^\gamma \leq L_{\gamma,n} \int V_+^{n/2+\gamma} dx \tag{15}$$

where $V_+ = \max\{0, V\}$ and $L_{\gamma,n}$ is independent of V.

Lemma (Critical Sobolev)

For $\alpha \in (0,2]$, $n > 2$, the space $\dot{\mathbb{H}}_{\alpha/2}$ is continuously embedded in $\mathbb{L}^{2n/(n-\alpha)}(\mathbb{R}^n)$, so there exists $S = S_{n,\alpha} > 0$ such that

$$\|V\|_{2n/(n-\alpha)} \leq S_{n,\alpha} \sqrt{\langle V, V \rangle_{\alpha/2}} \, .$$

Recall that $\lambda_j(V)$ are the eigenvalues of $-\Delta + W - aV$. Since $W \geq 0$ it follows that $\lambda_j(V) \geq \mu_j(aV)$. Hence $G_{\beta,a}(V) := \sum_{j=1}^k \beta_j \lambda_j(V) \geq -\sum_{j;\mu_j(aV)<0} \beta_j |\mu_j(aV)|$. By Holder inequality, for $\gamma \geq 1$, $\gamma' = \gamma/(\gamma-1)$ and (15)

$$G_{\beta,a}(V) \geq -\left(\sum_{j=1}^k |\beta_j|^{\gamma'}\right)^{1/\gamma'} \left(\sum_{j;\mu_j(aV)<0} |\mu_j(aV)|^\gamma\right)^{1/\gamma} \geq$$

$$-a^{1+n/2\gamma} L_{\gamma,n}^{1/\gamma} \|\vec{\beta}\|_{\gamma'} \left(\int V_-^{n/2+\gamma} dx\right)^{1/\gamma} .$$

Set now $\gamma = \frac{2n}{n-\alpha} - n/2 \equiv \frac{(4+\alpha)n-n^2}{2(n-\alpha)}$. Then, if $2 < n < 2+\sqrt{1+3\alpha}$ we get $\gamma'_{n,\alpha} \geq 1$ and

$$G_{\beta,a}(V) \geq -a^{\frac{4}{4+\alpha-n}} L_{\gamma,n}^{1/\gamma} \|\beta\|_{\gamma'_{n,\alpha}} \left(\int_{\mathbb{R}^n} V_+^{\frac{2n}{n-\alpha}}\right)^{\frac{2(n-\alpha)}{(4+\alpha)n-n^2}} .$$

Using the critical Sobolev inequality

$$G_{\beta,a}(V) \geq -a^{\frac{4}{4+\alpha-n}} L_{\gamma,n}^{1/\gamma} \|\beta\|_{\gamma'_{n,\alpha}} S_{n,\alpha}^{\frac{4}{(4+\alpha)-n}} \langle V, V\rangle_{\alpha/2}^{\frac{2}{(4+\alpha)-n}}$$

$$\mathcal{H}^{W,\alpha}_{\beta,a}(V) = \frac{a}{2}\langle V, V\rangle_{\alpha/2} + G_{\beta,a}(V) \geq$$

$$a\langle V, V\rangle_{\alpha/2}^{\frac{2}{4+\alpha-n}}\left(\frac{1}{2}\langle V, V\rangle_{\alpha/2}^{1-\frac{2}{4+\alpha-n}} - a^{\frac{n-\alpha}{4+\alpha-n}} L^{1/\gamma}_{\gamma,n}\|\beta\|_{\gamma'_{n,\alpha}} S_{n,\alpha}^{\frac{4}{(4+\alpha)-n}}\right) \quad (16)$$

It follows that $\mathcal{H}^{W,\alpha}_{\beta,a}$ is coersive for any $a > 0$ if $3 \leq n < 2+\alpha$. If $n = 2+\alpha$ then the functional is coersive if $a < \frac{S_{n,\alpha}^{\frac{4}{n-(4+\alpha)}}}{2L^{1/\gamma}_{\gamma,n}}|\beta|^{-1}_{\gamma'_{n,\alpha}}$. Note that $\gamma'_{n,\alpha} = \infty$ for $n = 3, \alpha = 1$ and $\gamma'_{n,\alpha} = 2$ for $n = 4, \alpha = 2$. Hence coersivity holds if

- $(\alpha, n) = (1, 3)$: $\quad a < \frac{S_{3,1}^{-2}}{2L^{1/\gamma}_{\gamma,3}}|\beta|^{-1}_{\infty} < a_c^{(3)}(\beta)$
- $(\alpha, n) = (2, 4)$: $\quad a < \frac{S_{4,2}^{-2}}{2L^{1/\gamma}_{\gamma,4}}|\beta|^{-1}_{2} < a_c^{(4)}(\beta)$

Equi-energy distribution

$$\vec{\beta} = \mathbf{1}_k := \underbrace{(k^{-1}, \dots, k^{-1})}_{k}$$

$$|\vec{\beta}|_\infty^{-1} = k \quad ; \quad |\vec{\beta}|_2^{-1} = \sqrt{k}$$

so

$$a_c^{(3)}(\mathbf{1}_k) > O(k) \quad ; \quad a_c^{(4)}(\mathbf{1}_k) > O(k^{1/2})$$

Publication

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