On s-stability of $W^{s,\frac{n}{s}}$-minimizing maps between spheres, in homotopy classes

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IASM BIRS Hangzhou
September 18, 2024

based on joint work with K. Mazowiecka

Funding: NSF Career DMS-2044898, Alexander von Humboldt Fellowship

Minimizing energies with geometry

Fix manifolds $\mathcal{N}, \mathcal{M}$, $p \in (1, \infty)$.

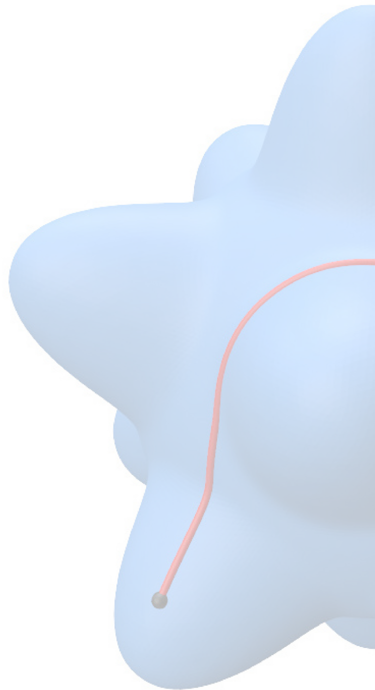
Basic Question:

What are the

- minimizers, properties of minimizers
- minimum energy

for energy

$$\int_{\mathcal{N}} |\nabla u|^p : \quad \text{subject to } u : \mathcal{N} \to \mathcal{M}$$



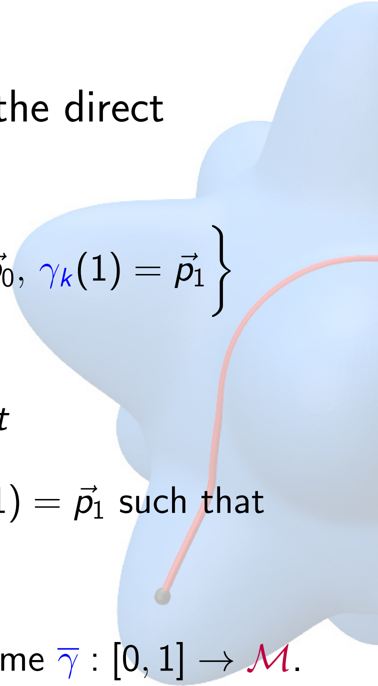
Warm-up in 1D: Minimizing curves - with boundary data

Fix manifold $\mathcal{M}$, $p \in (1, \infty)$. Find $\gamma : [0,1] \to \mathcal{M}$ that minimizes

$$\int_0^1 |\gamma'(t)|^p dt : \quad \gamma : [0,1] \to \mathcal{M}$$

subject to boundary data $\gamma(0) = \vec{p}_0$, $\gamma(1) = \vec{p}_1$.

- Minimizer exist, end of story (bit more work if $p = 1$), by the direct method of CalcVar.
  - Set
    $$X := \left\{ \gamma : [0,1] \to \mathcal{M} \text{ s.t. } \inf_{[0,1]} |\gamma'(t)|^p dt < \infty,\ \gamma_k(0) = \vec{p}_0,\ \gamma_k(1) = \vec{p}_1 \right\}$$
    Goal: find $\overline{\gamma} \in X$ such that
    $$\mathrm{INF} := \int_{[0,1]} |\overline{\gamma}'(t)|^p dt = \inf_{\gamma \in X} \int_{[0,1]} |\gamma'(t)|^p dt$$
  - Take a minimizing sequence $\gamma_k : [0,1] \to \mathcal{M}$, $\gamma_k(0) = \vec{p}_0$, $\gamma_k(1) = \vec{p}_1$ such that
    $$\mathrm{INF} = \lim_{k \to \infty} \int_{[0,1]} |\gamma_k{}'(t)|^p dt$$
    The energy is coercive, so up to subsequence convergent to some $\overline{\gamma} : [0,1] \to \mathcal{M}$.
  - Since $\overline{\gamma} \in X$ it is the minimizer, indeed:
    $$\int_{[0,1]} |\overline{\gamma}'(t)|^p dt \overset{l.s.c}{\leq} \lim_{k \to \infty} \int_{[0,1]} |\gamma_k{}'(t)|^p dt \overset{\text{minseq}}{=} \mathrm{INF} \overset{\overline{\gamma} \in X}{\leq} \int_{[0,1]} |\overline{\gamma}'(t)|^p dt$$
- These minimizers are just the geodesics (shortest curves)



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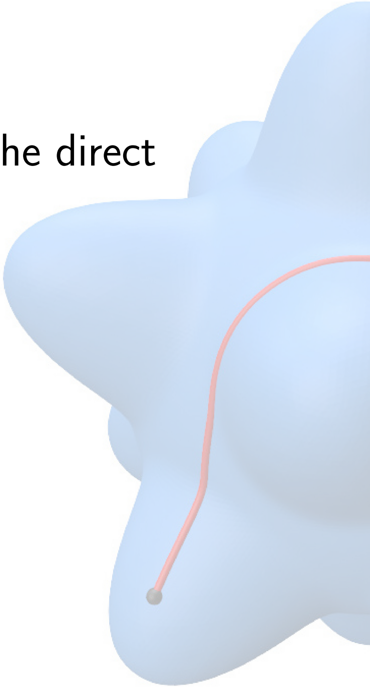
subject to ???.

- Minimizer exist, end of story (bit more work if $p = 1$), by the direct method of CalcVar.
- These minimizers are just the geodesics (shortest curves)
- If we are in the periodic setting "closed curves", i.e.

$$\gamma : \mathbb{S}^1 \to \mathcal{M},$$

  minimizers all constant and minimum energy is 0



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subject to topology.

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- If we are in the periodic setting "closed curves", i.e.

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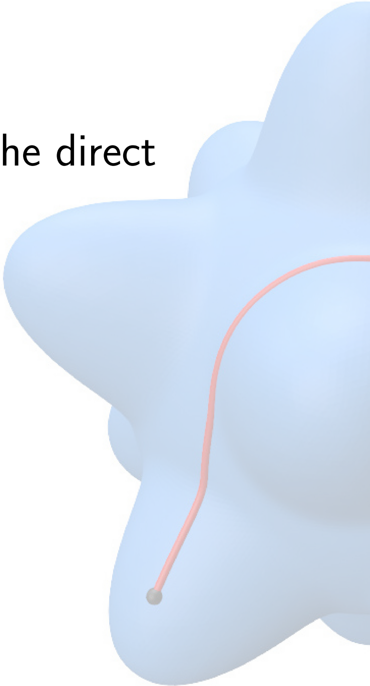
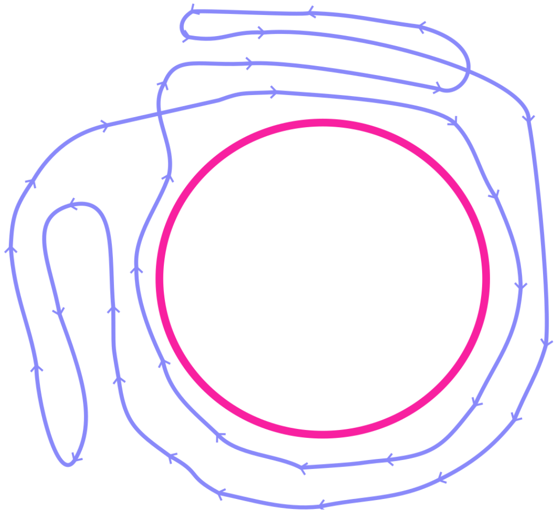
  minimizers all constant and minimum energy is 0 (boring)
- So let us introduce some topology:



Winding number a.k.a. degree

Let $\gamma : \mathbb{S}^1 \to \mathbb{S}^1$ continuous. Draw its image with orientation (clockwise).



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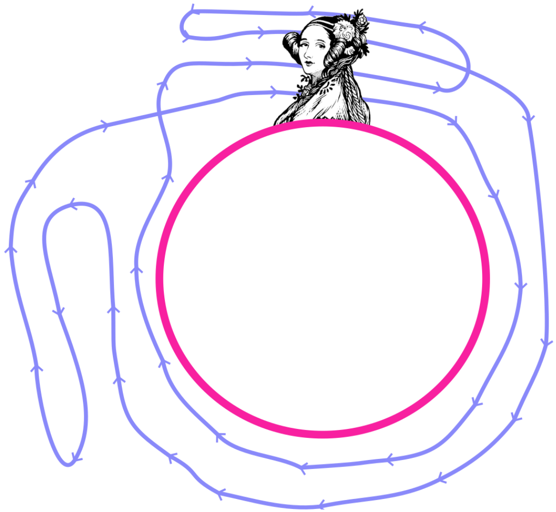

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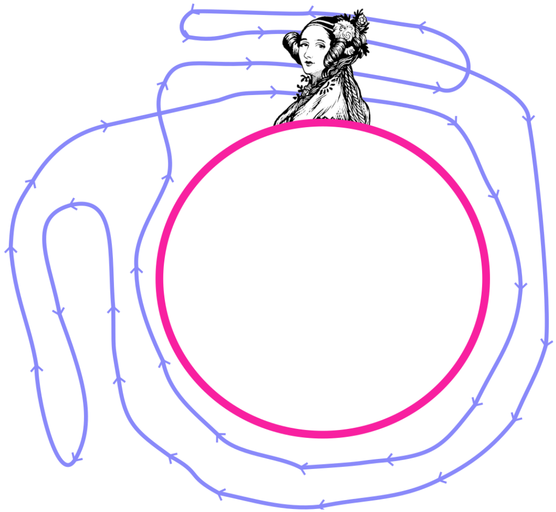

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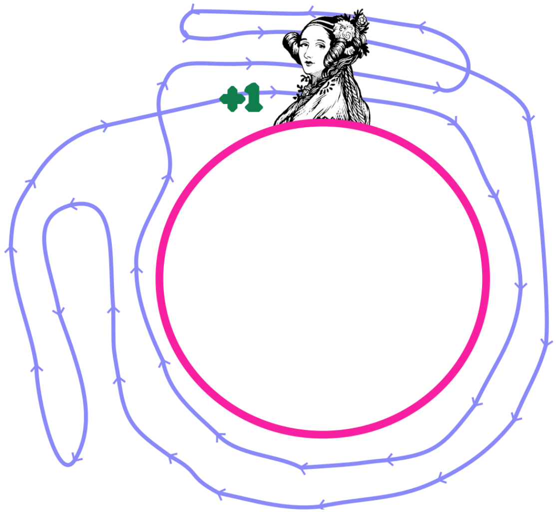

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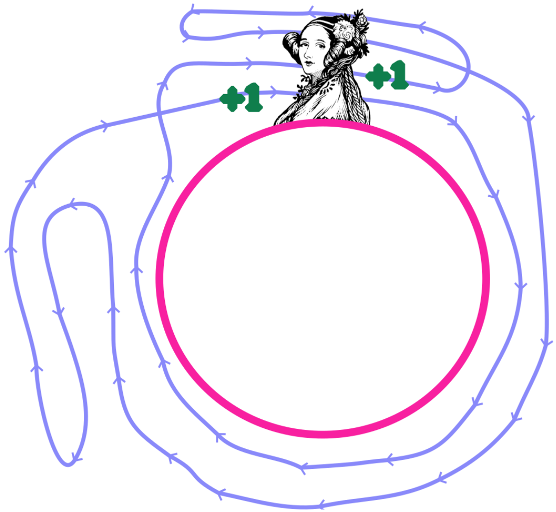

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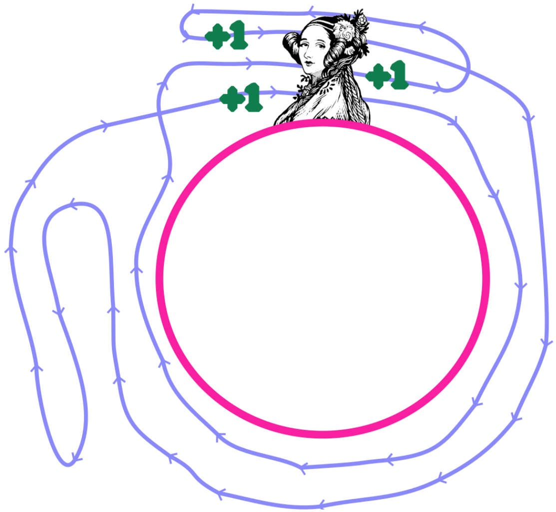

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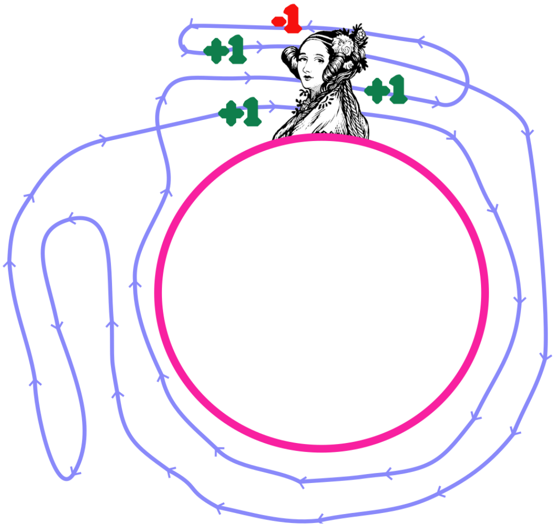

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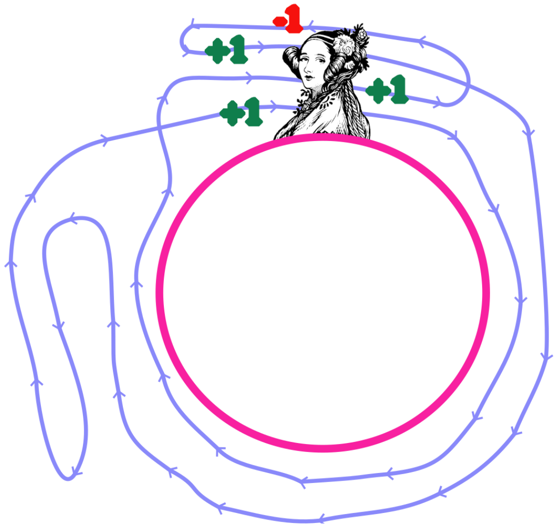

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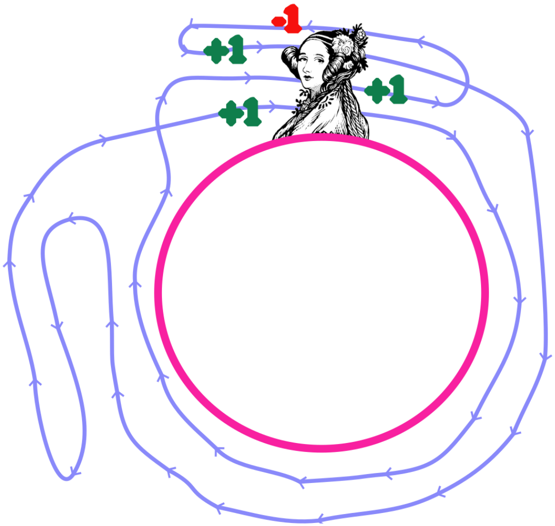

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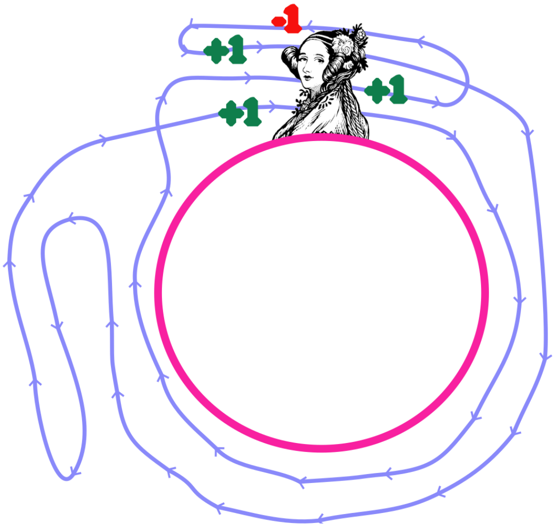

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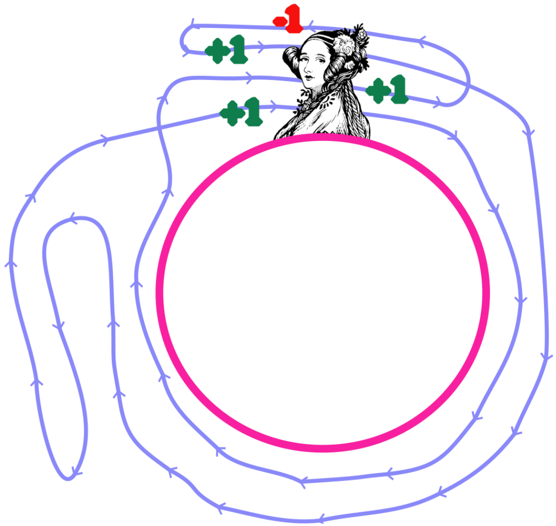

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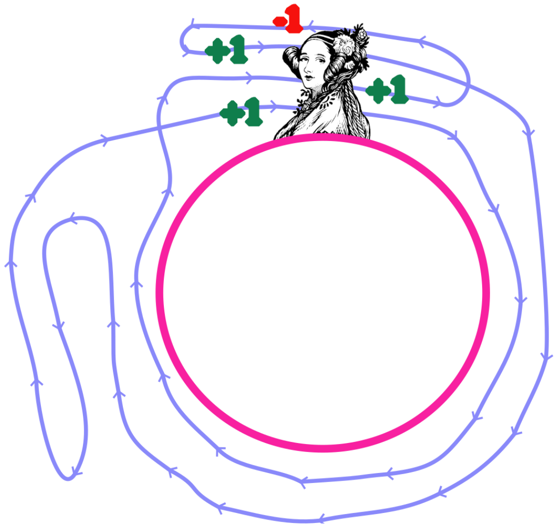

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- For fixed γ: If $\tilde{\gamma}$ is uniformly close to γ, then winding numbers are the same.
- More generally: Homotopy groups: $\alpha \in \pi_n(\mathcal{M})$ all maps $f : \mathbb{S}^n \to \mathcal{M}$ with $[f] = \alpha$.

Minimizing maps – with topology

Minimizing maps: Fix $\mathcal{M}$ compact manifold, $\alpha \in \pi_n(\mathcal{M})$.

$$\#_1\alpha := \min \int_{\mathbb{S}^n} |\nabla u|^p : \quad \text{s.t. } u : \mathbb{S}^n \to \mathcal{M},\ [u] \in \alpha \tag{H_1}$$

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  - Take a minimizer $u : \mathbb{S}^n \to \mathcal{M}$ in α (assume it exists)
  - We can mess with u without changing the energy:
  - Take $\tau_k : \mathbb{S}^n \to \mathbb{S}^n$ that maps most of the domain $\mathbb{S}^n$ to $\{\text{north pole} \pm \frac{1}{k}\}$
  - Consider the new minimizing map

    $$u_k := u \circ \tau_k \in \alpha$$

    but $u_k \xrightarrow{k\to\infty} const$ (i.e. it leaves the homotopy class).
  - We can choose τ_k conformal, the energy is conformally invariant: energy of $u \circ \tau_k$ is same as energy of u.
  - These "bubbles" could appear for any minimizing sequence!

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[1] $p = n = 2$: Sacks-Uhlenbeck. Many generalizations, Struwe,White, Kawai, Nakauchi, Takeuchi, Kuwert, Duzaar and many more: n-harmonic, polyharmonic. . .

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  - There exists a *generating set* $\{\alpha_1, \ldots, \alpha_N\} \subset \pi_n(\mathcal{M})$: minimizer of (H_1) exists for each α_i
  - Energy identity: for any $\alpha \in \pi_2(\mathcal{M})$ there exists $\beta_1, \ldots, \beta_L \in \pi_2(\mathcal{M})$ s.t.

$$\#_1\alpha = \sum_{j=1}^{L} \#_1\beta_j, \quad \text{and} \quad \alpha = \sum_{j=1}^{L} \beta_j$$

and $\#_1\beta_j$ is attained.

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    and $\#_1\beta_j$ is attained.
  - There are indeed examples of α where minimizers do not exist (FUTAKI)

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Question: How stable are these results as s changes?

- Fix α. If s-minimizer is attained, what about $\tilde{s} \approx s$ and the $\tilde{s}$-minimizer?
- Is $s \mapsto \#_s\alpha$ continuous?

Why not $W^{s,n/s}$-harmonic – a continuum of Energies

Minimizing maps: Fix $\mathcal{M}$ compact manifold, $\alpha \in \pi_n(\mathcal{M})$, $s \in (0,1)$

$$\#_s\alpha := \min \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}}}{|x-y|^{2n}}\, dx\, dy : \quad \text{s.t. } u \in \alpha \qquad (H_s)$$

The energy on the right is the $W^{s,\frac{n}{s}}$-seminorm, it is still conformally invariant.

We have Sacks-Uhlenbeck theory (Mazowiecka-S. 2023)

- There exists a *generating* set $\{\alpha_1,\ldots,\alpha_N\} \subset \pi_n(\mathcal{M})$ such that minimizer of (H_1) exists for each α_i
- Energy identity for any $\alpha \in \pi_n(\mathcal{M})$ there exists $\beta_1,\ldots,\beta_L \in \pi_n(\mathcal{M})$ s.t.

$$\#_s\alpha = \sum_{j=1}^{L} \#_s\beta_j \quad \text{and} \quad \alpha = \sum_{j=1}^{L} \beta_j$$

  and $\#_s\beta_j$ is attained.

Question: How stable are these results as s changes?

- Fix α. If s-minimizer is attained, what about $\tilde{s} \approx s$ and the $\tilde{s}$-minimizer?
- Is $s \mapsto \#_s\alpha$ continuous?
- Is the Sacks-Uhlenbeck generating set $\{\alpha_1,\ldots,\alpha_N\}$ the same for all s?

What to expect: mappings between spheres

Summary of what we know:

- $\mathbb{S}^2 \to \mathbb{S}^2$, $W^{1,2}$-minimizer exist for any degree[2]
- $\mathbb{S}^n \to \mathbb{S}^n$, $n \geq 3$, $W^{1,n}$-minimizer only exist at degree $1, -1, 0$[3]
- $\mathbb{S}^3 \to \mathbb{S}^2$: $W^{1,3}$ there exists infinitely many homotopy classes where minimizers are attained (Rivière)[4]
- $\mathbb{S}^1 \to \mathbb{S}^1$: $W^{\frac{1}{2},2}$: minimizers are attained for any degree Berlyand, Mironescu, Pisante, Rybalko, Sandier[5]

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[3] reason: minimizer are conformal

[4] power law for the energy w.r.t. Hopf degree

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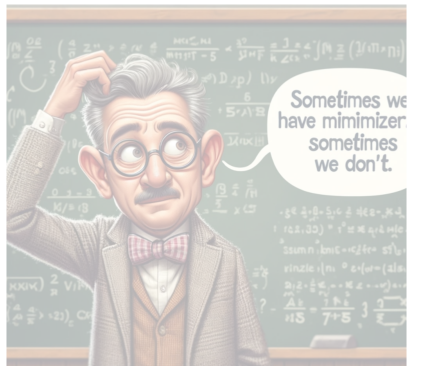

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Theorem (Mironescu)

There exists $\delta > 0$ such that degree 1 minimizer exists in $W^{s,\frac{1}{s}}(\mathbb{S}^1, \mathbb{S}^1)$ for $s \in [1/2, 1/2 + \delta]$.

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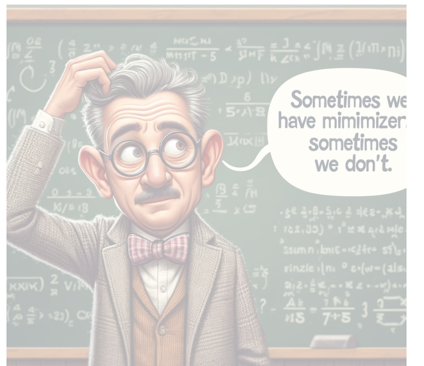

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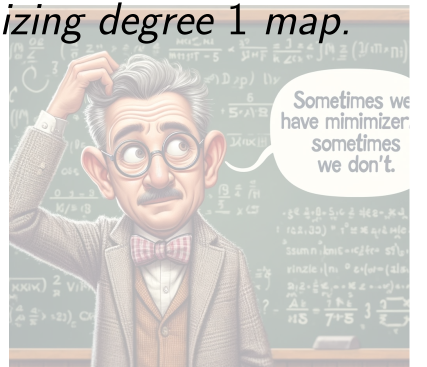
Conjecture (Natural conjecture?)

For any $s \in (0, 1]$, for any $n \in \mathbb{N}$, there exists a $W^{s,\frac{n}{s}}$-minimizing degree 1 map.



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continuous dependence of minimal energy

Theorem (Mazowiecka-S. (2023))

Fix $\alpha \in \pi_n(\mathbb{S}^\ell)$, i.e. consider maps $u : \mathbb{S}^n \to \mathbb{S}^\ell$. Then

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- By smooth approximation we get

$$\#_s \alpha \geq \limsup_{t \to s} \#_t \alpha$$

- Observe if $[u]_{W^{t,\frac{n}{t}}} < \infty$ then not necessarily $[u]_{W^{s,\frac{n}{s}}} < \infty$ for $s > t$!
- Proof is based on a new conformal regularity theorem (more: later).
- Let us first discuss some immediate consequences

Consequences (1): Progress on Mironescu's problem

Theorem (Mironescu)

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Contradiction.

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(Works with minimal energy homotopy classes – any dimension)

Consequences (2): Stability of generating homotopy groups

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For each s there exists generating set $X_s = \{\alpha_1, \ldots, \alpha_N\} \subset \pi_n(\mathbb{S}^\ell)$ *such that* $\#_s \alpha_i$ *is attained for each* $i = 1, \ldots, N$.

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Proof.

If $\#_t \alpha$ is not attained for some α and some $t \approx s$ then there exists (depending on t) generator $(\alpha_i)_{i=1}^N$ of α

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- We can repeat this argument for α_i on the right hand side, whenever for some $t \approx s$ there is no minimizer for α_i
- no term from the left-hand side can reappear again on the right-hand side
- finitely many choices of α_i, eventually stop or contradiction.

∎

Proof of continuity

Theorem (Mazowiecka-S. (2023))

Fix $\alpha \in \pi_n(\mathbb{S}^\ell)$. Then $s \mapsto \#_s\alpha$ is continuous.

Proof.

- Assume $u : \mathbb{S}^n \to \mathbb{S}^\ell$ is a $W^{s,\frac{n}{s}}$-minimizer of $\#_s\alpha$.
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- Interchanging roles of s and t we conclude.

∎

Conformal higher regularity for minimizers

Theorem (S.'15, Mazowiecka-S.'18)

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- This is usually ε-regularity: if $[u]_{W^{s,\frac{n}{s}}(B(0,R))} \ll 1$ then $u \in C^\alpha(B(0,R/2))$ with suitable estimates
- No way we have uniform higher regularity: Indeed, take any minimizer, rescale it conformally (almost bubble), the modulus of continuity is arbitarily bad.
- Idea: Obtain better regularity for another conformally invariant energy.

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- We show this for critical points, not only minimizers.
- No idea how to use minimizing property (no ε-regularity result!)
- Can't do it for general target manifolds

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- For classical $W^{1,2}$-harmonic maps $\mathbb{S}^2 \to \mathbb{S}^\ell$:

$$-\Delta u = u|\nabla u|^2$$

- Hèlein, Coifman-Lions-Meyer-Semmes:

$$u|\nabla u|^2 \in \mathcal{H}^1(\mathbb{R}^2).$$

-

$$\|\nabla^2 u\|_{L^1} \lesssim \|\Delta u\|_{\mathcal{H}^1} \lesssim \|\nabla u\|_{L^2}^3$$

This is a global (scaling-invariant!) estimate!

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Theorem (S., Mazowiecka-S.)

Critical points of the $[u]_{W^{s,\frac{n}{s}}}$-energy into spheres $\mathbb{S}^\ell$, then

$$[u]_{W^{s_0,\frac{n}{s_0}}(\mathbb{S}^n)} \lesssim C([u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n)}) \quad \text{for some } s_0 > s.$$

- Euler-Lagrange equations

$$(-\Delta)^{s}_{\frac{n}{s},\mathbb{S}^n} u \perp T_u \mathbb{S}^\ell.$$

where

$$(-\Delta)^{s}_{\frac{n}{s},\mathbb{S}^n} u[\varphi] =$$

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|u(x)-u(y)|^{\frac{n}{s}-2}(u(x)-u(y))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+sp}}\, d(x,y)$$

- In S.'16, I rewrote this as (for $t < s$)

$$(-\Delta)^{s}_{\frac{n}{s},\mathbb{S}^n} u = (-\Delta)^{\frac{t}{2}} T_t u$$

where $T_t v(z)$ roughly corresponds to $|\sqrt{(-\Delta)}^{\frac{sp-t}{p-1}} v|^{p-1}$,

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))\left(|x-z|^{t-n}-|y-z|^{t-n}\right)}{|x-y|^{n+sp}}\, d(x,y)$$

Conformal higher regularity for minimizers

Theorem (S., Mazowiecka-S.)

Critical points of the $[u]_{W^{s,\frac{n}{s}}}$-energy into spheres, then

$$[u]_{W^{s_0,\frac{n}{s_0}}(\mathbb{S}^n)} \precsim C([u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n)}) \quad \textit{for some } s_0 > s.$$

- Euler-Lagrange equations

$$(-\Delta)^{\frac{s}{2}} T_s u \perp T_u \mathbb{S}^\ell.$$

We observe even though $t < s$, since $T_t u$ is "somewhat tangential"

$$\|u \cdot T_t u\|_{L^{\frac{n}{n-t}}} \precsim C([u]_{W^{s,\frac{n}{s}}})$$

and by the PDE and compensation phenomena we have

$$\|u \wedge T_{t,\Omega} u\|_{L^{\frac{n}{n-t}}} \precsim C([u]_{W^{s,\frac{n}{s}}})$$

Thus

$$\|T_t u\|_{L^{\frac{n}{n-t}}} \precsim C([u]_{W^{s,\frac{n}{s}}})$$

- Iwaniec' stability then implies $u \in W^{r,\frac{n}{r}}(\mathbb{S}^n)$ for $r := s\frac{n-t}{n-s}$ with the corresponding estimate.

Summary

- Conformal higher regularity: Critical $W^{s,\frac{n}{s}}$-harmonic maps *into spheres* belong to $W^{s_0,\frac{n}{s_0}}$,

$$[u]_{W^{s_0,\frac{n}{s_0}}(\mathbb{S}^n)} \lesssim C([u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n)}) \quad \text{for some } s_0 > s.$$

Summary

- Conformal higher regularity: Critical $W^{s,\frac{n}{s}}$-harmonic maps *into spheres* belong to $W^{s_0,\frac{n}{s_0}}$,

$$[u]_{W^{s_0,\frac{n}{s_0}}(\mathbb{S}^n)} \lesssim C([u]_{W^{s,\frac{n}{s}}(\mathbb{S}^n)}) \quad \text{for some } s_0 > s.$$

- $\Rightarrow$ for any $\alpha \in \pi_n(\mathbb{S}^\ell)$

$$s \mapsto \#_s\alpha \equiv \inf_{[u]\in\alpha} [u]^{\frac{n}{s}}_{W^{s,\frac{n}{s}}(\mathbb{S}^n,\mathbb{S}^\ell)}$$

is continuous

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- $\Rightarrow$ If for some $\bar{s}$ and $\alpha \in \pi_n(\mathbb{S}^\ell)$ we have

$$\#_{\bar{s}}\alpha \leq \#_{\bar{s}}\beta \quad \text{for all } \beta \in \pi_n(\mathbb{S}^\ell) \setminus \{0\}$$

  Then $\#_s\alpha$ is attained for all $s \approx \bar{s}$

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  Then $\#_s\alpha$ is attained for all $s \approx \bar{s}$
- $\Rightarrow$ For any $\bar{s}$ there exists a generating set $\{\alpha_1, \ldots, \alpha_k\} \in \pi_n(\mathbb{S}^\ell)$ such that

$$\#_s\alpha_i \quad \text{is attained for all } s \approx \bar{s}$$

Things to do

- It would be very interesting to investigate the stability $s \to 1^-$ (for spheres this *might* be doable)
- What about $s \to 0^+$?
- What about general manifolds? Higher *conformal* regularity for *minimizers* into general manifolds?

Thank you for your attention

- Mazowiecka, S.: Minimal $W^{s,\frac{n}{s}}$-harmonic maps in homotopy classes (J. Lond. Math. Soc., 2023)
- Mazowiecka, S.: s-stability for $W^{s,n/s}$-harmonic maps in homotopy groups (preprint)