Dynamics of a nonlocal reaction-diffusion problem with memory[1]

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- J. Xu, T. Caraballo, J. Valero. *Asymptotic behavior of a semilinear problem in heat conduction with long time memory and non-local diffusion*, Journal of Differential Equations 327 (2022), 418447.
- J. Xu, T. Caraballo, J. Valero. *Asymptotic behavior of nonlocal partial differential equations with long time memory*. Discrete and Continuous Dynamical Systems, Series S (2022) doi:10.3934/dcdss.2021140.

Content

Introduction and overview

Main objective: To analyze a reaction-diffusion equation containing some nonlocal character as well as memory terms.

• Importance of the effects that memory terms (or the past history of a phenomenon) produce on the evolution of a dynamical system

1. T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuß, J. Valero, Global attractor for a non-autonomous integro-differential equation in materials with memory, *Nonlinear Analysis* 73 (2010), 183-201.
2. T. Caraballo, J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differential Equations* 205 (2004), 271-297.
3. M. Conti, V. Pata, M. Squassina, Singular limit of differential systems with memory, *Indiana U. Math. J.* 1 (2006), 169-215.
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6. C. Giorgi, Vittorino Pata, A. Marzochi, Asymptotic behavior of a semilinear problem in heat conduction with memory, *Nonlinear Differ. Equ. Appl.* 5 (1998), 333-354.

Introduction and overview

- Importance of nonlocal PDE:

1. P. M. Berná, J. D. Rossi, Nonlocal diffusion equations with dynamical boundary conditions, Nonlinear Anal,. 195 (2020), 111751.
2. Z. Szymańska, C. Morales-Rodrigo, M. Lachowicz, M. A. J. Chaplain, Mathematical modelling of cancer invasion of tissue: the role and effect of nonlocal interaction, Math. Models Methods Appl. Sci., 19 (2009), 257-281.
3. N. I. Kavallaris, Explosive solutions of a stochastic non-local reaction-diffusion equation arising in shear band formation, Math. Meth. Appl. Sci., 38 (2015), 3564-3574.
4. Chipot et al. (Rend. Sem. Mat. Univ. Padova, 110 (2003), 199-220; RAIRO Modél. Math. Anal. Numér., 26 (1992), 447-467; Asymptot. Anal., 45 (2005), 301-312.): population of bacteria with nonlocal term $a(\int_\Omega u)$ in a container, extended to a general nonlocal operator $a(l(u))$, where $l \in \mathcal{L}(L^2(\Omega);\mathbb{R})$, for instance, if $g \in L^2(\Omega)$, $l(u) = l_g(u) = \int_\Omega g(x)u(x)dx$.

$$\frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u), \tag{1}$$

5. P. Marín-Rubio, M. Herrera-Cobos, T.C: non-autonomous versions and their global dynamics (Proc. Roy. Soc. Edinburgh Sect. A 148 (2018), no. 5, 957981; Discrete Contin. Dyn. Syst. Ser. B 23 (2018), no. 3, 10111036; J. Math. Anal. Appl. 459 (2018), no. 2, 9971015; Discrete Contin. Dyn. Syst. Ser. B 22 (2017), no. 5, 18011816).

Introduction and overview

Motivated by some physical problems from thermal memory or materials with memory, V. Pata and collaborators studied a semilinear partial differential equation to model the heat flow in a rigid, isotropic, homogeneous heat conductor with linear memory,

$$\begin{cases} c_0\partial_t u - k_0\Delta u - \displaystyle\int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = h, & \text{in } \Omega\times(\tau,+\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega\times(\tau,+\infty), \\ u(x,\tau+t) = u_0(x,t), & \text{in } \Omega\times(-\infty,0], \end{cases} \quad (2)$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with regular boundary, $u:\Omega\times\mathbb{R}\to\mathbb{R}$ is the temperature field, $k:\mathbb{R}^+\to\mathbb{R}$ is the heat flux memory kernel, $\mathbb{R}^+$ denotes the interval $(0,+\infty)$, c_0 and k_0 denote the specific heat and the instantaneous conductivity, respectively.

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To solve (2) successfully, they made the past history of u from $-\infty$ to 0^- be part of the forcing term given by the causal function g, which is defined by

$$g(x,t) = h(x,t) + \int_{-\infty}^{\tau} k(t-s)\Delta u_0(x,s)ds, \qquad x \in \Omega, \quad t \geq \tau.$$

Thus, (2) becomes an initial value problem without delay or memory,

$$\begin{cases} c_0 \partial_t u - k_0 \Delta u - \int_{\tau}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \Omega \times (\tau, +\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (\tau, +\infty), \\ u(x,\tau) = u_0(x,0), & \text{in } \Omega. \end{cases} \tag{3}$$

But, it does not generate a dynamical system ((3) depends on the past history and we just fix an initial value at time τ).

Therefore, two alternatives are possible.

• Alternative 1: Based on Dafermos' idea, for linear viscoelasticity, in the 70's. Define new variables,

$$u^t(x,s) = u(x,t-s), \qquad s \geq 0, \quad t \geq \tau,$$

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^t u(x,r)dr, \qquad s \geq 0, \quad t \geq \tau. \quad (4)$$

Assume $k(\infty) = 0$, a ch. of variable and a formal integ. by parts

$$\int_{-\infty}^t k(t-s)\Delta u(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds.$$

Setting

$$\mu(s) = -k'(s),$$

the original eq. (2) becomes an autonomous system without delay,

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$$\begin{cases} c_0\frac{\partial u}{\partial t} - k_0\Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = h, & \text{in } \ \Omega\times(\tau,\infty), \\ \eta_t^t(s) = -\eta_s^t(s) + u(t), & \text{in } \ \Omega\times(\tau,\infty)\times\mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \ \partial\Omega\times\mathbb{R}\times\mathbb{R}^+, \\ u(x,\tau) = u_0(0), & \text{in } \ \Omega, \\ \eta^\tau(x,s) = \eta_0(s), & \text{in } \ \Omega\times\mathbb{R}^+, \end{cases} \tag{5}$$

where, η_s^t denotes the distributional derivative of $\eta^t(s)$ with respect to the internal variable s. From the definition of $\eta^t(x,s)$ (see (4)) we have

$$\eta_0(s) = \int_{\tau-s}^{\tau} u(r)dr = \int_{\tau-s}^{\tau} u_0(r-\tau)dr = \int_{-s}^{0} u_0(r)dr, \tag{6}$$

which is the initial integrated past history of u with vanishing boundary. Consequently, any solution to (2) is a solution to (5) for the corresponding initial values $(u_0(0), \eta_0)$ given by (6).

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However, problem (5) can be solved for arbitrary initial values (u_0, η_0) in a proper phase space $L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$, i.e., the second component η_0 does not necessarily depend on $u_0(\cdot)$.
Here $L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ is defined as follows:

Let μ satisfy the hypotheses:

(h_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0, \quad \forall s \in \mathbb{R}^+;$

(h_2) $\mu'(s) + \delta\mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+,$ for some $\delta > 0$.

Then $L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ is the Hilbert space of functions $w : \mathbb{R}^+ \to H^1_0(\Omega)$ endowed with the inner product,

$$((w_1, w_2))_\mu = \int_0^\infty \mu(s)(\nabla w_1(s), \nabla w_2(s))ds.$$

Then, the solutions of (5) permits to construct a dynamical system $S(t) : L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega)) \to L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$

$$S(t)(u_0, \eta_0) = (u(t), \eta^t).$$

and prove the existence of global attractors in this phase space.

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Notice:

- The transformed equation (5) is a generalization of problem (2).
- Not every solution to equation (5) possesses a corresponding one to (2).
- Both problems are equivalent if and only if the initial value η_0 belongs to a proper subspace of $L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$: the domain of the distributional derivative with respecto to s, denoted by $D(\mathbf{T})$.

$$D(\mathbf{T}) = \left\{\eta(\cdot) \in L^2_\mu(\mathbb{R}^+; H^1_0(\Omega)) \mid \eta_s(\cdot) \in L^2_\mu(\mathbb{R}^+; H^1_0(\Omega)),\ \eta(0) = 0\right\},$$

and $\mathbf{T}$ is defined by $\mathbf{T}\eta = -\eta_s, \qquad \eta \in D(\mathbf{T})$.

- Hence, it seems natural to construct a DS generated by (5) in $L^2(\Omega) \times D(\mathbf{T})$ and to prove the existence of attractors to the original problem, via the above relationship.
- Up to our knowledge, not possible to prove the existence of attractors in this space unless solutions are proved to have more regularity.
- Thus, we cannot (do not know how to) transfer the existence of attractors for system (5) to the original problem (2).

• Alternative 2: The idea comes from a simpler case, T. Caraballo et al. *Nonlinear Analysis* (2010) when the kernel is $k(t) = e^{-d_0 t}$, $d_0 > 0$ (non-singular kernel).

- It is proved that generates a dynamical system in the phase space $L^2_{H^1_0}$ given by $\varphi : (-\infty, 0] \to H^1_0(\Omega)$, such that $\int_{-\infty}^0 e^{\gamma s} \|\varphi(s)\|^2_{H^1_0} ds < +\infty$, for certain $\gamma > 0$.
- In this phase space, there exists a global attractor to this problem (in fact, the problem is non-autonomous and the attractor is of pullback type).
- For this kind of delay problems, the initial value at zero may not be related to the values for negative times.
- So (G. Sell's suggestion) the standard and more appropriate phase space is the cartesian product $L^2(\Omega) \times L^2_{H^1_0}$ (T.C & J. Real (2004)).

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For any initial values $u_0 \in L^2(\Omega)$ and $\varphi \in L^2_{H_0^1}$, there exists a unique solution to the following problem (set $\tau = 0$),

$$\begin{cases} c_0 \frac{\partial u}{\partial t} - k_0 \Delta u - \int_{-\infty}^{t} k(t-s)\Delta u(s) ds + f(u) = g, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = \varphi(x,t), & \text{in } \Omega \times (-\infty, 0). \end{cases} \tag{7}$$

We can define a dynamical system $S(t) : L^2(\Omega) \times L^2_{H_0^1} \to L^2(\Omega) \times L^2_{H_0^1}$ by the relation

$$S(t)(u_0, \varphi) := (u(t; 0, u_0, \varphi), u_t(\cdot; 0, u_0, \varphi)),$$

where $u(\cdot; 0, u_0, \varphi)$ denotes the solution of problem (7) (T.C & J. Real, JDE (2004)), and u_t denotes the history up to time t:

$$u_t(s; 0, u_0, \varphi)) = u(t+s; 0, u_0, \varphi)), \quad s \leq 0.$$

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- We emphasize: the two components of DS are the current state of the solution and the past history up to present– more sensible in a problem with delays or memory.
- The method in T. Caraballo et al. *Nonlinear Analysis* (2010) can be successfully applied to prove the existence of attractors to problem (7) when k is of exponential type (non-singular kernel– JDE 2021 for non-local terms).

It is a big restriction on the kernel k (and consequently, on μ): real situations often have singularities, e.g. $\mu(t) = e^{-d_0 t} t^{-\alpha}, \alpha \in (0,1)$. Our aim is to handle it in the phase space $L^2(\Omega) \times L^2_{H_0^1}$. (Already studied in the space $L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))$ in the paper DCDS-S (2022) for non-local case).

- We will obtain this result as a consequence of the analysis performed in this paper even for the more general case of non-local problems as described below.

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Chipot et al. studied a population of bacteria with non-local term $a(\int_\Omega u)$ in a container. Later, extended to a non-local operator $a(l(u))$, where $l \in \mathcal{L}(L^2(\Omega); \mathbb{R})$, for instance, if $g \in L^2(\Omega)$,

$$l(u) = l_g(u) = \int_\Omega g(x)u(x)dx.$$

Thus, we combined in (J.Xu et al. JDE 2021) the non-local feature with the memory or delay effects to study the dynamics of the following non-autonomous non-local PDE with delay and memory by using the Galerkin method and energy estimations,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u = f(u) + h(t, u_t) & \text{in} \quad \Omega \times [\tau, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times [\tau, \infty), \\ u_\tau(x, \theta) = \varphi(x, \theta) & \text{in} \quad \Omega \times (-\rho, 0], \end{cases} \tag{8}$$

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- $\Omega \subset \mathbb{R}^N$ is a bounded open set, $\tau \in \mathbb{R}$,
- $a \in C(\mathbb{R};\mathbb{R}^+)$ is locally Lipschitz s.t. $0 < m \leq a(s)$ for all $s \in \mathbb{R}$,
- $f \in C(\mathbb{R})$ and h contains hereditary characteristics and delays.
- $0 < \rho \leq \infty$, which includes bounded and unbounded delays.
- The functions $u_t : (-\infty, 0] \to X$ defined by

$$u_t(\theta) = u(t+\theta), \qquad \theta \in (-\infty, 0]$$

- Typical situations of delay and memory included:

$$h(t, u_t) = G(u(t-\tau(t))), \int_{-\rho}^{0} k(-s)\Delta u(t+s)\, ds, \int_{-\rho}^{0} k(t+s)u(t+s)ds$$

$$h(t, \phi) = G(\phi(-\tau(t))), \int_{-\rho}^{0} k(-s)\Delta\phi(s)\, ds, \int_{-\rho}^{0} k(t+s)\phi(s)ds$$

- BUT only valid for non-singular kernels (e.g., $k(t) = k_1 e^{-d_0 t}$, $k_1 \in \mathbb{R}, d_0 > 0$)

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A new model with long time memory and non-local diffusion,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \int_{-\infty}^{t} k(t-s)\Delta u(s)ds + f(u) = g, & \text{in } \Omega \times (\tau, \infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(t+\tau) = \varphi(t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{9}$$

$\Omega \subset \mathbb{R}^N$ bounded domain with regular boundary, function $a \in C(\mathbb{R}; \mathbb{R}^+)$ satisfies

$$0 < m \leq a(r), \qquad \forall r \in \mathbb{R}. \tag{10}$$

$k : \mathbb{R}^+ \to \mathbb{R}$ with or without singularities, $g \in L^2(\Omega)$.
The memory term in (9) can be interpreted as an infinite delay,

$$\begin{aligned} h(u_t) &:= \int_{-\infty}^{0} k(-s)\Delta u_t(x,s)ds = \int_{-\infty}^{0} k(-s)\Delta u(x,t+s)ds \\ &= \int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds. \end{aligned} \tag{11}$$

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- This model is an autonomous non-local PDE with memory (can be done for non-autonomous as well).
- In DCDS-S (2022) we proved the existence and uniqueness of solutions to (9) by Dafermos transformation.
- Next, constructed an autonomous DS in the phase space $L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ and proved the existence of a global attractor in this space.
- As in the local heat equation, the same lack of enough regularity does not allow us to obtain an appropriate attractor for the original problem (9) in the phase space $L^2(\Omega) \times L^2_{H^1_0}$.
- Our aim is to overcome this difficulty proceeding in this way.

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- Idea of the procedure:
  - Consider problem (9) with initial values $u(\tau) = u_0$ and $u(t+\tau) = \varphi(t)$ for $t < 0$, where $(u_0, \varphi) \in L^2(\Omega) \times L^2_{H_0^1}$.
  - For those kernels $\mu(\cdot)$ guaranteeing that, when $\varphi \in L^2_{H_0^1}$ the corresp. η_φ, given by $\eta_\varphi(s) = \int_{-s}^0 \varphi(r)\,dr,\ (s > 0)$ belongs to $L^2_\mu(\mathbb{R}^+; H_0^1)$, use Dafermos to obtain an IVP as in DCDS-S (2022).
  - Consequently, we have the existence, uniqueness and regularity of solutions in a straightforward way.
  - Thanks to this result, we construct the dynamical system in the phase space $L^2(\Omega) \times L^2_{H_0^1}$ thanks to some additional technical results.
  - The existence of global attractor is proved thanks to the existence of a bounded absorbing set and the asymptotic compactness property (appropriate adaptation of technique in Nonlinear Anal. (2010)).
  - These results improve Nonlinear Anal. (2010) when a is const.
  - Also improve the previous literature on the local case (V. Pata et al.), where it is only provided the existence of attractors for the transformed equation (5) but not for the original one (2).

Well-posedness to a non-local PDE with memory

Consider the non-local PDE associated with singular memory

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \int_{-\infty}^{t} k(t-s)\Delta u(x,s)ds + f(u) = g, & \text{in } \Omega \times (\tau, \infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,\tau) = u_0(x), & \text{in } \Omega \\ u(x,t+\tau) = \phi(x,t), & \text{in } \Omega \times (-\infty, 0], \end{cases} \tag{12}$$

where $\Omega \subset \mathbb{R}^N$ is a fixed bounded domain with regular boundary. The function $a \in C(\mathbb{R};\mathbb{R}^+)$ satisfies

$$0 < m \leq a(r), \qquad \forall r \in \mathbb{R}, \tag{13}$$

$k : \mathbb{R}^+ = (0, +\infty) \to \mathbb{R}$ is the memory kernel, whose properties will be specified later. The initial values are $u_0 \in L^2(\Omega)$ and $\phi \in L^2_{H_0^1(\Omega)}$.

Well-posedness to a non-local PDE with memory

Recalling the new variables

$$u^t(x,s) = u(x,t-s), \qquad s \geq 0,$$

$$\eta^t(x,s) = \int_0^s u^t(x,r)dr = \int_{t-s}^t u(x,r)dr, \qquad s \geq 0. \qquad (14)$$

Assuming $k(\infty) = 0$, a formal integration by parts yield

$$\int_{-\infty}^t k(t-s)\Delta u(s)ds = -\int_0^\infty k'(s)\Delta\eta^t(s)ds.$$

Setting

$$\mu(s) = -k'(s), \qquad (15)$$

Well-posedness to a non-local PDE with memory

we obtain the problem,

$$\begin{cases} \frac{\partial u}{\partial t} - a(l(u))\Delta u - \int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g, & \text{in } \Omega \times (\tau,\infty), \\ \frac{\partial}{\partial t}\eta^t(s) = u - \frac{\partial}{\partial s}\eta^t(s), & \text{in } \Omega \times (\tau,\infty) \times \mathbb{R}^+, \\ u(x,t) = \eta^t(x,s) = 0, & \text{on } \partial\Omega \times \mathbb{R} \times \mathbb{R}^+, \\ u(x,\tau) = u_0(x), & \text{in } \Omega, \\ \eta^\tau(x,s) = \eta_0(x,s), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \qquad (16)$$

where, by the definition of $\eta^t(x,s)$ (see (14)), it obviously follows

$$\eta^\tau(x,s) = \int_{\tau-s}^{\tau} u(x,r)dr = \int_{-s}^{0} \phi(x,r)dr := \eta_0(x,s), \qquad (17)$$

(initial integrated past history of u with vanishing boundary).

- We will consider solutions in the weak (variational) sense.

Well-posedness to a non-local PDE with memory

- **Assumptions:**
  - The nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of odd degree with positive leading coefficient,

$$f(u) = \sum_{k=1}^{2p} f_{2p-k} u^{k-1}, \qquad p \in \mathbb{N}. \tag{18}$$

    (Can be extended, to a more general function).
  - The variable μ satisfies the following hypotheses:

    (h_1) $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \quad \mu(s) \geq 0, \quad \mu'(s) \leq 0, \quad \forall s \in \mathbb{R}^+$;

    (h_2) $\mu'(s) + \delta\mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+$, for some $\delta > 0$.

Notice:

1. Conditions (h_1)-(h_2) are fulfilled by singular kernels as

$$\mu(t) = e^{-\delta t} t^{-\alpha}, \; t > 0, \quad \delta > 0, \alpha \in (0, 1).$$

2. Assumption (h_2) implies that $\mu(s)$ decays exponentially. Also, the memory kernel $k(\cdot)$ to have a singularity at $t = 0$ (our aim to study problem (16)).

Well-posedness to a non-local PDE with memory

- **Notation and set-up:**

Let Ω be a bounded domain in $\mathbb{R}^N$. Recall the Lebesgue space $L^p(\Omega)$, where $1 \leq p \leq \infty$, and the Sobolev space $W^{1,p}(\Omega)$. We denote $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ and $V^* = H^{-1}(\Omega)$. The norms in H, V and V^* will be denoted by $|\cdot|$, $\|\cdot\|$ and $\|\cdot\|_*$, respectively. Recall $L^2_\mu(\mathbb{R}^+; H)$ is the Hilbert space of functions $w : \mathbb{R}^+ \to H$ endowed with the inner product,

$$(w_1, w_2)_\mu = \int_0^\infty \mu(s)(w_1(s), w_2(s))ds,$$

and let $|\cdot|_\mu$ denote the corresponding norm. In a similar way, we introduce the inner products $((\cdot,\cdot))_\mu$, $(((\cdot,\cdot)))_\mu$ and relative norms $\|\cdot\|_\mu$, $|||\cdot|||_\mu$ on $L^2_\mu(\mathbb{R}^+; V)$, $L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega))$ respectively. It follows then that

$$((\cdot,\cdot))_\mu = (\nabla\cdot, \nabla\cdot)_\mu, \quad \text{and} \quad (((\cdot,\cdot)))_\mu = (\Delta\cdot, \Delta\cdot)_\mu.$$

Well-posedness to a non-local PDE with memory

We also define the Hilbert spaces

$$\mathcal{H} = H \times L^2_\mu(\mathbb{R}^+; V) \text{ and } \mathcal{V} = V \times L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega)),$$

which are respectively endowed with inner products

$$((w_1, \phi_1), (w_2, \phi_2))_{\mathcal{H}} = (w_1, w_2) + ((\phi_1, \phi_2))_\mu,$$

and

$$((w_1, \phi_1), (w_2, \phi_2))_{\mathcal{V}} = ((w_1, w_2)) + (((\phi_1, \phi_2)))_\mu,$$

where $(w_i, \phi_i) \in \mathcal{H}$ or $\mathcal{V}$ $(i = 1, 2)$ and usual norms.

Eventually, $\mathcal{D}(I; X)$ is the space of inf. diff. X-valued functions with compact support in $I \subset \mathbb{R}$, whose dual space is the distribution space $\mathcal{D}'(I; X^*)$. We define L^2_V the space of functions $u(\cdot) : (-\infty, 0) \to V$ satisfying

$$\int_{-\infty}^0 e^{\gamma s} \|u(s)\|^2 \, ds < \infty,$$

where $0 < \gamma < \min\{m\lambda_1, \delta\}$ and δ comes from (h_2).

Notice: $L^2((-\infty, 0); V) \subset L^2_V$.

Well-posedness to a non-local PDE with memory

- A technical result: define the operator $\mathcal{J} : L^2_V \to L^2_\mu(\mathbb{R}^+; V)$ by

$$(\mathcal{J}\phi)(s) = \int_{-s}^{0} \phi(r)\,dr, \quad s \in \mathbb{R}^+. \tag{19}$$

Lemma (Technical)

Assume (h_1)-(h_2) hold. Then, the operator $\mathcal{J}$ defined by (19) is a linear and continuous mapping. In particular, there exists a positive constant K_μ such that, for any $\phi \in L^2_V$, it holds

$$\|\mathcal{J}\phi\|^2_{L^2_\mu(\mathbb{R}^+;V)} \le K_\mu \|\phi\|^2_{L^2_V}, \tag{20}$$

where $K_\mu = e^\gamma \int_0^1 \mu(s)ds + \mu(1)e^\delta(\gamma-\delta)^{-2}$.

Notice: If we fix an initial value $\phi \in L^2_V$ for problem (12), the corresponding one for the second component of (16) becomes $\eta_0 := \mathcal{J}\phi$, which belongs to $L^2_\mu(\mathbb{R}^+; V)$.

Well-posedness to a non-local PDE with memory

First, we recall a general result proved in DCDS-S (2022) for problem (16) with general initial data in $H \times L^2_\mu(\mathbb{R}^+; V)$. Denote

$$z(t) = (u(t), \eta^t) \qquad \text{and} \qquad z_0 = (u_0, \eta_0) \in H \times L^2_\mu(\mathbb{R}^+; V).$$

Set

$$\mathcal{L}z = \left(a(l(u))\Delta u + \int_0^\infty \mu(s)\Delta\eta(s)ds, \quad u - \eta_s \right),$$

and

$$\mathcal{G}(z) = (-f(u) + g, \quad 0).$$

Then problem (16) can be written in the following compact form,

$$\begin{cases} z_t = \mathcal{L}z + \mathcal{G}(z), & \text{in } \Omega \times (\tau, \infty), \\ z(x,t) = 0, & \text{on } \partial\Omega \times (\tau, \infty), \\ z(x,\tau) = z_0, & \text{in } \Omega. \end{cases} \tag{21}$$

Well-posedness to a non-local PDE with memory

Theorem (DCDS-S (2022))

Suppose (13), (18) *and* (h_1)-(h_2) *hold, let* $g \in H$, *assume* $a(\cdot)$ *loc. Lipschitz, and there exists* $\tilde{m} > 0$ *such that,*

$$a(s) \leq \tilde{m}, \qquad \forall s \in \mathbb{R}. \tag{22}$$

(*i*) *For any* $z_0 \in \mathcal{H}$, *there exists a unique* $z(\cdot) = (u(\cdot), \eta^{\cdot})$ *solution to* (21) *s.t.*

$$u(\cdot) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)), \ \forall T > \tau,$$
$$\eta^{\cdot} \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)), \ \forall T > \tau.$$

Furthermore, $z(\cdot) \in C(\tau, T; \mathcal{H})$ *for each* $T > \tau$, *and the mapping* $F : z_0 \in \mathcal{H} \to z(t) \in \mathcal{H}$ *is continuous for every* $t \in [\tau, T]$.

(*ii*) *For any* $z_0 \in \mathcal{V}$, *there exists a unique* $z(\cdot) = (u(\cdot), \eta^{\cdot})$ *solution to* (21) *s.t.*

$$u(\cdot) \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\Omega)), \qquad \forall T > \tau,$$
$$\eta^{\cdot} \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega))), \qquad \forall T > \tau.$$

In addition, $z(\cdot) \in C(\tau, T; \mathcal{V})$ *for every* $T > \tau$.

Straightforwardly we have the corresponding result for (12).

Theorem (Existence and Uniqueness)

Assume (13), (18), and (h_1)-(h_2) hold. Let $a(\cdot)$ be locally Lipschitz satisfying (22), $g \in H$, $u_0 \in H$ and $\phi \in L^2_V$. Then, there exists a unique function $z(\cdot) = (u(\cdot), \eta^\cdot)$ satisfying

$$u(\cdot) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{2p}(\tau, T; L^{2p}(\Omega)), \qquad \forall T > \tau,$$
$$\eta^\cdot \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V)), \qquad \forall T > \tau,$$

such that $\partial_t z = \mathcal{L}z + \mathcal{G}(z)$ in the weak sense, and $z|_{t=\tau} = (u_0, \mathcal{J}\phi)$. Furthermore, $z(\cdot) \in C(\tau, T; \mathcal{H})$ for each $T > \tau$, and the mapping

$$z_0 \in \mathcal{H} \mapsto z(t) \in \mathcal{H} \text{ is continuous,}$$

for every $t \in [\tau, T]$. If we also assume that $u_0 \in V$, $\phi \in L^2_{V \cap H^2(\Omega)}$, then

$$u \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V \cap H^2(\Omega)), \qquad \forall T > \tau,$$
$$\eta^\cdot \in L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+; V \cap H^2(\Omega))), \qquad \forall T > \tau.$$

In addition, $z(\cdot) \in C(\tau, T; \mathcal{V})$ for every $T > \tau$.

Existence of global attractor

- **Construction of dynamical system:**
  - First, we construct the DS generated by (12) assuming that g does not depend on t (autonomous case)
  - The non-autonomous case can also be studied (either pullback attractor or uniform attractor).
  - The phase space is $X = H \times L^2_V$, endowed with the norm

$$\|(w_1, w_2)\|_X^2 = |w_1|^2 + \|w_2\|_{L^2_V}^2.$$

  - Thanks to previous Theorem, we define $S : \mathbb{R}^+ \times X \to X$ by

$$S(t)\,(u_0, \phi) = (u(t; 0, (u_0, \mathcal{J}\phi)), u_t(\cdot; 0, (u_0, \mathcal{J}\phi))),$$

    where $(u(\cdot; 0, (u_0, \mathcal{J}\phi)), \eta^\cdot)$ is the unique solution to problem (16) with $u\,(0) = u_0$, $\eta_0 = \mathcal{J}\phi$.

Existence of global attractor

Lemma ($S(t)$ is well defined)

Under assumptions of Theorem (Existence and uniqueness), if $(u_0, \phi) \in X$, *then* $S(t)\,(u_0, \phi) \in X$.

Proof. Let $(u_0, \phi) \in X$ and $(u(\cdot), \eta^{\cdot})$ the corresponding solution to (16) for $(u_0, \mathcal{J}\phi)$. Then, $u(t)$ belongs to H; prove now $u_t(\cdot) \in L^2_V$.

$$\begin{aligned}
\int_{-\infty}^{0} e^{\gamma s} \|u_t(s)\|^2 \, ds &= \int_{-\infty}^{0} e^{\gamma s} \|u(t+s)\|^2 \, ds \\
&= \int_{-\infty}^{t} e^{\gamma(\sigma - t)} \|u(\sigma)\|^2 \, d\sigma \\
&= e^{-\gamma t} \int_{-\infty}^{t} e^{\gamma \sigma} \|u(\sigma)\|^2 \, d\sigma \\
&= e^{-\gamma t} \int_{-\infty}^{0} e^{\gamma \sigma} \|\phi(\sigma)\|^2 \, d\sigma + \int_{0}^{t} e^{\gamma(\sigma - t)} \|u(\sigma)\|^2 \, d\sigma \\
&< +\infty,
\end{aligned}$$

since $\phi \in L^2_V$ and $u \in L^2(0, T; V)$ for all $T > 0$. □

- Thanks to Theorem (Existence & uniqueness), $S(t)$ is a DS in X.

- **Existence of bounded absorbing sets:**

Lemma (Absorbing set)

Under assumptions of Theorem (Existence & uniqueness), there exist two positive constants K_1 and K_2, such that

$$\|S(t)(u_0,\phi)\|_X^2 \leq K_1 \|(u_0,\phi)\|_X^2 e^{-\gamma t} + K_2, \ \forall t \geq 0, \ (u_0,\phi) \in X. \quad (23)$$

Therefore, the ball $B_0 = \{v \in X : \|v\|_X^2 \leq 2K_2\}$ is absorbing for the semigroup S.

(Lemma (Technical) is crucial in the proof)

Existence of global attractor

Proof. Let $(u_0, \phi) \in X$, $z(\cdot) = (u(\cdot), \eta^{\cdot})$ the solution to (16) corresponding to $(u_0, \mathcal{J}\phi)$. Multiply the first eq. in (16) by $u(t)$ in H and second eq. by η^t in $L^2_\mu(\mathbb{R}^+; V)$. Energy estimations give (noticing $f(s)s \geq \frac{1}{2} f_0 s^{2p} - a_0$),

$$\frac{d}{dt} \|z\|^2_{\mathcal{H}} + m\lambda_1 |u|^2 + m \|u\|^2 + f_0 |u|^{2p}_{2p} + 2(((\eta^t)', \eta^t))_\mu$$
$$\leq 2a_0 |\Omega| + \frac{2}{\sqrt{\lambda_1}} |g| \|u\| \leq 2a_0 |\Omega| + \frac{2}{m\lambda_1} |g|^2 + \frac{m}{2} \|u\|^2 .$$

Since $2(((\eta^t)', \eta^t))_\mu = -\int_0^\infty \mu'(s)|\nabla\eta^t(s)|^2 ds \geq \delta \int_0^\infty \mu(s)|\nabla\eta^t(s)|^2 ds$, it follows that

$$\frac{d}{dt} \|z\|^2_{\mathcal{H}} + \gamma \|z\|^2_{\mathcal{H}} + \frac{m}{2} \|u\|^2 + f_0 |u|^{2p}_{2p} \leq K_0, \tag{24}$$

where $K_0 = 2a_0|\Omega| + \frac{2}{m\lambda_1}|g|^2$ and we recall that $\gamma < \min\{m\lambda_1, \delta\}$. Multiplying by $e^{\gamma t}$ and integrating over $(0, t)$, neglecting the last term of the left hand side of (24),

$$\|z(t)\|^2_{\mathcal{H}} + \frac{m}{2} \int_0^t e^{-\gamma(t-s)} \|u(s)\|^2 ds \leq \|z(t)\|^2_{\mathcal{H}} + \frac{m}{2} \int_{-t}^0 e^{\gamma s} \|u_t(s)\|^2 ds$$
$$\leq \|z_0\|^2_{\mathcal{H}} e^{-\gamma t} + \frac{K_0}{\gamma}. \tag{25}$$

Existence of global attractor

Then

$$\frac{m}{2}\|u_t\|^2_{L^2_V} = \frac{m}{2}\int_{-\infty}^{0} e^{-\gamma(t-s)}\|\phi(s)\|^2 ds + \frac{m}{2}\int_0^t e^{-\gamma(t-s)}\|u(s)\|^2 ds$$
$$\leq \frac{m}{2} e^{-\gamma t}\|\phi\|^2_{L^2_V} + \|(u_0, \mathcal{J}\phi)\|^2_{\mathcal{H}} e^{-\gamma t} + \frac{K_0}{\gamma}.$$

In view of Lemma (Technical), we have that

$$\|z_0\|^2_{\mathcal{H}} \leq |u_0|^2 + \|\mathcal{J}\phi\|^2_{L^2_\mu(\mathbb{R}^+;V)} \leq |u_0|^2 + K_\mu\|\phi\|^2_{L^2_V}. \tag{26}$$

Hence, (25)-(26) imply the existence of positive constants K_1 and K_2, such that

$$\|S(t)(u_0,\phi)\|^2_X := |u(t)|^2 + \|u_t\|^2_{L^2_V} \leq K_1\left(|u_0|^2 + \|\phi\|^2_{L^2_V}\right) e^{-\gamma t} + K_2.$$

The proof of this lemma is complete. □

Existence of global attractor

- **Asymptotic compactness:** First state the next auxiliary result.

Lemma (Auxiliary)

Assume the hypotheses in Theorem 3. Let $\{(u_0^n, \phi^n)\}$ be a sequence, such that $(u_0^n, \phi^n) \to (u_0, \phi)$ weakly in X as $n \to \infty$. Then, $S(t)(u_0^n, \phi^n) = (u^n(t), u_t^n)$ fulfills:

$$u^n(\cdot) \to u(\cdot) \quad in \quad C([r,T],H) \quad for\ all \quad 0 < r < T; \tag{27}$$

$$u^n(\cdot) \to u(\cdot) \quad weakly\ in \quad L^2(0,T;V) \quad for\ all \quad T > 0; \tag{28}$$

$$u^n \to u \quad in \quad L^2(0,T;H) \quad for\ all \quad T > 0; \tag{29}$$

$$\limsup_{n\to\infty} \|u_t^n - u_t\|_{L_V^2}^2 \le K_5\ e^{-\gamma t} \limsup_{n\to\infty} \left(|u_0^n - u_0|^2 + \|\phi^n - \phi\|_{L_V^2}^2 \right) \quad for\ all \quad t \ge 0, \tag{30}$$

where $K_5 = \frac{1}{m}((\gamma+\delta)^2+1)$. Moreover, if $(u_0^n, \phi^n) \to (u_0, \phi)$ strongly in X as $n \to \infty$, then

$$u^n(\cdot) \to u(\cdot) \quad in \quad L^2(0,T;V) \quad for\ all \quad T > 0; \tag{31}$$

$$u_t^n(\cdot) \to u_t(\cdot) \quad in \quad L_V^2 \quad for\ all \quad t \ge 0. \tag{32}$$

Existence of global attractor

Corollary (Continuity with respect initial values)

Assume conditions of Theorem (Existence & uniqueness). Then, for any $t \geq 0$, the mapping $(u_0, \phi) \mapsto S(t)(u_0, \phi)$ is continuous.

Lemma (Asymptotic compactness of $S(t)$)

Under assumptions of Theorem (Existence & uniqueness), the semigroup S is asymptotically compact.

Theorem

Under the assumptions of Theorem (Existence & uniqueness), the semigroup S possesses a global connected attractor $\mathcal{A} \subset X$.