I Optimal Rearrangement

The University of Notting

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RS-IASM Workshop: Nonlocal Problems

thermatical Physics, Analysis and Geomet

Hangzhou, PR China

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BIRS-IASM Workshop: Nonlocal Problems in

Mathematical Physics, Analysis and Geometry

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Let us consider the stationary heat equation

the stationary heat equation

\n
$$
\begin{cases}\n\frac{\partial_t u}{\partial t} - \Delta u = f(x) & \text{in } \Omega, \\
u = u_0 & \text{on } \partial \Omega.\n\end{cases}
$$
\nmodeled by $f(x)$

\ndo not depend on time

\nrate on the boundary of Ω

\nactions f result different heat distributic

\n
$$
f
$$
 makes u be as equally distributed as\n
$$
\int f(x) dx = B
$$
\nand $0 \leq f(x) \leq M$?

- force function modeled by $f(x)$
- $u(x)$ and $f(x)$ do not depend on time
- constant temperature on the boundary of Ω

Different force functions *f* result different heat distributions *u^f* .

Problem: Which *f* makes *u* be as equally distributed as possible, given that

$$
\int_{\Omega} f(x)dx = B \text{ and } 0 \le f(x) \le M?
$$

Let *u^f* be the unique solution of the boundary value problem

$$
\begin{cases}\n-\Delta u_f(x) = f(x) & \text{in } \Omega, \\
u_f = 0 & \text{on } \partial\Omega.\n\end{cases}
$$

Consider the minimization/maximization of the functional

$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx
$$

over the class of admissible force functions

the unique solution of the boundary value problem
\n
$$
f(x) = f(x) \text{ in } \Omega,
$$
\n0 on $\partial\Omega$.
\nthe minimization/maximization of the functional
\n
$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx
$$
\nas of admissible force functions
\n
$$
\mathcal{R}_{\beta} = \{f : f = 0 \text{ or } f = 1, \text{ and } \int_{\Omega} f dx = \beta\}.
$$
\nproblem relates to the minimization/maximization
\nare of \mathcal{R}_{β}

The relaxed problem relates to the minimization/maximization over the weak-^{*} closure of \mathcal{R}_β

$$
\bar{\mathcal{R}}_{\beta}=\{f\ :\ 0\leq f\leq 1, \text{ and }\int_{\Omega}fdx=\beta\}.
$$

Theorem

There exists a unique minimizer \hat{f} of

$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx
$$

ique minimizer \hat{f} of
 $\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx$
 $\bar{\mathcal{R}}$, and $\alpha > 0$ such that for the function
 $\bullet 0 < \hat{u} \leq \alpha$ in Ω ,
 $\bullet f = \chi_{\{\hat{u} < \alpha\}} \in \mathcal{R}$,
 $\bullet \hat{u} = \alpha$ in $\{\hat{f} = 0\}$.

action $U = \alpha - \hat{u}$ is the over the set $f \in \overline{\mathcal{R}}$, and $\alpha > 0$ such that for the function $\hat{u} = u_{\hat{f}}$ the following is true

\n- $$
0 < \hat{u} \leq \alpha
$$
 in Ω ,
\n- $f = \chi_{\{\hat{u} < \alpha\}} \in \mathcal{R}$,
\n- $\hat{u} = \alpha$ in $\{\hat{f} = 0\}$.
\n

Moreover, the function $U = \alpha - \hat{u}$ is the minimizer of the functional

$$
J(w) = \int_{\Omega} |\nabla w|^2 + 2 \max(w, 0) dx,
$$

among functions $w \in W^{1,2}(\Omega)$ with boundary values α on $\partial\Omega$, and solves the obstacle problem

$$
\Delta U = \chi_{\{U>0\}}.
$$

The obstacle problem

Consider the minimization of the functional

$$
J(w) = \int_{\Omega} |\nabla w|^2 + 2 \max(w, 0) dx,
$$

among functions $w \in W^{1,2}(\Omega)$ with boundary values $\alpha > 0$ on $\partial \Omega$.

There is a unique minimizer *U* to this problem, which solves the following equation

Theorem

There exists a maximizer ˆ*f* of

$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx
$$

wimizer \hat{f} of
 $\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx$
 \overline{R} , and $\alpha > 0$ such that for the function
 $f = \chi_{\{\hat{u} > \alpha\}} \in \mathcal{R}$.

action $U = \alpha - \hat{u}$ is the minimizer of th
 $J(w) = \int_{\Omega} |\nabla w|^2 - 2 \max(w, 0) dx$,
 $w \in W^{1,2}(\Omega)$ with b over the set $f \in \overline{\mathcal{R}}$, and $\alpha > 0$ such that for the function $\hat{u} = u_{\hat{f}}$ the following is true

$$
f = \chi_{\{\hat{u} > \alpha\}} \in \mathcal{R}.
$$

Moreover, the function $U = \alpha - \hat{u}$ is the minimizer of the non-convex functional

$$
J(w) = \int_{\Omega} |\nabla w|^2 - 2 \max(w, 0) dx,
$$

among functions $w \in W^{1,2}(\Omega)$ with boundary values α on $\partial \Omega$, and solves the unstable obstacle problem

$$
\Delta U = \chi_{\{U<0\}}.
$$

Fractional setting

For a function u defined in \mathbb{R}^n let us define the Gagliardo semi-norm

defined in
$$
\mathbb{R}^n
$$
 let us define the Gagliard
\n
$$
[u]_s^2 := \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n + 2s}} \, dxdy,
$$
\n
$$
\therefore
$$
\n
$$
\text{ractical Sobolev spaces in } \mathbb{R}^n
$$
\n
$$
H^s(\mathbb{R}^n) = \{ v \in L^2(\mathbb{R}^n) \colon [v]_s^2 < \infty \},
$$
\n
$$
\text{d domain } D
$$
\n
$$
I_0^s(D) = \{ v \in H^s(\mathbb{R}^n) \colon v = 0 \text{ a.e. in } D^c
$$
\n
$$
\text{paces}
$$
\n
$$
s(\mathbb{R}^n) = (H^s(\mathbb{R}^n))', \quad H^{-s}(D) = (H_0^s(L))
$$

where $0 < s < 1$.

We also define fractional Sobolev spaces in \mathbb{R}^n

$$
H^{s}(\mathbb{R}^{n}) = \{ v \in L^{2}(\mathbb{R}^{n}) \colon [v]_{s}^{2} < \infty \},\
$$

and in a bounded domain *D*

$$
H_0^s(D)=\{v\in H^s(\mathbb{R}^n)\colon v=0\text{ a.e. in }D^c\},
$$

as well as dual spaces

$$
H^{-s}({\mathbb R}^n)=(H^{s}({\mathbb R}^n))', \quad H^{-s}(D)=(H^{s}_0(D))'.
$$

Fractional setting

 $\in H^{-s}(D)$ we say $u_f \in H_0^s(D)$ solves t
roblem in *D* with homogeneous Dirichle
 $\begin{cases} (-\Delta)^s u_f = f & \text{in } D, \\ u_f = 0 & \text{in } D^c, \end{cases}$
satisfied in the sense of distributions
 $\frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x - y|^{n + 2s}} dx dy = \frac{1}{\Delta}$
p).
ange For a function $f \in H^{-s}(D)$ we say $u_f \in H_0^s(D)$ solves the fractional boundary value problem in *D* with homogeneous Dirichlet boundary condition

$$
\begin{cases}\n(-\Delta)^s u_f = f & \text{in } D, \\
u_f = 0 & \text{in } D^c,\n\end{cases}
$$

if the equation is satisfied in the sense of distributions

$$
\frac{1}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy = \int_D fv dx
$$

for any $v \in H_0^s(D)$.

Fractional Rearrangement Problem

Minimize/maximize

$$
\Phi_s(f) = [u_f]_s^2
$$

over $f \in \overline{\mathcal{R}}$.

Theorem (Bonder, Cheng, Mikayelyan, 2020)

There exists a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_{\beta} \setminus \mathcal{R}_{\beta}$ such that

 $\Phi_s(\hat{f}) \leq \Phi_s(f)$

er, Cheng, Mikayelyan, 2020)

ique minimizer $\hat{f} \in \bar{\mathcal{R}}_{\beta} \setminus \mathcal{R}_{\beta}$ such that
 $\Phi_s(\hat{f}) \leq \Phi_s(f)$

Moreover, for some $\alpha > 0$ the function

ditions
 $0 \leq \hat{u} \leq \alpha$ in D,
 $\{\hat{f} < 1\} \subset \{\hat{u} = \alpha\}, \quad \{\hat{u} < \alpha\$ for any $f \in \bar{\mathcal{R}}_{\beta}$. Moreover, for some $\alpha > 0$ the function $\hat{u} = u_{\hat{f}}$ satisfies the following conditions

$$
0\leq \hat{u}\leq \alpha \text{ in } D,
$$

and

$$
\hat{f} > 0
$$
, $\{\hat{f} < 1\} \subset \{\hat{u} = \alpha\}$, $\{\hat{u} < \alpha\} \subset \{\hat{f} = 1\}$.

The minimization problem

Moreover, the function $\hat{U} := \alpha - \hat{u}$ minimizes the functional

$$
J(v) = [v]_s^2 + \int_D v^+ dx
$$

notion $\hat{U} := \alpha - \hat{u}$ minimizes the function
 $J(v) = [v]_s^2 + \int_D v^+ dx$
 $= \{v \in H_{loc}^s(\mathbb{R}^n) : v - \alpha \in H_0^s(D)\}$, and

alities
 $\chi_{\{U > 0\}} \le -(-\Delta)^s U \le \chi_{\{U \ge 0\}}$ in D

stributions.

iizer of J in H_α^s is unique and is ${\rm over}$ the set $H^s_\alpha = \{ v \in H^s_{loc}(\mathbb{R}^n) \colon v - \alpha \in H^s_0(D) \},$ and the function $\hat U$ verifies the inequalities

$$
\chi_{\{U>0\}}\le -(-\Delta)^sU\le \chi_{\{U\ge 0\}}\quad\text{in }D
$$

in the sense of distributions.

Finally, the minimizer of J in H^s_α is unique and is the unique solution to the inequalities above.

Fractional normalized obstacle problem

Theorem (Bonder, Cheng, Mikayelyan, 2020) The function $U \in H_{\alpha}$ satisfies

$$
\chi_{\{U>0\}} \le -(-\Delta)^s U \le \chi_{\{U\ge 0\}} \quad \text{in } D
$$

if and only if it satisfies the equation

orem (Bonder, Cheng, Mikayelyan, 2020)

\nfunction
$$
U \in H_{\alpha}
$$
 satisfies

\n
$$
\chi_{\{U>0\}} \le -(-\Delta)^{s}U \le \chi_{\{U\ge 0\}} \quad \text{in } D
$$
\nof only if it satisfies the equation

\n
$$
\begin{cases}\n -(-\Delta)^{s}U - \chi_{\{U\le 0\}} \min\{-(-\Delta)^{s}U^{+};1\} = \chi_{\{U>0\}}, & \text{in } D, \\
 U = \alpha & \text{in } D^c, \\
 \text{and functions} & H_{sub}^s(D) = \{ u \in H_{loc}^s(\mathbb{R}^n) : (-\Delta)^{s}u \le 0 \text{ in } D \}.\n \end{cases}
$$
\nis

\n
$$
\begin{cases}\n \chi_{\{V\}}(D) = \{ u \in H_{loc}^s(\mathbb{R}^n) : (-\Delta)^{s}u \le 0 \text{ in } D \}.\n \end{cases}
$$

among functions

$$
H^s_{sub}(D)=\left\{u\in H^s_{loc}(\mathbb{R}^n)\colon (-\Delta)^su\leq 0\textit{ in }D\right\}.
$$

Here

$$
\min\{-(-\Delta)^s U^+; 1\} = 1 - (1 + (-\Delta)^s U)^+.
$$

Theorem (Bonder, Cheng, Mikayelyan, 2021)

There exists a maximizer $\hat{f} \in \mathcal{R}_{\beta}$ such that

 $\Phi_s(\hat{f}) \geq \Phi_s(f)$

er, Cheng, Mikayelyan, 2021)
 $ximize \hat{f} \in \mathcal{R}_{\beta} \text{ such that}$
 $\Phi_s(\hat{f}) \ge \Phi_s(f)$

Moreover, for some $\alpha > 0$ the function

ation
 $(-\Delta)^s \hat{u} = \chi_{\{\hat{u} > \alpha\}} = \hat{f}.$

(including the local case)

nique. for any $f \in \bar{\mathcal{R}}_{\beta}$. Moreover, for some $\alpha > 0$ the function $\hat{u} = u_{\hat{f}}$ satisfies the following equation

$$
(-\Delta)^s \hat{u} = \chi_{\{\hat{u} > \alpha\}} = \hat{f}.
$$

Open Problem: (including the local case)

D convex $\Rightarrow \hat{f}$ unique.

Reinforced membrane problem

Hernot and Maillot have considered the following problem, where *f* ≥ 0 is the external load and $\omega \subset D$ is the subset with increased stiffness:

For a fixed function $f \in L^2(D)$, let $\omega \subset D$ and $u_{\omega} \in W_0^{1,2}(D)$ be the unique solution of the following problem in a domain *D*

alllot have considered the following problem,
\nload and
$$
\omega \subset D
$$
 is the subset with increased
\naction $f \in L^2(D)$, let $\omega \subset D$ and $u_{\omega} \in W_0^{1,2}$
\nIn of the following problem in a domain D

\n
$$
\begin{cases}\n-\Delta u_{\omega}(x) + \chi_{\omega}(x)u_{\omega}(x) = f(x) & \text{in } D, \\
u_{\omega}(x) = 0 & \text{on } \partial D.\n\end{cases}
$$

\nfunctional

\n
$$
\begin{aligned}\n\text{Equation: } \text{Equation: } \mathcal{L}^2 &= \int_D |\nabla u_{\omega}|^2 + \chi_{\omega} u_{\omega}^2 dx \\
\text{Equation: } \text{Equation: } \mathcal{L}^2 &= \int_D f u_{\omega} dx \\
\text{Equation: } \text{Equation: } \mathcal{L}^2 &= \int_D f u_{\omega} dx \\
\text{Equation: } \text{Equation: } \mathcal{L}^2 &= \int_D f u_{\omega} dx \\
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\text{Equation: } \text{Equation: } \mathcal{L}^2 &= \int_D f u_{\omega} dx \\
\text{Equation: } \text{Equation: } \mathcal{L}^2 &= \int_D f u_{\omega} dx \\
\text{Equation: } \text
$$

Minimize the functional

$$
F(\omega)=\int_D|\nabla u_\omega|^2+\chi_\omega u_\omega^2dx\ \ \left(=\int_D fu_\omega dx\right)
$$

over all subsets with given volume $|\omega| = \beta$.

Q.: For which functions *f* does the optimal set *ω* exist?

Reinforced membrane problem

is examples, one has to consider the rela
 k^{2*} closure of the set of characteristic functional
 $\dot{U}_{\beta} = \{l : 0 \le l \le 1, \text{ and } \int_D l dx = \beta\}$
 $U = \int_D |\nabla u_l|^2 + l u_l^2 dx \quad \left(= \int_D f u_l dx \right)$
 $\vdots \bar{R}_{\beta}$, where function $f \in L^2(D)$ As in the previous examples, one has to consider the relaxed problem. Consider the weak-* closure of the set of characteristic functions *χ^ω* of given *L* 1 -norm *β*:

$$
\bar{\mathcal{R}}_{\beta} = \{l \; : \; 0 \le l \le 1, \text{ and } \int_{D} l dx = \beta \}.
$$

Minimize the functional

$$
F(l) = \int_D |\nabla u_l|^2 + l u_l^2 dx \ \left(= \int_D f u_l dx \right)
$$

 ϕ over functions $l \in \bar{\mathcal{R}}_{\beta}$, where function $f \in L^2(D)$, and $u_l \in W^{1,2}_0(D)$ be the unique solution of the BVP

$$
\begin{cases}\n-\Delta u_l(x) + l(x)u_l(x) = f(x) & \text{in } D, \\
u_l(x) = 0 & \text{on } \partial D.\n\end{cases}
$$

Reinforced membrane problem

Henrot and Maillot has shown the existence of the minimizer, as well as proven some properties.

Moreover, using the auxiliary function u_0

$$
\begin{cases}\n-\Delta u_0(x) = f(x) & \text{in } D, \\
u_0(x) = 0 & \text{on } \partial D,\n\end{cases}
$$

they have proven that the minimizer is a characteristic function, provided the function *f* satisfies one of the following conditions

Henrot and Maillot has shown the existence of the minimizer, as well proven some properties.

\nMoreover, using the auxiliary function
$$
u_0
$$

\n
$$
\begin{cases}\n-\Delta u_0(x) = f(x) & \text{in } D, \\
u_0(x) = 0 & \text{on } \partial D,\n\end{cases}
$$
\nthey have proven that the minimizer is a characteristic function, prove the function *f* satisfies one of the following conditions

\n(i) $u_0 \leq f$ in *D*,

\n(ii) $f \leq -\Delta f$ in *D*,

\n(iii) $|\{x \in D : u_0 > \gamma\}| < \beta$, where $\gamma = \inf\{f(x) : f(x) < u_0(x)\}$.

\nFurthermore, they prove that the minimizer is a characteristic function *f*.

Furthermore, they prove that the minimizer is a characteristic function, in case of the ball and a non-increasing radial symmetric function *f*.

Fractional reinforced membrane

on $f \in L^2(D)$, let $l \in \overline{\mathcal{R}}_{\beta}$ and $u_l \in H_0^s$
f the following fractional analogue of the
m in D
 $\Delta)^s u_l(x) + l(x)u_l(x) = f(x)$ in D ,
 $(x) = 0$ in $\mathbb{R}^n \setminus$
on is satisfied in the sense of distribution
 $\frac{-u_f(y))(v(x) - v(y$ For a fixed function $f \in L^2(D)$, let $l \in \overline{\mathcal{R}}_{\beta}$ and $u_l \in H^s_0(D)$ be the unique solution of the following fractional analogue of the reinforced membrane problem in *D*

$$
\begin{cases}\n(-\Delta)^s u_l(x) + l(x)u_l(x) = f(x) & \text{in } D, \\
u_l(x) = 0 & \text{in } \mathbb{R}^n \setminus D,\n\end{cases}
$$

where the equation is satisfied in the sense of distributions

$$
\iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy + \int_D l u_l v dx = \int_D fv dx
$$

for any $v \in H_0^s(D)$.

Fractional reinforced membrane

Consider the minimization of the design function

inimization of the design function

\n
$$
F_s(l) := [u_l]_s^2 + \int_D l u_l^2 dx \quad \left(= \int_D f u_l dx \right)
$$
\nUsing the equation (1.5), we get:

\n
$$
\vec{\mathcal{R}}_{\beta}.
$$
\nUsing the equation (2024).

\nand is weak*-continuous in $\{f \in L^{\infty}(\Omega) : f$ is exists \hat{l} in \mathcal{R}_{β} such that

\n
$$
u) = \min_{l \in \mathcal{R}_{\beta}} F_s(l) = F_s(\hat{l}).
$$

over the set $l \in \bar{\mathcal{R}}_{\beta}$.

Theorem (Cheng, Mikayelyan, 2024)

F^{*s*} is convex and is weak^{*}−continuous in ${f \in L^{\infty}(\Omega) : f \geq 0}$ a.e.}. In particular, there exists \hat{l} in $\bar{\mathcal{R}}_{\beta}$ such that

$$
\inf_{\omega \in \mathcal{R}_{\beta}} F_s(\omega) = \min_{l \in \bar{\mathcal{R}}_{\beta}} F_s(l) = F_s(\hat{l}).
$$

Fractional reinforced membrane

Theorem (Cheng, Mikayelyan, 2024)

Let \hat{u} solve (**) with a design function $\hat{l} \in \bar{\mathcal{R}}_{\beta}$, and

$$
\Omega_0 = \left\{ x \in \Omega : \hat{l}(x) = 0 \right\},\
$$

$$
\Omega_1 = \left\{ x \in \Omega : \hat{l}(x) = 1 \right\},\
$$

$$
\Omega_* = \left\{ x \in \Omega : 0 < \hat{l}(x) < 1 \right\}.
$$

g, Mikayelyan, 2024)

with a design function $\hat{l} \in \mathcal{R}_{\beta}$, and
 $\Omega : \hat{l}(x) = 0$,
 $\Omega : \hat{l}(x) = 1$,
 $\Omega : 0 < \hat{l}(x) < 1$,
 F_s if and only if the following two cone
 $\inf_{x \in \Omega_1} \hat{u}(x)$.
 $\hat{u}(x) = \gamma_{\hat{l}}$ a.e. in Ω_* Then \hat{l} minimizes F_s if and only if the following two conditions hold $\gamma_{\hat{l}} = \sup_{x \in \Omega_0}$ $\hat{u}(x) = \inf_{x \in \Omega_1} \hat{u}(x).$ $\text{If } |\Omega_*| > 0$, then $\hat{u}(x) = \gamma_{\hat{l}}$ a.e. in Ω_* .

Theorem (Cheng, Mikayelyan, 2024)

Let $\Omega = B_1$. Assume that $f = f(r)$ is non-negative, radially symmetric and decreasing in $r = |x|$. Then, for every $R \in [0, 1]$, the characteristic f unction $\hat{l} = \chi_{B_R}$ is a minimizer of F_s over $\bar{\mathcal{R}}_{\beta}$ with $\beta = |B_R|.$

Cylindrical rearrangement problem

ne cylindrical domain $\Omega = D_{x'} \times (0, 1)_x$
functions which are independent of x_n
 $\overline{\mathcal{R}}_{\beta}^D = \{f \in \overline{\mathcal{R}}_{\beta} : f(x) = f(x')\}.$
que solution of the boundary value prob
= $f(x')$ in Ω,
on ∂Ω.
mization of the functional
 $\Phi(f$ Let us consider the cylindrical domain $\Omega = D_{x'} \times (0,1)_{x_n}$ and the subclass of force functions which are independent of *xⁿ*

$$
\bar{\mathcal{R}}_{\beta}^D = \{ f \in \bar{\mathcal{R}}_{\beta} : f(x) = f(x') \}.
$$

Let *u^f* be the unique solution of the boundary value problem

$$
\begin{cases}\n-\Delta u_f(x) = f(x') & \text{in } \Omega, \\
u_f = 0 & \text{on } \partial\Omega.\n\end{cases}
$$

Consider the minimization of the functional

$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx
$$

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over the class of admissible force functions $\bar{\mathcal{R}}^D_\beta.$

Cylindrical rearrangement problem

Theorem (Mikayelyan, 2018)

The minimization problem

 $\min_{f \in \bar{\mathcal{R}}_D} \Phi(f)$

has a unique solution $\hat{f} \in \bar{\mathcal{R}}_D \setminus \mathcal{R}_D$, $\hat{f} > 0$ in D , and there exists a constant *α >* 0 such that for the function

$$
\hat{v}(x') = v_{\hat{f}}(x') = \int_0^1 u_{\hat{f}}(x', t) dt
$$

Theorem (Mikayelyan, 2018)

Moreover, the function $\hat{U}(x) = \alpha - \hat{u}(x)$ is the minimizer of the convex functional

kayelyan, 2018)
\nfunction
$$
\hat{U}(x) = \alpha - \hat{u}(x)
$$
 is the minimizer
\n
$$
J(U) = \int_{\Omega} |\nabla U|^2 dx + 2 \int_{D} \max(V, 0) dx',
$$
\nas in $U \in W^{1,2}(\Omega)$ such that $U = \alpha$ on $\partial \Omega$
\n
$$
V(x') = \int_{0}^{1} U(x', t) dt.
$$
\nkayelyan, 2018)
\n
$$
\hat{u} = u_{\hat{f}} \in W^{2,2}(D' \times (0, 1))
$$
\nD.

among functions in $U \in W^{1,2}(\Omega)$ such that $U = \alpha$ on $\partial\Omega$, where

$$
V(x') = \int_0^1 U(x', t) dt.
$$

Theorem (Mikayelyan, 2018)

$$
\hat{u} = u_{\hat{f}} \in W^{2,2}(D' \times (0,1))
$$

for any $D' \in D$.

Theorem (Mikayelyan, 2018)

yelyan, 2018)

imization of the convex functional in the
 $\sqrt{2}u|^2 dx + 2 \int_D v^+ dx'$

with prescribed boundary values $u = g :$
 $u(x', x_n)dx_n$.

and J has a unique minimizer u, and
 $u(x) = \chi_{\{v>0\}} + 2\partial_\nu u(x', 0)\chi_{\{v=0\}}$ in ! Consider the minimization of the convex functional in the domain $\Omega = D \times (0,1)$

$$
J(u) = \int_{\Omega} |\nabla u|^2 dx + 2 \int_{D} v^+ dx'
$$

among functions with prescribed boundary values $u = g = const$ on $\partial\Omega$, where $v(x') = \int_0^1 u(x', x_n) dx_n$.

Then the functional *J* has a unique minimizer *u*, and

$$
\Delta u(x)=\chi_{\{v>0\}}+2\partial_\nu u(x',0)\chi_{\{v=0\}} \ \ \hbox{in} \ \ \Omega.
$$

Numerical simulations

joint work with Zhilin Li (North Carolina State University)

Lemma

Let u_1 and u_2 minimize

$$
J(u)=\int_{\Omega}|\nabla u|^{2}dx+2\int_{D}v^{+}dx'
$$

inimize
 $J(u) = \int_{\Omega} |\nabla u|^2 dx + 2 \int_{D} v^+ dx'$

with constant boundary data α_1 and α_2

Then the comparison principle does no
 u_2 .
 $u_1(x) \le u_2(x)$ is not true for all $x \in \Omega$.
 $u_3(x',t)dt, j = 1, 2$,
 $u_1(x') \le v_2(x')$ in among functions with constant boundary data α_1 and α_2 respectively, and $0 < \alpha_1 < \alpha_2$. Then the comparison principle does **not** hold for the functions u_1 and u_2 .

 $u_1(x) \le u_2(x)$ *is not true for all* $x \in \Omega$ *.*

Conjecture

For
$$
v_j(x') = \int_0^1 u_j(x', t) dt
$$
, $j = 1, 2$,
 $v_1(x') \le v_2(x')$ in D.

Comparison principle

ot, Mikayelyan, 2022)

and $v_j(x') = \int_0^1 u_j(x',t)dt$, $j = 1, 2$,
 $v_1(x') \le v_2(x')$ in D.
 \Downarrow
 $u_2 - \alpha_2 \le u_1 - \alpha_1$
 \Downarrow
 $\partial_{x_n x_n}^2 (u_2 - u_1) \ge 0$ **Theorem (Chipot, Mikayelyan, 2022)** $For\ 0 < \alpha_1 < \alpha_2$ and $v_j(x') = \int_0^1 u_j(x',t) dt, \ j = 1,2,$ $v_1(x') \le v_2(x')$ in *D*. ⇕ $u_2 - \alpha_2 \leq u_1 - \alpha_1$

⇕

$$
\partial_{x_nx_n}^2(u_2-u_1)\geq 0
$$

Theorem

Consider the minimizer *u* of the convex functional

$$
J(u) = \int_{\Omega} |\nabla u|^2 dx + 2 \int_{D} v^+ dx'.
$$

imizer u of the convex functional
 $|\nabla u|^2 dx + 2 \int_D v^+ dx'.$
 $= D \times (0, 1)$, where $v(x') = \int_0^1 u(x', x_n)$
 $\Delta v = h(x') \chi_{\{v > 0\}},$
 $h(x') = 1 - 2 \partial_\nu u(x', 0) \in C^\alpha(D),$
 $h \ge 0$ in $\{v \ge 0\}.$ in the domain $\Omega = D \times (0,1)$, where $v(x') = \int_0^1 u(x',x_n) dx_n$.

Then

$$
\Delta v = h(x') \chi_{\{v > 0\}},
$$

where

$$
h(x') = 1 - 2\partial_{\nu}u(x', 0) \in C^{\alpha}(D),
$$

and

$$
h \ge 0 \ \text{in} \ \{v \ge 0\}.
$$

Remarks on free boundary regularity

and $h(x') > 0$, then we have same regu
problem.
 b have $h(x') = 0$ on $\partial \{v > 0\}$?

if $h(x') = 0$? If $x' \in \partial \{v > 0\}$ and $h(x') > 0$, then we have same regularity as for the classical obstacle problem.

Open questions:

- 1. Is it possible to have $h(x') = 0$ on $\partial \{v > 0\}$?
- 2. What happens if $h(x') = 0$?

Lemma

ma
\n
$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx = \int_{\Omega} f u_f dx = \sup_{u \in W_0^{1,2}(\Omega)} \int_{\Omega} 2fu - |\nabla u|^2 dx.
$$
\nma
\nfunctional Φ is
\nweakly sequentially continuous in L^2 ,
\nstrictly convex,
\nGâteaux differentiable, and $\Phi'(f)$ can be identified with $2u_f$.

Lemma

The functional Φ is

(*i*) weakly sequentially continuous in L^2 ,

(ii) strictly convex,

(iii) Gâteaux differentiable, and $\Phi'(f)$ can be identified with $2u_f$.

Step 1:

The minimizer \hat{f} of Φ over $\bar{\mathcal{R}}$ exists and is unique. **Step 2:**

The minimality condition is

$$
0\in \partial \Phi(\hat{f})+\partial \xi_{\bar{\mathcal{R}}}(\hat{f}),
$$

where *∂*Φ is the sub-differential of Φ and

$$
\xi_{\bar{\mathcal{R}}}(g) = \begin{cases} 0 & \text{if } g \in \bar{\mathcal{R}} \\ \infty & \text{if } g \notin \bar{\mathcal{R}} \end{cases}.
$$

Thus

tep 1:

\nthe minimizer
$$
\hat{f}
$$
 of Φ over $\bar{\mathcal{R}}$ exists and is unique.

\n**tep 2:**

\nthe minimality condition is

\n
$$
0 \in \partial \Phi(\hat{f}) + \partial \xi_{\bar{\mathcal{R}}}(\hat{f}),
$$
\nhere $\partial \Phi$ is the sub-differential of Φ and

\n
$$
\xi_{\bar{\mathcal{R}}}(g) = \begin{cases} 0 & \text{if } g \in \bar{\mathcal{R}} \\ \infty & \text{if } g \notin \bar{\mathcal{R}} \end{cases}.
$$
\nthus

\n
$$
-2\hat{u} \in \partial \xi_{\bar{\mathcal{R}}}(\hat{f}) = \left\{ w \in L^{2}(\Omega) \; : \; \xi_{\bar{\mathcal{R}}}(f) - \xi_{\bar{\mathcal{R}}}(\hat{f}) \geq \int_{\Omega} (f - \hat{f}) w dx' \right\}
$$
\nand for any $f \in \bar{\mathcal{R}}$

\n
$$
\int_{\Omega} \hat{f}(\hat{f}) \, d\mathcal{L}(\hat{f}) \, d\mathcal{L}(\hat{f}) = \int_{\Omega} \hat{f}(\hat{f}) \, d\mathcal{L}(\hat{
$$

and for any $f \in \overline{\mathcal{R}}$

$$
\int_{\Omega} \hat{f} \hat{u} dx \le \int_{\Omega} f \hat{u} dx.
$$

Lemma

For $f, g \in L^2_+(D)$ there exists $\widetilde{f} \in ext(\bar{\mathcal{R}}(f))$ such that functional

$$
\int_D \widetilde{f}gdx \le \int_D hgdx,
$$

for all $h \in \overline{\mathcal{R}}(f)$.

Step 3:

There exists $\tilde{f} \in \mathcal{R}$ such that for any $f \in \bar{\mathcal{R}}$

there exists
$$
\tilde{f} \in ext(\bar{R}(f))
$$
 such that $\int_D \tilde{f}g dx \le \int_D hg dx$,

 \mathcal{R} such that for any $f \in \bar{R}$

 $\int_{\Omega} \hat{f} \hat{u} dx = \int_{\Omega} \tilde{f} \hat{u} dx \le \int_{\Omega} f \hat{u} dx$.

 $\tilde{f} = \hat{f}$.

Step 4: Prove that

$$
\widetilde{f} = \widehat{f}.
$$

Technique (constrained case)

For
$$
f \in \overline{\mathcal{R}}_D
$$
 we have $f(x) = f(x')$ and thus

$$
f \in \bar{\mathcal{R}}_D \text{ we have } f(x) = f(x') \text{ and thus}
$$
\n
$$
\Phi(f) = \int_{\Omega} |\nabla u_f|^2 dx = \int_{\Omega} f u_f dx = \int_{D} f(x') v_f(x') dx',
$$
\n
$$
v_f(x') = \int_{0}^{1} u_f(x', t) dt.
$$
\nWe can consider Φ in $L_D^2(\Omega)$ or in $L^2(D)$.\n\nma\nfunctional Φ is\nreakly sequentially continuous in $L_D^2(\Omega)$ and in $L^2(D)$,\ntrictly convex,\nGâteaux differentiable. Moreover, $\Phi'(f)$ can be identified

where

$$
v_f(x') = \int_0^1 u_f(x',t)dt.
$$

We can consider
$$
\Phi
$$
 in $L^2_D(\Omega)$ or in $L^2(D)$.

Lemma

The functional Φ is

(i) weakly sequentially continuous in $L^2_D(\Omega)$ and in $L^2(D)$,

(ii) strictly convex,

(iii) Gâteaux differentiable. Moreover, $\Phi'(f)$ can be identified with $2u_f$ if we consider Φ in $L^2(\Omega)$ or $2v_f$ if we consider Φ in $L^2(D)$.

Lemma

$$
\text{Let } \Omega = D_{x'} \times (0,1)_{x_n} \text{ and }
$$

$$
m = \Omega
$$
\n
$$
\Omega = D_{x'} \times (0, 1)_{x_n} \text{ and}
$$
\n
$$
\begin{cases}\n-\Delta u(x) = f(x') & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.\n\end{cases}
$$
\n
$$
u(x', x_n) = u(x', 1 - x_n),
$$
\n
$$
u(x', x_n) = u(x', 1 - x_n),
$$
\n
$$
u(x', x_n) = \int_0^1 u(x', x_n) dx_n \text{ satisfies the following equation}
$$
\n
$$
\begin{cases}\n-\Delta_{x'} v = f(x') + 2\partial_\nu u(x', 0) & \text{in } D, \\
v = 0 & \text{on } \partial D.\n\end{cases}
$$
\n(3)

Then

$$
u(x', x_n) = u(x', 1 - x_n),
$$
\n(2)

and the function $v(x') = \int_0^1 u(x',x_n) dx_n$ satisfies the following equation

$$
\begin{cases}\n-\Delta_{x'}v = f(x') + 2\partial_{\nu}u(x',0) & \text{in } D, \\
v = 0 & \text{on } \partial D.\n\end{cases}
$$
\n(3)

How does the proof work?

Steps 1-3 are similar to unconstrained case

There exists a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_D$ of Φ .

There exists $\widetilde{f} = \chi_{D_0}(x') \in ext(\overline{\mathcal{R}}_D) = \mathcal{R}_D$ such that

How does the proof work?
\nre similar to unconstrained case
\na unique minimizer
$$
\hat{f} \in \bar{\mathcal{R}}_D
$$
 of Φ .
\n $\tilde{f} = \chi_{D_0}(x') \in ext(\bar{\mathcal{R}}_D) = \mathcal{R}_D$ such that
\n
$$
\int_D \hat{f} \hat{v} dx' = \int_{\Omega} \tilde{f}(x') \hat{u}(x) dx \le \int_{\Omega} f(x') \hat{u}(x) dx
$$

\nion $f \in \bar{\mathcal{R}}_D$.
\nMain challenge: $\tilde{f} \neq \hat{f}$.

for any function $f \in \overline{\mathcal{R}}_D$.

Main challenge: $\tilde{f} \neq \hat{f}$.

Technique (constrained case)

Step 4:

Claim 1:

$$
\alpha = \sup_{D_0} \hat{v} \le \inf_{D \setminus D_0} \hat{v}.
$$

Claim 2:

$$
\hat{f} = \tilde{f} = 1, \text{ in } \{\hat{v} < \alpha\}.
$$

Claim 3:

$$
\alpha = \sup_{D_0} \hat{v} \le \inf_{D \setminus D_0} \hat{v}.
$$

$$
\hat{f} = \tilde{f} = 1, \text{ in } \{\hat{v} < \alpha\}.
$$

$$
\{\hat{v} > \alpha\} \subset D^{\#} := \{\hat{f} = 0\}.
$$

Claim 4:

 $D^{\#}$ has no interior. Thus $\hat{v} \leq \alpha$.

From [\(3\)](#page-31-0) and the Hopf's lemma it follows that

$$
\Delta_{x'}\hat{v}(x')=-2\partial_\nu u(x',0)>0\ \ \text{in}\ \ \text{int}(D^\#)
$$

 $D^{\#}$ has no interior. Thus $\hat{v} \leq \alpha$.

Hopf's lemma it follows that
 $x'\hat{v}(x') = -2\partial_{\nu}u(x',0) > 0$ in $int(D^{\#}D^{\#})$. This means that there exists $y \in \beta > \alpha$, which contradicts Claim 3 and $\hat{f} > 0$.

this only in $int({$ and $\hat{v} \ge \alpha$ in int($D^{\#}$). This means that there exists $y \in \partial$ (int($D^{\#}$)) such that $\hat{v}(y) = \beta > \alpha$, which contradicts Claim 3 and continuity of \hat{v} .

Claim 5:

 $\hat{f} > 0$.

We need to verify this only in $int({\hat{v} = \alpha})$ where

$$
0 = \Delta_{x'}\hat{v} = -\hat{f}(x') - 2\partial_{\nu}\hat{u}(x', 0).
$$

and the outer normal derivative of \hat{u} is not vanishing in D by Hopf lemma.

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