Blow-up criterion for a degenerate fully parabolic chemotaxis system

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The HLS inequality

A classical inequality, due to Hardy ('28), Littlewood ('30) and Sobolev ('38), states that

$$|\mathcal{H}[f,g]| = \left| \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \right| \le C(d,\lambda,p)\|f\|_p\|g\|_t$$

for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^t(\mathbb{R}^d)$, where $p,t > 1$, $\lambda \in (0,d)$, satisfy

$$\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{t} = 2.$$

The sharp constant for HLS inequality

- The sharp constant:

$$C_{HLS} = \sup_{f \neq 0} \left\{ \frac{\| |x|^{-\lambda} * f \|_q}{\|f\|_p} \,\middle|\, f \in L^p(\mathbb{R}^d) \right\},$$

  where $1/p + \lambda/d = 1 + 1/q$.
- There is a $f \in L^p$ with $\|f\|_p = 1$ that maximizes C_{HLS}. Moreover, every maximizing f is symmetrically decreasing following a translation and fulfills a pair of equations (Lieb, '83, Ann. Math.; '01, Analysis)

$$|x|^{-\lambda} * f = g^{t-1}, \quad |x|^{-\lambda} * g = f^{p-1},$$

  for some symmetric $g \in L^t$.

A variant to the HLS (VHLS) inequality

Let $d \geq 3$, and let $\lambda = d-2$ (Newtonian potential), $m_* = 2d/(d+2)$ and $m^* = 2 - 2/d$. For $m \in (m_*, m^*]$,

$$C_{HLS,m} := \sup_{h \neq 0} \left\{ \frac{\mathcal{H}[h,h]}{\|h\|_1^{1-\sigma} \|h\|_m^{1+\sigma}}, h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d),\ h \geq 0 \right\}$$

with $\sigma = [m(d-2)]/[(m-1)d] - 1 \in (0,1)$.

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Proposition

There exists a non-negative, radially symmetric and non-increasing function $V \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ with $\|V\|_1 = \|V\|_m = 1$ that attains the variational problem $C_{HLS,m}$.

- See (Blanchet/Carrillo/Laurençot, '09, CVPDE) for $m = m^*$ and (Kimijima/Nakagawa/Ogawa, '14, CVPDE) for $m \in (m_*, m^*)$.

Parabolic-elliptic (PE) Keller-Segel model

The Keller-Segel (Keller/Segel, '70, JTB) model describes the motion of cells by chemotactical attraction by means of the coupled system:

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \quad x \in \mathbb{R}^d, \\ -\Delta v = u, \quad x \in \mathbb{R}^d. \end{cases} \tag{1}$$

- $u = u(x,t)$: mass density;
- $v = v(x,t)$: the concentration of the chemical attractant;
- $m > 1$: degenerate diffusion;
- Note that $v = \mathcal{W} * u$ solves the elliptic problem, where

$$\mathcal{W}(x) = \frac{c_d}{|x|^{d-2}}, \text{ if } d > 2.$$

.

Lyapunov functional for PE system

The main implication for us is that there is a natural Lyapunov functional for (1) defined on the set of centered mass densities $\rho \in L^1_+(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ given by

$$\mathcal{E}[\rho] = \mathcal{G}[\rho] - \mathcal{I}[\rho],$$

$$\mathcal{G}[\rho] = \frac{1}{m-1}\int_{\mathbb{R}^d} \rho^m(x)dx, \;\; \mathcal{I}[\rho] = \frac{1}{2}\int_{\mathbb{R}^d} \rho(x)\mathcal{W} * \rho(x)dx,$$

$$\mathcal{E}[\rho](t) \leq \mathcal{E}[\rho_0], \quad t > 0,$$

$$\rho(x) \geq 0, \;\; \int_{\mathbb{R}^d} \rho(x)dx = M.$$

Different regimes

The functional $\mathcal{E}$ becomes

$$\mathcal{E}[\rho^\lambda] = \lambda^{d(m-1)}\mathcal{G}[\rho] - \lambda^{d-2}\mathcal{I}[\rho]$$

by taking dilations $\rho^\lambda(x) := \lambda^d \rho(\lambda x)$. By scaling considerations, one can find 3 different regimes:

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- Diffusion-dominated regime: $m > m^*$. The diffusion part dominates and the intuition is that solutions exist globally in time;
- Aggregation-dominated regime: $m < m^*$. Blow-up occurs for some initial data, but not for all initial data;
- Fair-Competition regime: $m = m^*$. The total mass of system is the critical quantity. There is a critical value separating the diffusive behavior from the blow-up behavior.

Dichotomy phenomenon for $m = m^*$

Turn back to

$$\mathcal{E}[u] = \frac{1}{m-1}\|u\|_m^m - \frac{c_d}{2}\mathcal{H}[u,u] \left(\le C_{HLS,m}\|u\|_1^{2/d}\|u\|_m^m\right).$$

Let

$$M_c = \left[\frac{2}{c_d(m-1)C_{HLS,m}}\right]^{d/2}.$$

Characterizing the infimum of the free energy reveals a dichotomous phenomenon (Blanchet/Carrillo/Laurençot, '09, CVPDE):

- $M = \|u\|_1 < M_c$ ($\mathcal{E} > 0$): Solutions are global;
- $M = M_c$ ($\mathcal{E} = 0$): Solutions with finite second moment diverge to a Dirac mass, but those with radial initial conditions remain uniformly bounded (Bedrossian/Kim, '13, SJMA);
- $M > M_c$ ($\mathcal{E} < 0$): Solutions blow up in finite time.

Threshold value for $m \in (m_*, m^*)$

Let the function $V \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ be the maximizer for $C_{HLS,m}$. Assume that

$$\|u_0\|_1^{\frac{m\alpha}{\beta}} \mathcal{E}[u_0] < \|V\|_1^{\frac{m\alpha}{\beta}} \mathcal{E}[V]$$

with

$$\begin{cases} \alpha := \frac{2}{2-m} - \frac{d}{m}, \\ \beta := d - \frac{2}{2-m}. \end{cases}$$

Then we have the sharp result (Kimijima/Nakagawa/Ogawa, '14, CVPDE):

- $\|u_0\|_1^{\alpha}\|u_0\|_m^{\beta} < \|V\|_1^{\alpha}\|V\|_m^{\beta} \Longrightarrow$ Global existence;
- $\|u_0\|_1^{\alpha}\|u_0\|_m^{\beta} > \|V\|_1^{\alpha}\|V\|_m^{\beta} \Longrightarrow$ Finite-time blow-up.

Parabolic-Parabolic (PP) KS model

The PP Keller-Segel model:

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = -f = \Delta v - v + u, \end{cases}$$

where $m > 1$ and $d \geq 3$.

The v can be decomposed by

$$v = \tilde{v} + \hat{v},$$

where

$$\tilde{v}(x) := (\mathscr{B} * u)(x), \ \ -\Delta\tilde{v} + \tilde{v} = u,$$

$$\hat{v}(x) := (\mathscr{B} * f)(x), \ \ -\Delta\hat{v} + \hat{v} = f.$$

Here $\mathscr{B}$ is the Bessel kernel.

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Question: What is sharp critical mass criterion for the fully parabolic Keller-Segel model?

Lyapunov functional for PP system

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases}$$

Lyapunov functional: Gradient flow of

$$\mathcal{F}[u, v] := \int_{\mathbb{R}^d} \left(\frac{1}{m-1} u^m + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - uv \right) dx.$$

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Note that

$$\mathcal{F}[u,v] = \mathcal{F}[u,\tilde{v}] + \frac{1}{2} \|\nabla (v - \tilde{v})\|_2^2 + \frac{1}{2} \|v - \tilde{v}\|_2^2,$$

where $\mathcal{F}[u,\tilde{v}]$ is Lyapunov functional for

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla \tilde{v}), \\ \tilde{v} = \mathscr{B} * u. \end{cases}$$

- $\mathcal{F}[u,\tilde{v}] > 0 \Longrightarrow \mathcal{F}[u,v] > 0$

Global well-posedness

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases}$$

- Let $m = m^*$ and $d \geq 3$. If $M < M_c$, then there exists a weak global solution to the parabolic-parabolic Keller-Segel system (Blanchet/Laurençot, '13, CPDE);

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- Let $m = m^*$ and $d \geq 3$. If $M < M_c$, then there exists a weak global solution to the parabolic-parabolic Keller-Segel system (Blanchet/Laurençot, '13, CPDE);
- Let $m \in (m_*, m^*)$ and $d \geq 3$. If $\|u_0\|^{\frac{m\alpha}{\beta}} \mathcal{F}[u_0, v_0] < \|V\|^{\frac{m\alpha}{\beta}} \mathcal{E}[V]$ and

  $$\|u_0\|_1^{\alpha} \|u_0\|_m^{\beta} < \|V\|_1^{\alpha} \|V\|_m^{\beta},$$

  the weak solution is global and uniformly bounded (L./Wang, preprint).

Sharp blow-up criterion

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases}$$

- Let $m = m^*$ and $d = 3, 4$. If $M > M_c$, radially symmetric solutions with negative initial energy blow up in finite time (Laurençot/Mizoguchi, '15, Poincaré);

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- Let $m \in (m_*, m^*)$ and $d \geq 3$. If $\|u_0\|_1^{\frac{m\alpha}{\beta}} \mathcal{F}[u_0, v_0] < \|V\|_1^{\frac{m\alpha}{\beta}} \mathcal{E}[V]$ and

$$\|u_0\|_1^{\alpha} \|u_0\|_m^{\beta} > \|V\|_1^{\alpha} \|V\|_m^{\beta},$$

there exists a radially symmetric weak solution blowing up in finite time (L./Wang, preprint).

Sharp blow-up criterion

Denoting the second moment by

$$M_2(t) = \int_{\mathbb{R}^d} |x|^2 u(x,t)dx$$

and computing we have

$$\frac{d}{dt}M_2(t) \leq 2(d-2)\underbrace{\mathcal{F}[u,v]}_{\leq \mathcal{F}[u_0,v_0]} - 2\underbrace{\left(\frac{d-2}{m-1} - d\right)}_{>0}\|u\|_m^m$$

$$+\begin{cases} C\|f\|_2 M_2^{1/4}(t), & \text{if } d=3, \\ C\|f\|_2, & \text{if } d=4, \\ C(\|f\|_2^\theta + \|f\|_2),\ \theta\in(1,2), & \text{if } d>4. \end{cases}$$

.

Sharp blow-up criterion

The functional $\mathcal{F}$ fulfills

$$\mathcal{F}[u,v](t) + \int_0^t \left(\mathcal{D}[u,v] + \|f\|_2^2 \right) d\tau \le \mathcal{F}[u_0, v_0],$$

$$\mathcal{F}[u,v] \ge -C(1 + \mathcal{D}[u,v])^{\delta}, \ \ \delta \in (0,1),$$

where the entropy dissipation $\mathcal{D}$ is given by

$$\mathcal{D}[u,v] = \int_{\mathbb{R}^d} u \left| \frac{m}{m-1} \nabla u^{m-1} - \nabla v \right|^2 dx.$$

If $T_{\max} = +\infty$, then $\int_0^\infty \|f\|_2^2 d\tau < \infty$. However, M_2 will be negative after some finite time and hence blow-up occurs.

Conclusions

- In the fair competition regime for degenerate parabolic-elliptic Keller-Segel model: dichotomy similar to two-dimensional setting.
- The sharp critical mass is related to the maximizer for the HLS inequality;
- This criterion for parabolic-parabolic Keller-Segel model is consistent with the parabolic-elliptic version;

Thanks for your attention!