**Blow-up criterion for a degenerate fully
parabolic chemotaxis system**
KE LIN
College of Mathematics
Southwestern University of Finance and Economics, Chengdu
(Joint work with Sheng Wang)
Nonlocal Problems in Mathematical Blow-up criterion for a degenerate fully parabolic chemotaxis system

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A classical inequality, due to Hardy ('28), Littlewood ('30) and Sobolev ('38), states that

ILS inequality
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\nlev (38), states that
\n
$$
|\mathcal{H}[f,g]| = \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x)g(y)}{|x - y|^{\lambda}} dxdy \right| \le C(d, \lambda, p) ||f||_p ||g||_t
$$
\n
$$
||f \in L^p(\mathbb{R}^d) \text{ and } g \in L^t(\mathbb{R}^d), \text{ where } p, t > 1, \lambda \in (0, d), \text{ satisfies}
$$
\n
$$
\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{t} = 2.
$$
\n**EXECUTE:** How-up criterion for chemotaxis system

for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^t(\mathbb{R}^d)$, where $p, t > 1, \lambda \in (0, d)$, satisfy

$$
\frac{1}{p} + \frac{\lambda}{d} + \frac{1}{t} = 2.
$$

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The sharp constant for HLS inequality

• The sharp constant:

$$
C_{HLS} = \sup_{f \neq 0} \left\{ \frac{|||x|^{-\lambda} * f||_q}{||f||_p} \middle| f \in L^p(\mathbb{R}^d) \right\},\,
$$

where $1/p + \lambda/d = 1 + 1/a$.

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The sharp constant:
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where $1/p + \lambda/d = 1 + 1/q$.

There is $a f \in L^p$ with $||f||_p = 1$ that maximizes C_{HLS} . Mover, every There is $af \in L^p$ with $||f||_p = 1$ that maximizes C_{HLS} . Moreover, every maximizing *f* is symmetrically decreasing following a translation and fulfills a pair of equations (Lieb, '83, Ann. Math.; '01, Analysis)

$$
|x|^{-\lambda} * f = g^{t-1}, \quad |x|^{-\lambda} * g = f^{p-1},
$$

for some symmetric $g \in L^t$.

A variant to the HLS (VHLS) inequality

Let *d* \geq 3, and let $\lambda = d - 2$ (Newtonian potential), $m_* = 2d/(d+2)$ and $m^* = 2 - \frac{2}{d}$. For $m \in (m_*, m^*]$,

variant to the HLS (VHLS) inequality
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$$
, and let $\lambda = d - 2$ (Newtonian potential), $m_* = 2d/(d+2)$
\nand $m^* = 2 - 2/d$. For $m \in (m_*, m^*]$,
\n $C_{HLS,m} := \sup_{h \ne 0} \left\{ \frac{\mathcal{H}[h, h]}{\|h\|_1^{1-\sigma} \|h\|_m^{1+\sigma}}, h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d), h \ge 0 \right\}$
\nwith $\sigma = [m(d-2)]/[(m-1)d] - 1 \in (0, 1)$.
\n**RELM Blow-up criterion for chemotaxis system**

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A variant to the HLS (VHLS) inequality

Let *d* \geq 3, and let $\lambda = d - 2$ (Newtonian potential), $m_* = 2d/(d+2)$ and $m^* = 2 - \frac{2}{d}$. For $m \in (m_*, m^*]$,

[D](#page-0-0)raft *CHLS*,*^m* := sup *h*6=0 (H[*h*, *h*] k*h*k 1−σ 1 k*h*k 1+σ *m* , *h* ∈ *L* 1 (R *d*) ∩ *L m* (R *d*), *h* ≥ 0) with σ = [*m*(*d* − 2)]/[(*m* − 1)*d*] − 1 ∈ (0, 1).

Proposition

There exists a non-negative, radially symmetric and non-increasing f unction $V \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ with $\|V\|_1 = \|V\|_m = 1$ that attains *the variational problem CHLS*,*m.*

See (Blanchet/Carrillo/Laurençot, '09, CVPDE) for $m = m^*$ and (Kimi- $\lim_{m \to \infty}$ /Nakagawa/Ogawa, '14, CVPDE) for $m \in (m_*, m^*)$.

Parabolic-elliptic (PE) Keller-Segel model

The Keller-Segel (Keller/Segel, '70, JTB) model describes the motion of cells by chemotactical attraction by means of the coupled system:

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\nre Keller-Segel (Keller/Segel, '70, JTB) model describes the motion

\nlls by chemotactical attraction by means of the coupled system:

\n
$$
\begin{cases}\n\frac{\partial_t u}{\partial t} = \Delta u^m - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^d, \\
-\Delta v = u, & x \in \mathbb{R}^d.\n\end{cases}
$$
\n(1)

\n
$$
u = u(x, t) : \text{mass density};
$$
\n
$$
v = v(x, t) : \text{the concentration of the chemical attractant};
$$
\n
$$
m > 1: \text{degenerate diffusion};
$$
\nNote that $v = W * u$ solves the elliptic problem, where

\n
$$
\mathcal{W}(x) = \frac{c_d}{|x|^{d-2}}, \text{ if } d > 2.
$$
\nRELM

\nBlow-up criterion for chemotaxis system

• $u = u(x, t)$: mass density;

- $v = v(x, t)$: the concentration of the chemical attractant;
- \bullet *m* > 1: degenerate diffusion;

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• Note that $v = W * u$ solves the elliptic problem, where

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$$

Lyapunov functional for PE system

The main implication for us is that there is a natural Lyapunov functional for (1) defined on the set of centered mass densities $\rho \in$ $L^1_+(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ given by

unov functional for PE system
\nthe main implication for us is that there is a natural Lyapunc-
\ntional for (1) defined on the set of centered mass densities
$$
\rho
$$

\n \mathbb{R}^d) $\cap L^m(\mathbb{R}^d)$ given by
\n
$$
\mathcal{E}[\rho] = \mathcal{G}[\rho] - \mathcal{I}[\rho],
$$
\n
$$
\mathcal{G}[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^d} \rho^m(x) dx, \quad \mathcal{I}[\rho] = \frac{1}{2} \int_{\mathbb{R}^d} \rho(x) \mathcal{W} * \rho(x) dx,
$$
\n
$$
\mathcal{E}[\rho](t) \leq \mathcal{E}[\rho_0], \quad t > 0,
$$
\n
$$
\rho(x) \geq 0, \int_{\mathbb{R}^d} \rho(x) dx = M.
$$
\n**RELS B MS RELS BS MS**

The functional $\mathcal E$ becomes

$$
\mathcal{E}[\rho^{\lambda}] = \lambda^{d(m-1)}\mathcal{G}[\rho] - \lambda^{d-2}\mathcal{I}[\rho]
$$

by taking dilations $\rho^{\lambda}(x) := \lambda^d \rho(\lambda x)$. By scaling considerations, one can find 3 different regimes:

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- Diffusion-dominated regime: $m > m^*$. The diffusion part dominates and the intuition is that solutions exist globally in time;
- Aggregation-dominated regime: $m < m^*$. Blow-up occurs for some initial data, but not for all initial data;
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can find 3 different regimes:
 [D](#page-0-0)iffusion-dominated regime: $m > m^*$. The diffusion part do
 Fair-Competition regime: $m = m^*$. The total mass of system is the critical quantity. There is a critical value separating the diffusive behavior from the blow-up behavior.

Dichotomy phenomenon for $m = m^*$

Chotomy phenomenon for
$$
m = m^*
$$

\nTurn back to
\n
$$
\mathcal{E}[u] = \frac{1}{m-1} ||u||_{m}^{m} - \frac{c_d}{2} \mathcal{H}[u, u] \left(\leq C_{HLS,m} ||u||_{1}^{2/d} ||u||_{m}^{m} \right).
$$
\nLet
\n
$$
M_c = \left[\frac{2}{c_d (m-1) C_{HLS,m}} \right]^{d/2}.
$$
\nCharacteristicing the infimum of the free energy reveals a dichotomo phenomenon (Blanchet/Carrillo/Laurengot, '09, CVPDE):
\n• $M = ||u||_{1} < M_c$ ($\mathcal{E} > 0$): Solutions are global;
\n• $M = M_c$ ($\mathcal{E} = 0$): Solutions with finite second moment diver
\nto a Dirac mass, but those with radial initial conditions rema uniformly bounded (Bedrossian/Kim, '13, SIMA);
\n• $M > M_c$ ($\mathcal{E} < 0$): Solutions blow up in finite time.

$$
M_c = \left[\frac{2}{c_d(m-1)C_{HLS,m}}\right]^{a/2}
$$

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hold value for $m \in (m_*, m^*)$

he function $V \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ be the maximizer for C_{HLS}

me that
 $||u_0||_1^{\frac{m\alpha}{\beta}} \mathcal{E}[u_0] < ||V||_1^{\frac{m\alpha}{\beta}} \mathcal{E}[V]$
 $\begin{cases} \alpha := \frac{2}{2-m} - \frac{d}{m}, \\ \beta := d - \frac{2}{2-m}. \end{cases}$

we Let the function $V \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ be the maximizer for $C_{HLS,m}$. Assume that

$$
||u_0||_1^{\frac{m\alpha}{\beta}}\mathcal{E}[u_0]<||V||_1^{\frac{m\alpha}{\beta}}\mathcal{E}[V]
$$

with

$$
\begin{cases} \alpha := \frac{2}{2-m} - \frac{d}{m}, \\ \beta := d - \frac{2}{2-m}. \end{cases}
$$

Then we have the sharp result (Kimijima/Nakagawa/Ogawa, '14, CVPDE):

- $||u_0||_1^{\alpha}||u_0||_m^{\beta} < ||V||_1^{\alpha}||V||_m^{\beta} \Longrightarrow$ Global existence;
- $||u_0||_1^{\alpha} ||u_0||_m^{\beta} > ||V||_1^{\alpha} ||V||_m^{\beta} \Longrightarrow$ Finite-time blow-up.

The PP Keller-Segel model:

[D](#page-0-0)raft ∂*tu* = ∆*u ^m* − ∇ · (*u*∇*v*), ∂*tv* = −*f* = ∆*v* − *v* + *u*,

where $m > 1$ and $d \geq 3$.

The *v* can be decomposed by

 $v = \tilde{v} + \hat{v}$.

where $\tilde{v}(x) := (\mathcal{B} * u)(x), -\Delta \tilde{v} + \tilde{v} = u,$

$$
\hat{v}(x) := (\mathcal{B} * f)(x), \ -\Delta \hat{v} + \hat{v} = f.
$$

Here \mathcal{B} is the Bessel kernel.

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KE LIN Blow-up criterion for chemotaxis system

The PP Keller-Segel model:

The
$$
v
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$$
\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = -f = \Delta v - v + u, \end{cases}
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[D](#page-0-0)ie-Parabolic (PP) KS model

P Keller-Segel model:

The *v* can be decomposed by
 $u = \Delta u^m - \nabla \cdot (u\nabla v)$, where
 $v = -f = \Delta v - v + u$, $\tilde{v}(x) := (\mathcal{B} * u)(x), -\Delta \tilde{v} +$
 $\hat{v}(x) := (\mathcal{B} * f)(x), -\Delta \tilde{v} +$

Here \mathcal{B} is the B where $\tilde{v}(x) := (\mathscr{B} * u)(x), -\Delta \tilde{v} + \tilde{v} = u,$ $\hat{v}(x) := (\mathcal{B} * f)(x), -\Delta \hat{v} + \hat{v} = f.$ Here \mathscr{B} is the Bessel kernel.

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Question: What is sharp critical mass criterion for the fully parabolic Keller-Segel model?

Lyapunov functional for PP system

$$
\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases}
$$

Lyapunov functional: Gradient flow of

mov functional for PP system

\n
$$
\begin{cases}\n\frac{\partial_t u}{\partial t} = \Delta u^m - \nabla \cdot (u \nabla v), \\
\frac{\partial_t v}{\partial t} = \Delta v - v + u.\n\end{cases}
$$
\nunov functional: Gradient flow of

\n
$$
\mathcal{F}[u, v] := \int_{\mathbb{R}^d} \left(\frac{1}{m-1} u^m + \frac{1}{2} |\nabla v|^2 + \frac{1}{2} v^2 - uv \right) dx.
$$
\nKE LIN

\nBlow-up criterion for chemotaxis system

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Lyapunov functional for PP system

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$$
\nthat

\n
$$
\mathcal{F}[u, v] = \mathcal{F}[u, \tilde{v}] + \frac{1}{2} ||\nabla (v - \tilde{v})||_2^2 + \frac{1}{2} ||v - \tilde{v}||_2^2,
$$
\nwe

\n
$$
\mathcal{F}[u, \tilde{v}] \text{ is Lyapunov functional for}
$$
\n
$$
\begin{cases}\n\frac{\partial_t u}{\partial t} = \Delta u^m - \nabla \cdot (u \nabla \tilde{v}), \\
\tilde{v} = \mathcal{B} * u.\n\end{cases}
$$
\n
$$
\mathcal{F}[u, \tilde{v}] > 0 \implies \mathcal{F}[u, v] > 0
$$
\nNow-up criterion for chemotaxis system

Note that

$$
\mathcal{F}[u, v] = \mathcal{F}[u, \tilde{v}] + \frac{1}{2} ||\nabla(v - \tilde{v})||_2^2 + \frac{1}{2} ||v - \tilde{v}||_2^2,
$$

where $\mathcal{F}[u, \tilde{v}]$ is Lyapunov functional for

$$
\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla \tilde{v}), \\ \tilde{v} = \mathcal{B} * u. \end{cases}
$$

 σ $\mathcal{F}[u, \tilde{v}] > 0 \Longrightarrow \mathcal{F}[u, v] > 0$

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Global well-posedness

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\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases}
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Let $m = m^*$ and $d \geq 3$. If $M < M_c$, then there exists a we

global solution to the parabolic-parabolic Keller-Segel syste

(Blanchet/Laurençot, ' Let $m = m^*$ and $d \geq 3$. If $M < M_c$, then there exists a weak global solution to the parabolic-parabolic Keller-Segel system (Blanchet/Laurençot, '13, CPDE);

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- **1 well-posedness**
 $\begin{cases}\n\partial_t u = \Delta u^m \nabla \cdot (u \nabla v), \\
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(Blanchet/Laurençot, '1 Let $m = m^*$ and $d \geq 3$. If $M < M_c$, then there exists a weak global solution to the parabolic-parabolic Keller-Segel system (Blanchet/Laurençot, '13, CPDE);
- Let $m\in (m_*,m^*)$ and $d\geq 3$. If $\|u_0\|^{\frac{m\alpha}{\beta}}\mathcal{F}[u_0,v_0]<\|V\|^{\frac{m\alpha}{\beta}}\mathcal{E}[V]$ and

 $\|u_0\|_1^{\alpha}\|u_0\|_m^{\beta} < \|V\|_1^{\alpha}\|V\|_m^{\beta},$

the weak solution is global and uniformly bounded (L./Wang, preprint).

Sharp blow-up criterion

$$
\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), \\ \partial_t v = \Delta v - v + u. \end{cases}
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Let $m = m^*$ and $d = 3, 4$. If $M > M_c$, radially symmetric slutions with negative initial energy blow up in finite time (Lateracyt/Mizoguchi, '15, Po Let $m = m^*$ and $d = 3, 4$. If $M > M_c$, radially symmetric solutions with negative initial energy blow up in finite time (Laurençot/Mizoguchi, '15, Poincaré);

Sharp blow-up criterion

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- **blow-up criterion**
 $\begin{cases}\n\frac{\partial_t u}{\partial t} = \Delta u^m \nabla \cdot (u \nabla v), \\
\frac{\partial_t v}{\partial t} = \Delta v v + u.\n\end{cases}$ $\begin{cases}\n\frac{\partial_t u}{\partial t} = \Delta u^m \nabla \cdot (u \nabla v), \\
\frac{\partial_t v}{\partial t} = \Delta v v + u.\n\end{cases}$ $\begin{cases}\n\frac{\partial_t u}{\partial t} = \Delta u^m \nabla \cdot (u \nabla v), \\
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- Let $m \in (m_*, m^*)$ and $d \geq 3$. If $\|u_0\|_1^{\frac{m\alpha}{\beta}} \mathcal{F}[u_0, v_0] < \|V\|_1^{\frac{m\alpha}{\beta}} \mathcal{E}[V]$ and

$$
||u_0||_1^{\alpha}||u_0||_m^{\beta} > ||V||_1^{\alpha}||V||_m^{\beta},
$$

there exists a radially symmetric weak solution blowing up in finite time (L./Wang, preprint).

Sharp blow-up criterion

Denoting the second moment by

$$
M_2(t) = \int_{\mathbb{R}^d} |x|^2 u(x, t) dx
$$

and computing we have

.

blow-up criterion
\n
$$
M_2(t) = \int_{\mathbb{R}^d} |x|^2 u(x, t) dx
$$
\ncomputing we have
\n
$$
\frac{d}{dt} M_2(t) \le 2(d-2) \underbrace{\mathcal{F}[u, v]}_{\le \mathcal{F}[u_0, v_0]} - 2 \underbrace{\left(\frac{d-2}{m-1} - d\right)}_{>0} ||u||_m^m
$$
\n
$$
+ \begin{cases} C||f||_2 M_2^{1/4}(t), & \text{if } d = 3, \\ C||f||_2, & \text{if } d = 4, \\ C(||f||_2^{\theta} + ||f||_2), & \theta \in (1, 2), \text{if } d > 4. \end{cases}
$$
\n**KELIN**

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The functional $\mathcal F$ fulfills

$$
\mathcal{F}[u,v](t)+\int_0^t \left(\mathcal{D}[u,v]+\|f\|_2^2\right)d\tau\leq \mathcal{F}[u_0,v_0],
$$

 $\mathcal{F}[u, v] \geq -C(1 + \mathcal{D}[u, v])^{\delta}, \ \delta \in (0, 1),$

where the entropy dissipation $\mathcal D$ is given by

$$
\mathcal{D}[u,v] = \int_{\mathbb{R}^d} u \left| \frac{m}{m-1} \nabla u^{m-1} - \nabla v \right|^2 dx.
$$

blow-up criterion

functional F fulfills
 $\mathcal{F}[u, v](t) + \int_0^t (\mathcal{D}[u, v] + ||f||_2^2) d\tau \leq \mathcal{F}[u_0, v_0],$ $\mathcal{F}[u, v](t) + \int_0^t (\mathcal{D}[u, v] + ||f||_2^2) d\tau \leq \mathcal{F}[u_0, v_0],$ $\mathcal{F}[u, v](t) + \int_0^t (\mathcal{D}[u, v] + ||f||_2^2) d\tau \leq \mathcal{F}[u_0, v_0],$
 $\mathcal{F}[u, v] \geq -C(1 + \mathcal{D}[u, v])^\delta, \ \delta \in (0, 1),$

e the entropy dissipation D is given by
 $\mathcal{D}[u, v] = \int_{\mathbb{R}^d} u \left|$ If $T_{\text{max}} = +\infty$, then $\int_0^\infty ||f||_2^2 d\tau < \infty$. However, M_2 will be negative after some finite time and hence blow-up occurs.

- In the fair competition regime for degenerate parabolic-ellip
Keller-Segel model: dichotomy similar to two-dimensional sting.
The sharp critical mass is related to the maximizer for the HI
inequality;
This criterion for pa • In the fair competition regime for degenerate parabolic-elliptic Keller-Segel model: dichotomy similar to two-dimensional setting.
- The sharp critical mass is related to the maximizer for the HLS inequality;
- This criterion for parabolic-parabolic Keller-Segel model is consistent with the parabolic-elliptic version;

Thanks for your attention! *Thanks for your attention!*

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