Geometry and Physics of Quantum Toroidal Algebra

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1 Overview of the Field

This workshop focused on quantum toroidal algebras (QTAs) and double affine Hecke algebras (DAHAs), which are central objects in modern mathematical physics and representation theory. These algebras serve as powerful tools for understanding a variety of structures arising in both geometry and quantum field theory, while simultaneously providing deep connections between areas such as integrable systems, algebraic geometry, and representation theory.

QTAs is a toroidal generalization of quantum groups, extending the structure of quantum affine algebras [21, 12]. These algebras are the key to understand exact solvable models. They now play a pivotal role in mathematical frameworks that describe topological string theory and 5-dimensional supersymmetric gauge theories. The rational degeneration of QTA, called affine Yangian, plays an essential role in the proof of the AGT conjecture [46]. Another key area of interest is the connection between quantum toroidal algebras and the elliptic Hall algebras, which offers new insights into both combinatorial and geometric representation theory. Furthermore, their relations to deformed W-algebras and the study of quantum integrable systems have opened new avenues of research [13, 14], particularly in the context of Bethe ansatz and the quantization of moduli spaces of sheaves on algebraic surfaces. The study of these algebras connects to several advanced topics, including the enumerative geometry, integrable systems, and the theory of derived categories.

DAHAs, introduced by I. Cherednik in the context of solving conjectures on Macdonald polynomials [8, 9], have also become essential tools in the study of symmetric functions and representation theory. DAHA and QTA are related by Schur-Weyl duality [49, 43]. Furthermore, DAHAs are connected with QTAs via the large rank limit [45]. Recent developments in DAHA have demonstrated the importance of these algebras in the study of vertex operator algebras, affine Grassmannians, the moduli space of Higgs bundles and the theory of 4d Coulomb branches.

They have concrete implications for understanding the various moduli spaces, and related enumerative invariants in algebraic geometry. In particular, the interplay between DAHAs and quantum toroidal algebras is crucial for understanding the geometric representation theory of intriguing geometry, such as quiver varieties, and the moduli space of BPS states. The link between these algebras and supersymmetric gauge theories has also led to advances in the study of quantum integrable systems, BPS states and geometry, further bridging the gap between mathematics and theoretical physics.

In summary, QTAs and double affine Hecke algebras form the foundation of a rich and active area of research at the intersection of algebra, geometry, and physics. QTAs and DAHA represent algebras of the 21st century that deserve further in-depth investigation. Their applications are vast, influencing fields from quantum field theory to algebraic geometry, and their intricate structures continue to inspire new directions in the mathematical sciences.

2 Recent Developments and Open Problems

The field of QTAs and DAHAs has seen substantial progress in recent years, with several exciting developments emerging at the intersection of algebra, geometry, and physics. Although the BIRS-IASM workshop was postponed for four years due to the pandemic, this area of research has steadily advanced through the dynamic interaction between mathematics and physics. Many of these advancements were highlighted during the workshop, emphasizing not only the growing influence of these algebras in mathematical physics but also the increasing number of connections they reveal to various other domains. However, despite these achievements, the field continues to present significant open problems that invite further exploration. Although the developments and open questions in this rapidly evolving area are too numerous to list exhaustively, we outline some of the key advancements and promising research directions below.

QTAs in Gauge Theory and String Theory: One of the most significant areas of recent progress is the deepening connection between QTAs and gauge theory, particularly within the framework of 5-dimensional supersymmetric gauge theories and string theory. QTAs have emerged as hidden symmetries in the equivariant K-theory of moduli spaces of instantons [33] and in the moduli space of BPS states in string theory. This relationship forms part of a broader program known as the *BPS/CFT correspondence* [37], which links the algebraic structures of QTAs to conformal field theories through BPS state counting.

In addition, the *topological vertex*a key tool in string theorycan be formulated using representations of QTAs [2]. Moreover, *Corner Vertex Algebras* [16], which arise at the junction of fivebranes with D3-branes in Type IIB string theory, provide another important link to affine Yangian. It was later discovered that by ”gluing” these trivalent verticesakin to the construction of the topological vertexone can generate a larger class of W-algebras, referred to as the *web of W-algebras* [41]. Given a quiver diagram and a superpotential, one can associate *quiver quantum toroidal algebras* [29, 17, 39] to the system, extending the interplay between algebra and geometry. This structure has greatly enriched the understanding of QTAs and their cousins and their relationship to string theory.

String theory has thus provided a powerful framework for generalizing QTAs and connecting them to other algebraic structures. However, this is just the beginning. There remains much to explore, particularly in the representation theory of these algebras and the study of associated partition functions and enumerative invariants. Furthermore, the realization of QTAs in terms of vertex operator algebras and deformed W-algebras has opened up new avenues for the study of *integrable systems* and *conformal field theory*, offering fresh perspectives and tools for advancing both mathematics and physics.

Enumerative Invariants and Representation Theory: All of the algebras discussed here can be understood as ”algebras of BPS states.” In physics, a general principle suggests that a particular subspace of BPS states forms an algebra, which then acts on a broader (or potentially the entire) space of BPS states, thus providing a representation. The character of this representation serves as the generating function for enumerative invariantssuch as Gopakumar-Vafa, Donaldson-Thomas, and Pandharipande-Thomas invariantsassociated with the corresponding Calabi-Yau manifold. While this framework provides a comprehensive picture, only a limited number of explicit examples have been developed to fully illustrate this idea. Consequently, linking the representation theory of quantum toroidal algebras to the enumerative geometry of Calabi-Yau manifolds remains a highly promising direction for future research [3, 28, 44, 51, 47, 42, 11, 10, 25].

The cohomological Hall algebra (COHA) [26] associated with quiver data (Q, W) is anticipated to form the mathematical foundation for algebras of BPS states. Building on the structure of the COHA, geometric constructions of corner vertex operator algebras (VOAs), QTAs, and shifted Yangians have been developed. These constructions establish a rich interplay between algebra and geometry, offering new perspectives on these algebras. Moreover, the representation theory of these algebras is further enriched by the ”correspondence” framework, which provides deep connections to enumerative invariants. Through this approach, the representation theory of quantum toroidal algebras can be linked to the enumerative geometry of moduli spaces, offering fertile ground for future exploration.

Applications in Quantum Integrable Systems: One of the most important features of QTAs is their structure as quantum groups equipped with a universal R-matrix. This R-matrix plays a central role in the construction of integrable models. By employing the quantum inverse scattering method, it is possible to construct an infinite family of mutually commuting conserved charges and Bethe states through the transfer matrix and Bethe ansatz equations. These conserved charges correspond to integrals of motion in integrable field theories, such as those governed by intermediate long-wave equations.

Another significant connection between supersymmetric gauge theories and integrable systems arises through the Bethe/Gauge correspondence. In this context, we introduce the physical observable known as the *qq-character*, which represents a key manifestation of the BPS/CFT correspondence. The qq-character provides a unifying framework that incorporates quantum toroidal algebras, the Bethe/Gauge correspondence, and BPS defects. From the perspective of integrable systems, the qq-character can be viewed as the second quantization of the TQ-relation in quantum integrable models, such as spin chain systems [38], offering an algebraic explanation of the Bethe/Gauge correspondence.

Another direction is to find a closed formula of the R-matrix. Recently, a closed recursion formula was derived in the context of Macdonald functions [34, 19, 20], suggesting a deeper connection to integrable models. While generalizations to other quiver quantum toroidal algebras are yet to be fully explored, future work in this direction would be highly desirable. In particular, investigating the relationship between these results and the Maulik-Okounkov R-matrix [30] within the full Ω-background framework may provide a systematic understanding of the connections between QTAs and the various integrable systems discussed above.

Double Affine Hecke Algebras and Coulomb branch DAHAs were initially constructed to provide an algebraic framework for understanding q-orthogonal symmetric polynomials and difference operators. Since their introduction, however, numerous connections to diverse areas of mathematics and physics have emerged. In particular, recent research has explored DAHAs in the context of the geometry of affine Grassmannians, character varieties, and 4d Coulomb branches [50, 40, 5], unveiling deep relationships between algebra, geometry, and physics.

One significant research direction involves the deformation quantization of the coordinate rings of 4d Coulomb branches [6, 7], which provides a vast generalization of DAHAs. A promising avenue for future investigation is the study of the representation theory of these quantized algebras in relation to the geometry of Coulomb branches. The interpretation of these structures in physics reveals further connections to the algebra of BPS states [23], 4d mirror symmetry [48], and vertex operator algebras arising from Higgs branches [4]. These connections are only beginning to be uncovered, and much remains to be explored, presenting many open questions that call for deeper investigation.

Conclusion: The recent progress in the study of QTAs and DAHAs has greatly expanded the scope of these algebras and their applications. However, the field remains rich with open problems that invite further inquiry. These challenges lie at the interface of algebra, geometry, and physics, and their resolution will likely lead to significant breakthroughs in our understanding of representation theory, gauge theory, and integrable systems. The continued exploration of these topics promises to deepen the connections between mathematics and theoretical physics, offering new insights into both disciplines.

3 Workshop highlight

The workshop brought together a diverse group of 39 in-person and 21 online participants, creating a truly international environment with researchers joining from Asia, Europe, North America, and Australia. This broad geographic representation underscored the global interest and collaborative spirit in exploring the cutting-edge connections between mathematics and physics.

A unique aspect of the workshop was the mixture of 37 mathematicians and 23 physicists, which naturally fostered cross-disciplinary discussions. In this field, the lines between mathematics and physics are often blurred, and this mix led to lively exchanges and collaborative thinking, especially in areas like geometry, representation theory, integrable systems, quantum field theory and string theory.

The program included three online talks by the leading researcher in the field, which allowed remote participants to stay involved, as well as two presentations by students, highlighting the growing interest in this field from younger researchers. Additionally, two female researchers presented their work, contributing valuable perspectives to the conversations. This diversityacross both experience levels and backgroundsadded depth and vibrancy to the workshop, making it an enriching experience for everyone involved.

Across the presentations, participants explored various facets of QTAs and DAHAs (as well as their degeneration algebras), highlighting their wide-ranging applications. Talks by Evgeny Mukhin, Hitoshi Konno, and Yaping Yang focused on the connections to integrable systems, emphasizing solvable structures that are fundamental for understanding these quantum algebras and their physical implications.

In contrast, Yutaka Matsuo, Hidetoshi Awata, Hiroaki Kanno, Noshita Go, and Yegor Zenkevich approached QTAs from a more physics-oriented perspective. Their presentations explored connections to string theory, branes, and supersymmetric theories, showcasing how these abstract algebraic structures are deeply integrated with modern theoretical physics. Sergei Gukov talk the connection between DAHA and quantum topology of 3-manifolds. This balance between algebraic and physical perspectives offered participants a comprehensive understanding of the current state of the field.

The workshop also placed a strong emphasis on the deep connections between QTAs, DAHAs, and geometry. Andrei Negut, Michael McBreen, Konstantin Jakob, Gufang Zhao, and Eric Vasserot approach these algebras through a geometric and categorical perspective [35, 31, 24, 52, 32], focusing on the moduli space of sheaves on surfaces and the moduli space of Higgs bundles. Their talks underscored how the geometry of these moduli spaces is intricately tied to the structure of QTAs and DAHAs.

Cohomological Hall algebras (COHAs) were another major focus of the workshop. In particular, Eric Vasserot discussed COHAs attached to quivers and algebraic surfaces [32]. Wei Li described how COHAs describe BPS spectra and their connection to spherical shuffle algebras, offering insights into 1/2-BPS line defects in supersymmetric theories [15].

Additionally, Dylan Allegretti and Harold Williams explored the geometry of Coulomb branches and their associated quantized algebras, further exploring the interplay between geometry and quantum algebras [1, 18]. Tomoyuki Arawaka gave an overview talk on the relation between vertex operator algebras and Higgs branch of supersymmetric theories with eight-supercharges. Meanwhile, Oleksandr Tsymbaliuk, Jean-Emile Bourgine, Duncan Laurie, and Syu Kato approached QTAs from an algebraic perspective [36, 27], focusing on their representations and highlighting the rich algebraic structure of these theories.

Workshop Schedule

Monday, September 2

- **Evgeny Mukhin**: Integrable systems related to quantum toroidal algebras
- **Yutaka Matsuo**: Some applications of W-infinity/affine Yangian/QTA to physics
- **Michael McBreen**: The Hamiltonian reduction of hypertoric mirror symmetry
- **Wei Li**: BPS algebras for 4D N=2 theories and their line defects
- **Andrei Negut**: Elliptic $\mathrm{gl}_1^{\wedge\wedge}$ and moduli spaces of sheaves on surfaces (Online)

Tuesday, September 3

- **Tomoyuki Arakawa**: Symplectic singularities and vertex algebras
- **Konstantin Jakob**: A Deligne-Simpson problem for irregular G-connections on $\mathbb{P}^1$
- **Hidetoshi Awata**: Quantum deformation of the N=2 superconformal algebra I
- **Hiroaki Kanno**: Quantum deformation of the N=2 superconformal algebra II
- **Gufang Zhao**: A Langlands Duality of Elliptic Hecke Algebras
- **Hitoshi Konno**: Vertex Operators and L-operators of Elliptic Quantum Toroidal Algebras

Wednesday, September 4

- **Eric Vasserot**: W-algebras for surfaces

- **Noshita Go**: Gauge origami and quiver W-algebras
- **Duncan Laurie**: Quantum toroidal algebras in untwisted types

Thursday, September 5

- **Harold Williams**: Differential operators on the base affine space and quantized Coulomb branches (Online)
- **Sergei Gukov**: Toroidal structures in quantum topology
- **Yaping Yang**: Higher spin representations of the Yangian of sl_2 and R-matrices
- **Yegor Zenkevich**: Spiraling branes and quantum toroidal algebras
- **Dylan Allegretti**: Skein algebras and quantized Coulomb branches
- **Oleksandr Tsymbaliuk**: Lyndon words and fused currents in shuffle algebra (Online)

Friday, September 6

- **Jean-Emile Bourgine**: Free field representations of quantum groups and q-deformed W-algebras through cluster algebras
- **Syu Kato**: Kostka polynomials of $\mathrm{G}(\ell, 1, n)$

Areas for Improvement: First, it would have been beneficial to incorporate two or three discussion sessions in place of some research talk slots to enhance participant engagement. These sessions could be moderated by selected experts who would prepare guiding questions and discussion topics in advance. This would provide an opportunity for attendees to exchange new ideas, explore emerging research directions, and foster collaborative insights.

Additionally, international participants found navigating the travel logistics to the workshop venue challenging, especially traveling from Shanghai to the hotel. Some participants were unaware that Hangzhou airport is the closest option and therefore flew into Shanghai instead. Ideally, comprehensive travel information for reaching BIRS-IASM could be provided with the initial invitation letter, sent at least three months before the workshop, to allow participants to make well-informed travel arrangements. Since the primary organizer and some attendees are based in Shanghai, offering international participants the option to stay in Shanghai before and after the workshop could further ease travel logistics. This added guidance would help ensure a smoother and more accessible experience for those traveling from abroad.

4 Outcomes of the Workshop and Future Works

While the primary focus of this 5-day workshop was to share recent research results and promote collaboration in QTAs and DAHAs, the nature of the presentations and discussions means that the outcomes will unfold over time as participants continue their research. The most immediate and significant achievement was the successful organization of the workshop at IASM in Hangzhou, after a 4-year delay due to the COVID-19 pandemic. Despite the challenges, we brought together many of the worlds leading experts in the field, fostering a highly collaborative environment. The interactions and exchanges that took place during the workshop are expected to catalyze new developments and accelerate progress in this rapidly evolving research area.

The future of QTAs and DAHAs holds a wealth of opportunities. Many of the open problems and challenges outlined in Section 2 remain open, providing rich directions for future research. In addition, the presentations at the workshop pose many new questions and problems, and the workshop itself fosters discussions on these questions, as participants explore new ideas and potential research directions. These open problems span across algebraic, geometric, and physical perspectives, providing fertile ground for future research. As participants move forward with these questions in mind, the field expects further developments and breakthroughs. We hope that the workshop has made a lasting impact on both the participants and the broader research community, and we expect that the collaborations initiated during the event will play a role in advancing this exciting area of study.

The ongoing study of QTAs and DAHAs continues to present a wealth of research challenges and opportunities. While it is impossible to address all the open problems and future directions that emerged during the workshop, we highlight some of the key areas of research below.

Mathematical Formulations and Interrelation of Algebras The workshop highlighted the emergence of several significant algebras, including QTAs, corner vertex algebras, the web of W-algebras, quiver W-algebras, COHAs, and shuffle algebras. A key challenge for future research lies in understanding the equivalences and relationships between these algebras. Clarifying these connections will not only advance our knowledge of the underlying algebraic frameworks but also strengthen the links between algebra, geometry, and physics. Many of the physics insights and predictions presented at the workshop now require translation into mathematically rigorous formulations.

The study of the representation theory of these algebras presents a particularly promising direction. Approaches discussed at the workshop include links to enumerative invariants of Calabi-Yau manifolds, integrable systems, and the geometry of various moduli spaces. Addressing these open problems will help bridge the gap between algebraic structures and their geometric and physical applications, offering fresh perspectives in both mathematics and theoretical physics.

Connection Between DAHAs and Mirror Symmetry The spherical DAHA can be constructed as the deformation quantization of coordinate ring of the Hitchin moduli space over a curve. Thus, the representation theory of DAHA can be studied by the Fukaya category of the Hitchin moduli space [22]. It is important to study DAHA in the context of mirror symemtry and geometric Langlands duality. Although the SYZ mirror symmetry in the Hitchin moduli space leads to geometric Langlands duality, the precise relationship between DAHA and Langlands duality is still not fully understood. Further research is needed to clarify this connection, particularly in the context of higher-dimensional varieties and their mirror duals. This problem also intersects with the study of homological mirror symmetry and derived categories.

BPS States, Quivers, and Spherical Shuffle Algebras Both cohomological Hall algebras (COHAs) and quantum toroidal algebras (QTAs) are deeply connected to BPS state counting in supersymmetric gauge theories. A prominent open problem in this area is the conjecture that the BPS algebra for certain 4d $\mathcal{N} = 2$ gauge theories with a quiver description is a spherical shuffle algebra, which shares significant structural parallels with both QTAs and COHAs. The algebraic framework of COHAs on the moduli space of quiver representations often reflects the representation-theoretic structure of QTAs, presenting a rich avenue for future exploration into their shared properties and interactions.

Moreover, spherical shuffle algebras are closely linked to QTAs and arise naturally in the study of COHAs when viewed through the moduli spaces of quiver or sheaf representations. For certain 4d $\mathcal{N} = 2$ gauge theories, the BPS algebra is conjectured to take the form of a spherical shuffle algebra. Understanding the precise relationship between spherical shuffle algebras, QTAs, and COHAs offers significant potential for new insights into the algebraic structures governing BPS states. This connection between algebra, geometry, and supersymmetric theories provides fertile ground for further research, and deeper investigation into these relationships could yield breakthroughs in the study of both QTAs and COHAs.

Moduli Spaces of Sheaves or Quiver Varieties: A promising direction for future research lies in the study of quantum toroidal algebras and their actions on the cohomology of moduli spaces of sheaves on surfaces, such as the Hilbert schemes of points. Similarly, COHAs are constructed from the cohomology of moduli spaces of quiver representations. The overlap in the types of moduli spaces associated with both quantum toroidal algebras and COHAs suggests a rich area for further exploration. This shared geometric setting provides a natural bridge between these two algebraic structures, and understanding the precise nature of this connection could yield significant insights into both fields. Exploring these moduli spaces and their algebraic symmetries promises to uncover deeper relationships between algebra, geometry, and physics.

Categorification of QTAs: A key open problem is the construction of a comprehensive categorification for higher-rank QTAs, which would involve a deeper understanding of their homological properties. Additionally, developing the categorical representation theory of QTAs in connection with geometric representation theory remains a significant challenge. This intersection between categorification and geometry is expected to yield new insights into the structure and representations of QTAs, making it an exciting area for future exploration.

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