Quasi-neutral limit in Euler-Poisson systems

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Outline

1. One-fluid Euler-Poisson system

2. Justification of the quasi-neutral limit

3. Quasi-neutral limit in two-fluid Euler-Poisson system

1. One-fluid Euler-Poisson system

plasma model

$$\begin{cases} \partial_t n + \mathrm{div}(nv) = 0 \\ \partial_t(nv) + \mathrm{div}(nv \otimes v) + \nabla p(n) = -n\nabla\phi - \underbrace{n\,v}_{\text{damping}} \ , \quad t > 0,\ x \in \mathbb{R}^d \\ -\lambda^2 \Delta\phi = n - b, \quad \lim_{|x|\to+\infty} \phi(t,x) = 0 \\ t = 0\ :\ (n, v) = \big(n_{0\lambda}(x), v_{0\lambda}(x)\big) \end{cases}$$

$n > 0$: density of electrons

v : velocity of electrons

p : pressure, $p'(n) > 0$ for $n > 0$, example : $p(n) = n^\gamma$, $\gamma \geq 1$

ϕ : electric potential

$b > 0$: density of ions, given function of (t, x), for simplicity, take $b = 1$

$q = nv$: momentum

$\lambda > 0$: Debye length, small parameter compared to the size of (n, v)

$\star$ F.Chen, Introduction to Plasma Physics and Controlled Fusion, 1984 the quasi-neutrality of a plasma means

$$\text{electron density} = \text{ion density} \iff n = 1 \quad (b = 1)$$

It is formally realized from the Poisson equation as $\lambda \to 0$

$$-\lambda^2 \Delta\phi = n - 1$$

Therefore, $\lambda \to 0$ is called the quasi-neutral limit

Goal : Study the limit behavior of smooth solutions to EPS as $\lambda \to 0$

EPS : Euler equations coupled to a Poisson equation

Another form of the Poisson equation

$$-\lambda^2 \Delta \phi = n - 1 \implies -\lambda^2 \Delta \partial_t \phi = \partial_t n = -\operatorname{div} q$$

Introduce the electric field $E = \nabla \phi$ and a projection operator

$$\mathcal{P} = \nabla \Delta^{-1} \operatorname{div}$$

Then

$$\operatorname{div}\left(\lambda^2 \partial_t E\right) = \operatorname{div} q \Longrightarrow \lambda^2 \partial_t E = \mathcal{P} q$$

We obtain a first-order PDEs

$$\begin{cases} \partial_t n + \operatorname{div}(nv) = 0 \\ \partial_t (nv) + \operatorname{div}(nv \otimes v) + \nabla p(n) = -nE - nv \\ \lambda^2 \partial_t E = \mathcal{P} q \end{cases}$$

with

$$\|\mathcal{P} q\| \leq \|q\|, \qquad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}$$

Hyperbolic structure of EPS

$$\mathcal{E}(n,q,E)=\frac{|q|^2}{2n}+H(n)+\frac{\lambda^2}{2}|E|^2,\qquad \mathcal{F}(n,q)=\frac{|q|^2q}{2n^2}+h(n)q$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^d$ and

$$h'(n)=\frac{p'(n)}{n},\qquad H'(n)=h(n)$$

$(\mathcal{E},\mathcal{F})$ is a pair of entropy-entropy flux satisfying an entropy equality

$$\partial_t\mathcal{E}(n,q,E)+\operatorname{div}\mathcal{F}(n,q)+\frac{|q|^2}{n}=(\mathcal{P}-I)qE$$

and an energy equality

$$\int_{\mathbb{R}^d}\mathcal{E}(n,q,E)dx+\int_0^t\int_{\mathbb{R}^d}\frac{|q|^2}{n}dxdt'=\int_{\mathbb{R}^d}\mathcal{E}(n_{0\lambda},q_{0\lambda},E_{0\lambda})dx$$

where

$$q_{0\lambda}=n_{0\lambda}v_{0\lambda},\qquad E_{0\lambda}=\nabla\phi_{0\lambda},\qquad -\lambda^2\Delta\phi_{0\lambda}=n_{0\lambda}-1$$

Moreover

$$\forall\, n > 0, \qquad \mathcal{E}''(n,q,E) = \frac{1}{n^3}\begin{pmatrix} |q|^2 + n^2 p'(n) & -nq^\top & 0 \\ -nq & n^2 I_d & 0 \\ 0 & 0 & \lambda^2 n^3 I_d \end{pmatrix}$$

which is symmetric positive definite $\Longrightarrow \mathcal{E}$ is a strictly convex entropy

A system of balance laws admitting a strictly convex entropy is symmetrizable hyperbolic (S.K. Godunov 1961)

Consequence : EPS is a symmetrizable hyperbolic system

Local existence of smooth solutions (Lax 1973, Kato 1975, Majda 1984

$$m > d/2 + 1 \text{ is an integer}, \quad n_{0\lambda} - 1, v_{0\lambda}, \nabla\phi_{0\lambda} \in H^m = H^m(\mathbb{R}^d)$$

Then $\exists$ a maximum time $T_\lambda > 0$ and a smooth solution $(n_\lambda, v_\lambda, \phi_\lambda)$ satisfying

$$(n_\lambda - 1, v_\lambda, \nabla\phi_\lambda) \in C([0, T_\lambda); H^m) \cap C^1([0, T_\lambda); H^{m-1})$$

Global existence with small initial data

⋆ Ali-Bini-Rionero 2000

⋆ H.L.Li-M.Mei-Markowich 2002

⋆ Hsiao-Markowich-Wang 2003

$\exists\, \delta_\lambda > 0$ such that if

$$\|n_{0\lambda} - 1\|_m + \|v_{0\lambda}\|_m + \|\nabla\phi_{0\lambda}\|_m \le \delta_\lambda$$

the smooth solution is defined for all time

$$(n_\lambda - 1, v_\lambda, \nabla\phi_\lambda) \in C(\mathbb{R}^+; H^m) \cap C^1(\mathbb{R}^+; H^{m-1})$$

2. Justification of the quasi-neutral limit

Let $(n_\lambda, v_\lambda, \phi_\lambda)$ be the smooth solution to

$$\begin{cases} \partial_t n_\lambda + \operatorname{div}(n_\lambda v_\lambda) = 0 \\ \partial_t (n_\lambda v_\lambda) + \operatorname{div}(n_\lambda v_\lambda \otimes v_\lambda) + \nabla p(n_\lambda) = -n_\lambda \nabla \phi_\lambda - n_\lambda v_\lambda \\ -\lambda^2 \Delta \phi_\lambda = n_\lambda - 1 \end{cases}$$

Formal limit leads, as $\lambda \to 0$

$$\lim_{\lambda \to 0} (n_\lambda, v_\lambda, \nabla \phi_\lambda) = (n, v, \nabla P)$$

and

$$\begin{cases} n = 1 \\ \operatorname{div} v = 0 \text{ (incompressibility condition)} \\ \partial_t v + (v \cdot \nabla) v + \nabla P = -v \end{cases}$$

Therefore, v satisfies incompressible Euler equations with damping

Justification

$$(n_\lambda, v_\lambda, \phi_\lambda) \longrightarrow (1, v, \phi) \ ?$$

Local-in-time convergence : assume

$$\|n_{0\lambda} - 1\|_m + \|v_{0\lambda} - v(0)\|_m + \lambda\|\nabla(\phi_{0\lambda} - \phi(0))\|_m \leq C\lambda$$

$\exists$ constants $T > 0$ and $C > 0$ (independent of λ) such that

$$\sup_{t\in[0,T]} \big(\|n_\lambda(t) - 1\| + \|v_\lambda(t) - v(t)\| + \lambda\|\nabla(\phi_\lambda(t) - \phi(t))\|\big) \leq C\lambda$$

Method : energy estimates for $U = (n_\lambda - 1, v_\lambda - v, \nabla(\phi_\lambda - \phi))$

⋆ Cordier and Grenier 2000, from 1-d model to compressible Euler equations

$$\begin{cases} \partial_t n + \partial_x(nv) = 0 \\ \partial_t(nv) + \partial_x(nv^2) + a^2\partial_x n = -n\partial_x\phi \\ -\lambda^2\partial_{xx}\phi = n - e^\phi \end{cases}$$

$$\lambda \to 0 \implies \phi = \ln n \quad \text{(Boltzmann relation)}$$

⋆ S.Wang 2004, for multi-d model

⋆ P. and Y.G.Wang 2005, for multi-d model with variable ion density

⋆ Gérard-Varet, Han-Kwan and Rousset 2013, in a bounded domain ⋯⋯

Global convergence with small initial data : P. 2015, $d = 2, 3$ $(m \geq 3)$

Method : energy estimates and passage to the limit in the system

Uniform global existence : $\exists$ constants $\delta > 0$ and $C > 0$ (independent of λ) such that for all $\lambda \in (0, 1]$, if

$$\|n_{0\lambda} - 1\|_m + \|v_{0\lambda}\|_m + \lambda \|\nabla \phi_{0\lambda}\|_m \leq \delta$$

the smooth solution is defined for all time and satisfies

$$\sup_{t \in \mathbb{R}^+} \big(\|n_\lambda(t) - 1\|_m + \|v_\lambda(t)\|_m + \lambda \|\nabla \phi_\lambda(t)\|_m \big) \leq C\delta$$

Consequence :

$$\|n_\lambda(t) - 1\|_{m-1} \leq C\delta\lambda$$

$$n_\lambda \longrightarrow 1 = n \quad \text{strongly in} \;\; L^\infty\big(\mathbb{R}^+; H^{m-1}\big)$$

and up to a subsequence

$$v_\lambda \rightharpoonup v \quad \text{weakly-}* \;\; \text{in} \;\; L^\infty\big(\mathbb{R}^+; H^m\big)$$

$$\nabla \times v_\lambda \longrightarrow \nabla \times v \quad \text{strongly in} \;\; C\big([0, T]; H^{m-2}\big), \quad \forall\, T > 0$$

This allows to pass to the limit in the system

3. Quasi-neutral limit in two-fluid Euler-Poisson system

$$\begin{cases} \partial_t n_e + \operatorname{div}(n_e v_e) = 0 \\ \partial_t (n_e v_e) + \operatorname{div}(n_e v_e \otimes v_e) + \nabla p_e(n_e) = n_e \nabla \phi - n_e v_e \\ \partial_t n_i + \operatorname{div}(n_i v_i) = 0 \\ \partial_t (n_i v_i) + \operatorname{div}(n_i v_i \otimes v_i) + \nabla p_i(n_i) = -n_i \nabla \phi - n_i v_i \\ -\lambda^2 \Delta \phi = n_i - n_e, \qquad \lim_{|x| \to +\infty} \phi(t,x) = 0 \\ t = 0 : \quad (n_\nu, v_\nu) = \big(n_{0\nu}^\lambda(x), v_{0\nu}^\lambda(x)\big), \quad \nu = e, i \end{cases}$$

$$-\lambda^2 \Delta \phi_0^\lambda = n_{0i}^\lambda - n_{0e}^\lambda$$

$$p_e'(n_e) > 0, \qquad p_i'(n_i) > 0$$

As $\lambda \to 0$, we have formally $n_e \to n$, $n_i \to n$

$$\begin{cases} \partial_t n + \operatorname{div}(n v_e) = 0 \\ \partial_t (n v_e) + \operatorname{div}(n v_e \otimes v_e) + \nabla p_e(n) = n \nabla \phi - n v_e \\ \partial_t n + \operatorname{div}(n v_i) = 0 \\ \partial_t (n v_i) + \operatorname{div}(n v_i \otimes v_i) + \nabla p_i(n) = -n \nabla \phi - n v_i \end{cases}$$

Existence of solution $(n, u_e, u_i, \nabla\phi)$?

One can prove that if $u_e = u_i$ at $t = 0$, then

$$u_e(t) = u_i(t), \qquad \forall\, t \in [0, T], \quad T > 0$$

Therefore

$$\begin{cases} \partial_t n + \operatorname{div}(nv) = 0 \\ \partial_t(nv) + \operatorname{div}(nv \otimes v) + \nabla p_e(n) = n\nabla\phi - nv \\ \partial_t(nv) + \operatorname{div}(nv \otimes v) + \nabla p_i(n) = -n\nabla\phi - nv \end{cases}$$

Adding and subtracting two momentum equations yields compressible Euler equations with damping

$$\begin{cases} \partial_t n + \operatorname{div}(nv) = 0 \\ \partial_t(nv) + \operatorname{div}(nv \otimes v) + \nabla p_+(n) = -nv, \quad p_+(n) = \dfrac{1}{2}\big(p_e(n) + p_i(n)\big) \end{cases}$$

and

$$\nabla\phi = \nabla h_-(n), \qquad h_-(n) = \frac{1}{2n}\big(p_e'(n) - p_i'(n)\big)$$

Justification of local convergence

Q.C. Ju, H.L. Li, Y. Li and S. Jiang 2010

If

$$\|n_{0\nu}^{\lambda} - n(0)\|_m + \|v_{0\nu}^{\lambda} - v(0)\|_m + \lambda\|\nabla(\phi_{0\lambda} - \phi(0))\|_m \leq C\lambda$$

$\exists\ T > 0$ such that

$$\sup_{t\in[0,T]} \big(\|n_{\nu}^{\lambda}(t) - n(t)\| + \|u_{\nu}^{\lambda}(t) - u(t)\| + \lambda\|\nabla(\phi_{\lambda}(t) - \phi(t))\|\big) \leq C\lambda$$

Justification of global convergence (P. and C.M.Liu 2023)

Let $(n_\nu^\lambda, v_\nu^\lambda, \phi^\lambda)$, $\nu = e, i$, be the solution to the original system, near an equilibrium state $(1, 0, 0)$ for $\nu = e, i$. We need to show

$$n_e^\lambda - n_i^\lambda \to 0, \quad v_e^\lambda - v_i^\lambda \to 0$$

The first limit follows from Poisson equation

$$\lambda^2 \Delta \phi^\lambda = n_e^\lambda - n_i^\lambda$$

For the second one, introduce

$$u_-^\lambda = u_e^\lambda - u_i^\lambda$$

$$\mu^\lambda = h_e(n_e^\lambda) - h_i(n_i^\lambda) - 2\phi^\lambda$$

The momentum equations give

$$\partial_t u_-^\lambda + \big((u_-^\lambda \cdot \nabla) u_e^\lambda + (u_i^\lambda \cdot \nabla u_-^\lambda)\big) + u_-^\lambda + \nabla \mu^\lambda = 0$$

Using

$$\lambda^2 \partial_t \nabla \phi^\lambda = \mathcal{P}(n_i^\lambda u_i^\lambda - n_e^\lambda u_e^\lambda)$$

we obtain

$$\partial_t^2 u_-^\lambda + \partial_t u_-^\lambda + 2\lambda^{-2} \mathcal{P} u_-^\lambda = F$$

$$\partial_t (I - \mathcal{P}) u_-^\lambda + (I - \mathcal{P}) u_-^\lambda = -(I - \mathcal{P})\big((u_-^\lambda \cdot \nabla) u_e^\lambda + (u_i^\lambda \cdot \nabla) u_-^\lambda\big)$$

which imply an error estimate of

$$u_-^\lambda = \mathcal{P} u_-^\lambda + (I - \mathcal{P}) u_-^\lambda$$

Finally, assume

$$\|n_{0\nu}^\lambda - 1\|_m + \|v_{0\nu}^\lambda\|_m + \|\nabla \phi_0^\lambda\|_m \quad \text{is uniformly small}$$

$$\|n_{0e}^\lambda - n_{0i}^\lambda\|_{m-1} + \|v_{0e}^\lambda - v_{0i}^\lambda\|_{m-1} + \|\nabla \mu^\lambda(0)\|_{m-2} = O(\lambda^2)$$

Then the solution is globally defined, and for all $t > 0$

$$\|n_\nu^\lambda(t) - 1\|_m + \|v_\nu^\lambda(t)\|_m + \|\nabla \phi^\lambda(t)\|_m \quad \text{is uniformly small}$$

$$\|n_e^\lambda(t) - n_i^\lambda(t)\|_{m-1} + \|v_e^\lambda(t) - v_i^\lambda(t)\|_{m-1} + \|\nabla \mu^\lambda(t)\|_{m-2} = O(\lambda)$$

Thank you for your attention !