Further *a priori* estimates in gas dynamics through Compensated Integrability

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Mathematical gas dynamics

($t \in \mathbb{R}$ the time, $y \in \mathbb{R}^d$ the space variable)

Gas dynamics is modeled by *conservation laws* :

$$\begin{aligned} \partial_t \rho + \mathrm{div}_y(\rho u) &= 0, \\ \partial_t(\rho u) + \mathrm{Div}_y(\rho u \otimes u) &= \mathrm{Div}_y \Sigma, \\ \partial_t\left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon\right) + \mathrm{div}_y\left(\left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon\right)u\right) &\leq \mathrm{div}_y(q + \Sigma u). \end{aligned}$$

Basic estimates are conservation of mass and decay of energy :

$$\int_{\mathbb{R}^d} \rho(t,y)\,dy \equiv M := \int_{\mathbb{R}^d} \rho_0(y)\,dy, \qquad \int_{\mathbb{R}^d} \left(\frac{1}{2}\rho|u|^2 + \rho\varepsilon\right) dy \leq E_0.$$

Our context :

1. Physical domain $\mathbb{R}^d$ (pure Cauchy problem). Initial data $\rho_0, u_0, \dots$
2. $M, E_0 < +\infty$.

From $O(\mathbb{R}^d)$-invariance,

$$\Sigma(t,y) \in \mathbf{Sym}_d$$

$\iff$ cons. of angular momentum.

Euler :

$$\Sigma = -pI_d$$

where $p \geq 0$ (the pressure).

Navier-Stokes :

$$\Sigma = (-p + \nu \, \mathrm{div}_y u) I_d + \mu(\nabla_y u)^{\mathrm{sym}}.$$

Other relevant models $\longrightarrow$

Boltzman. The unknown is a kinetic density $f(t,y,\xi) \geq 0$,

$$(\partial_t + \xi \cdot \nabla_y) f = Q[f].$$

First momenta satisfy the conservation laws of mass, momentum and energy. For instance

$$\rho(t,y) := \int_{\mathbb{R}^d} f(t,y,\xi)\, d\xi, \qquad m(t,y) := \int_{\mathbb{R}^d} f(t,y,\xi)\xi\, d\xi$$

satisfy

$$\partial_t \rho + \mathrm{div}_y m = 0,$$

from which, again

$$\int_{\mathbb{R}^d} \rho(t,y)\, dy \equiv M.$$

Vlasov-type. Here

$$(\partial_t + \xi \cdot \nabla_y) f + F \cdot \nabla_\xi f = 0,$$

where $F(t,y) = F[f]$ is a self-induced force.

Goal

To establish new Natural estimates.
By this, we mean estimates involving only M, E_0, and possibly of the moment of inertia

$$I_0 := \int_{\mathbb{R}^d} \rho_0(y) \frac{|y|^2}{2} \, dy \qquad \text{(not a conserved quantity !).}$$

New estimates look like *Strichartz inequalities*, involving space-time integrals and expressing a gain of integrability.

For instance

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \, dt \leq_d M^{\frac{1}{d}} \sqrt{ME_0}$$

where $\leq_d (\cdots)$ means $\leq c_d(\cdots)$ for an explicit (and not so big) constant depending only upon the space dimension.

Tools

Exploit the symmetric structure of the mass-momentum claws

$$\mathrm{Div}_{t,y} A = 0, \qquad A := \begin{pmatrix} \rho & \rho u^T \\ \rho u & \rho u \otimes u - \Sigma \end{pmatrix}.$$

Mind that row-wise Divergence is not an elliptic DO :

$$\Lambda_{\mathrm{Div}} = \{S \in \mathbf{Sym}_{1+d} \,|\, \det S = 0\} \neq \{0\}.$$

This is why discontinuites (shock waves, contacts) may occur in Euler system.

Slightly better situation if A is positive semi-definite, because Λ_{Div} does not intersect the interior of $\mathbf{Sym}^+_{1+d}$. Discontinuities still occur, but improved integrability could happen :

Compensated Integrability

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What is Compensated Integrability ?

$U \subset \mathbb{R}^n$ an open domain. (Later on, we shall let $n = 1 + d$, $x = (t, y)$.)

Definition 1

A symmetric tensor over U is an $n \times n$ symmetric matrix A whose entries a_{jk} are distributions over $U \subset \mathbb{R}^n$.
Its (row-wise) Divergence is a vector of distributions :

$$(\mathrm{Div}\, A)_j = \sum_{k=1}^{n} \partial_k a_{jk}.$$

When A is positive semi-definite, the entries are Radon measures.

Definition 2

The tensor A is Div-BV if its entries a_{jk}, as well as the coordinates $(\mathrm{Div}\, A)_i$ are finite measures.

Normal trace

Let U have a Lipschitz boundary. Denote $\vec{\nu}$ be the outer unit normal vector field to ∂U.

If A is Div-BV over U, then $A\vec{\nu}$ is defined by duality (Green formula)

$$\langle A\vec{\nu}, \vec{\phi}\rangle_{\partial U} = \langle \mathrm{Div}\; A, \vec{\phi}\rangle_U + \langle A, \nabla\vec{\phi}\rangle_U.$$

This trace is a (vector-valued) distribution of order -1, at worst.

Proposition 1

For a Div-BV tensor $A : U \to \mathbf{Sym}_n$, the extension $\widetilde{A}$ by 0_n to U^c is Div-BV over $\mathbb{R}^n$ iff $A\vec{\nu}$ is a finite measure over ∂U. And we have

$$\|\,\mathrm{Div}\;\widetilde{A}\|_{\mathcal{M}} = \|\,\mathrm{Div}\;A\|_{\mathcal{M}} + \|A\vec{\nu}\|_{\mathcal{M}}.$$

Two examples of Div-free ($\mathrm{Div}\, A \equiv 0$) tensors

Diagonal tensors ($U = I_1 \times \cdots \times I_n$).
Given n functions of $n-1$ variables $f_j = f_j(\widehat{x_j})$,

$$A := \mathrm{diag}(f_1, \ldots, f_n).$$

Since $\partial_j f_j = 0$, A is Div-free.

C.I. is reminiscent to the Gagliardo Inequality : the function

$$f(x) = \det A = \prod_1^n f_j(\widehat{x_j})$$

satisfies

$$\|f\|_{L^1(U)} \le \prod_1^n \|f_j\|_{L^{n-1}(U_j)}.$$

Special tensors.

Given a potential $\theta : U \to \mathbb{R}$, the matrix of cofactors

$$A = \widehat{\mathrm{D}^2\theta}$$

is Div-free (because of Piola's identity).

- If θ is convex, then $A : U \to \mathbf{Sym}_n^+$.
- If $n = 2$, every Div-free tensor is special. False if $n \geq 3$.

Notice the formula

$$\det A = (\det \mathrm{D}^2\theta)^{n-1}.$$

Div-free/BV tensors are ubiquitous

Second example : **Relativistic GD**.

Warning : the tensor involves the claw of energy, instead of that of the mass :

$$\begin{aligned}
\partial_t\left(\frac{\rho c^2+p}{c^2-|v|^2}-\frac{p}{c^2}\right)+\mathrm{div}_y\left(\frac{\rho c^2+p}{c^2-|v|^2}\,v\right) &= 0,\\
\partial_t\left(\frac{\rho c^2+p}{c^2-|v|^2}\,v\right)+\mathrm{Div}_y\left(\frac{\rho c^2+p}{c^2-|v|^2}\,v\otimes v\right)+\nabla_y p &= 0.
\end{aligned}$$

The (symmetric !) energy-momentum tensor

$$A=\begin{pmatrix} \frac{\rho c^2+p}{c^2-|v|^2}-\frac{p}{c^2} & \frac{\rho c^2+p}{c^2-|v|^2}\,v \\ \frac{\rho c^2+p}{c^2-|v|^2}\,v & \frac{\rho c^2+p}{c^2-|v|^2}\,v\otimes v+pI_3 \end{pmatrix}$$

is Div-free.

Third example : **Maxwell system** in vacuum.

The *Electro-magnetic field* is a closed 2-form

$$\alpha = (\vec{E} \cdot dy) \wedge dt + B_1\, dy_2 \wedge dy_3 + B_2\, dy_3 \wedge dy_1 + B_3\, dy_1 \wedge dy_2 .$$

Its closedness expresses the Gauß–Faraday law

$$\partial_t \vec{B} + \operatorname{curl} \vec{E} = 0, \qquad \operatorname{div} \vec{B} = 0.$$

The electric/magnetic inductions are defined in terms of a Lagrangian $L(\vec{B}, \vec{E})$:

$$\vec{D} = \frac{\partial L}{\partial \vec{E}}\,, \qquad \vec{H} = -\frac{\partial L}{\partial \vec{B}}\,.$$

The Div-free energy-momentum tensor ($c = 1$) is

$$A = \begin{pmatrix} L - \vec{E}\cdot\vec{D} & \vec{H}\times\vec{E} \\ \vec{D}\times\vec{B} & (L + \vec{B}\cdot\vec{H})\, I_3 - \vec{E}\otimes\vec{D} - \vec{H}\otimes\vec{B} \end{pmatrix}.$$

The symmetry amounts to the identities

$$\vec{H}\times\vec{E} = \vec{D}\times\vec{B}, \qquad \vec{E}\otimes\vec{D} + \vec{H}\otimes\vec{B} = \vec{D}\otimes\vec{E} + \vec{B}\otimes\vec{H}$$

which are equivalent to the Lorentz invariance :

$$L = \ell\left(\vec{E}\cdot\vec{B}, \frac{|\vec{B}|^2 - |\vec{E}|^2}{2}\right).$$

$\mathrm{Div}\, A \equiv 0$ follows from Noether's thm and Lorentz invariance.

HOW do we treat Div-BV tensor ?

$$BV_{\mathrm{Div}}(\mathbb{R}^n) = \{A \in \mathcal{M}(\mathbb{R}^n; \mathbf{Sym}_n) \,|\, \mathrm{Div}\, A \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^n)\}$$

mimics the space

$$BV(\mathbb{R}^n) = \{f \in \mathcal{M}(\mathbb{R}^n) \,|\, \nabla f \in \mathcal{M}(\mathbb{R}^n)\},$$

for which we have (Gagliardo–Nirenberg–Sobolev)

$$BV(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n),$$

with a functional inequality

$$\|f\|_{\frac{n}{n-1}} \leq c_n \|\nabla f\|_{\mathcal{M}}.$$

But Div is not elliptic, unlike ∇ ...

In the spirit of Compensated Compactness, we expect that some non-linear quantity $D(A)$ behaves better than the entries a_{ij} do individually ...

Which quantity ?

Examin the case of a special tensor, in a periodic setting

$$\fint (\det A)^{\frac{1}{n-1}}\,dx = \fint \underbrace{\det \mathrm{D}^2\theta}_{\text{null}-\text{Lagr.}}\,dx = \left(\det \fint A\,dx\right)^{\frac{1}{n-1}}.$$

This calculation suggests – the Gagliardo inequality does too, – that this nice quantity is

$$A \stackrel{D}{\longmapsto} (\det A)^{\frac{1}{n-1}}.$$

Notice that $\det^{\frac{1}{n-1}}$ is super-linear over $\mathbf{Sym}_n^+$, hence not concave, unlike $\det^{\frac{1}{n}}$. We shall use the latter to study the former...

Main result : Compensated Integrability

Observe that $(\det A)^{\frac{1}{n}}$ is a well-defined measure,

$$0_n \le (\det A)^{\frac{1}{n}} \overset{\text{(AGM)}}{\le} \frac{1}{n}\,\mathrm{Tr}\,A.$$

Theorem 1 (Comp. Int. in $\mathbb{R}^n$ (D.S., JMPA 2019).)

Let $A \succ 0_n$ be a Div-BV tensor over $\mathbb{R}^n$. Then $(\det A)^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and we have

$$\int_{\mathbb{R}^n} (\det A)^{\frac{1}{n-1}}\,dx \le c_n \|\,\mathrm{Div}\,A\|_{\mathcal{M}}^{\frac{n}{n-1}}.$$

Dual structure : the "2nd" BVP for the Monge-Ampère equation

$$\det \mathrm{D}^2 u = f \quad (> 0,\ u \text{ convex}). \tag{MAE}$$

The proof exploits Brenier's theorem in Optimal Transport.

- The constant c_n is explicit and sharp ! Equality happens when $A = \chi_B I_n$ and B is a ball.
- For general domains Ω, the choice $A = \chi_\Omega I_n$ yields the Isoperimetric Inequality

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_n)} \leq \left(\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Area}(\partial B_n)}\right)^{\frac{n}{n-1}}.$$

- With $A = f(x) I_n$, one recovers the Sobolev embedding

$$BV(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n).$$

Periodic framework

Theorem 2 (D.S., Ann. IHP 2018.)

Let $A \succ 0_n$ be a periodic Div-free tensor. Then

$$\fint (\det A)^{\frac{1}{n-1}}\,dx \le \left(\det \fint A\,dx\right)^{\frac{1}{n-1}}.$$

Looks like Jensen's Inequality ... but $\det^{\frac{1}{n-1}}$ is **not** concave over $\mathbf{Sym}_n^+$, contrary to $\det^{\frac{1}{n}}$.

Similar proof : Duality with periodic MAE, whose existence theory is due to Yan Yan Li (1990).

This is Div-quasi-concavity (terminology of Fonseca, Müller, De Philippis).

Whence a weak-star upper semi-continuity result :

Theorem 3 (L. De Rosa, D. S. & R. Tione, JFA 2020.)

Let $A_m \succ 0_n$ be a sequence of Div-BV tensors, such that $\mathrm{Div}\, A_m$ is bounded in $\mathcal{M}(U)$ and $A_m \overset{*}{\rightharpoonup} A$ in L^p with $p > \frac{n}{n-1}$. Then up to a subsequence

$$* \lim_{m\to\infty} (\det A_m)^{\frac{1}{n-1}} \le (\det A)^{\frac{1}{n-1}}.$$

Related results by

- Skipper & Wiedemann (2021),
- Guerra, Raiță & Schrecker (2021, 2022).

Sound improvement when $p = \frac{n}{n-1}$ by De Rosa & Tione (2023) ; concentration phenomenon.

Variation on C.I.

Multi-linearization : Denote $D_n : \mathbf{Sym}_n \times \cdots \times \mathbf{Sym}_n \to \mathbb{R}$ the symmetric n-linear form such that

$$D_n(A, \dots, A) = \det A.$$

For instance

$$D_2(A, B) = \frac{1}{2} \left(a_{11} b_{22} + a_{22} b_{11} - 2 a_{12} b_{12} \right).$$

Recall that $\det$ is a *hyperbolic polynomial* over $\mathbf{Sym}_n$, with forward cone $\mathbf{Sym}_n^+$. Thus (Gårding)

$$D_n \geq 0 \quad \text{over} \quad \mathbf{Sym}_n^+ \times \cdots \times \mathbf{Sym}_n^+.$$

In particular

$$D_n(A_1, \dots, A_n) \leq \frac{1}{n!} \det(A_1 + \cdots + A_n).$$

Applying C.I. to $A_1 + \cdots + A_n$, then rescaling, we infer

Theorem 4

Let $A_1, \ldots, A_n \succ 0_n$ *be Div-BV tensors over* $\mathbb{R}^n$*. Then* $(D_n(A_1, \ldots, A_n))^{\frac{1}{n}} \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$ *and we have*

$$\int_{\mathbb{R}^n} (D_n(A_1, \ldots, A_n))^{\frac{1}{n-1}}\, dx \le c_n \prod_1^n \| \operatorname{Div} A_j \|_{\mathcal{M}}^{\frac{1}{n-1}}.$$

Application to Functional Analysis

It is known that

$$\|\frac{1}{|x|}\star f\|_\infty \leq_n TV(f), \qquad \forall f \in BV(\mathbb{R}^n).$$

Hint (L. Tartar) : $BV(\mathbb{R}^n)$ actually embeds into $L^{\frac{n}{n-1},1}$ (Alvino 1977), while $x \mapsto \frac{1}{r}$ belongs to $L^{n,\infty}$.

Choosing

$$A_1 = \underbrace{\phi(r)}_{\text{truncation}} \quad g(\frac{x}{r}) \; \underbrace{\frac{x \otimes x}{r^{n+1}}}_{\text{Div}-\text{free}}$$

and $A_2 = \cdots = A_n = f(x)\, I_n$, we obtain the following improvement

$\longrightarrow$

A Radon-like transform

Theorem 5

Define

$$Rf(z;\omega) := \int_{\mathbb{R}} r^{n-2} f(z+r\omega)\, dr, \qquad z \in \mathbb{R}^n,\ \omega \in S^{n-1}.$$

Then BV-functions satisfy

$$\sup_z \|Rf(z;\cdot)\|_{L^{\frac{n-1}{n-2}}(S^{n-1})} \leq_n TV(f).$$

The "classical" inequality is an estimate Rf in $L^\infty_z L^1_\omega$ only.

Another variant : Evolution problems

Again $n = 1 + d$ and $x = (t, y)$. A splits accordingly

$$A = \begin{pmatrix} \rho & m^T \\ m & \frac{1}{\rho} m \otimes m + \sigma \end{pmatrix}, \qquad \det A = \rho \det \sigma.$$

The positiveness of A amounts to that of ρ and σ, its Schur complement.

Denote $M \equiv \int \rho(t, y)\, dy$. The following involves again a scaling argument :

Theorem 6 (D.S. 2021.)

Let $A \succ 0_n$ be a Div-free tensor over $(0, T) \times \mathbb{R}^d$. Then

$$\int_0^T dt \int_{\mathbb{R}^d} (\rho \det \sigma)^{\frac{1}{d}}\, dx \leq_d M^{\frac{1}{d}} \left(\|m(0, \cdot)\|_{\mathcal{M}} + \|m(T, \cdot)\|_{\mathcal{M}} \right).$$

Various applications of C.I.

C.I. applies to models that involve a positive Div-BV (often Div-free) tensor :

- Compressible Euler, (classical as well relativistic)
- Boltzmann equation,
- Particle dynamics or mean field models, under a radial, repulsive interaction force,
- Hard spheres dynamics,
- Multi-D scalar conservation laws (coll. with L. Silvestre).

It does not if the Div-free tensor is indefinite (or can be so) :

- Navier-Stokes system,
- Incompressible Euler equation,
- Maxwell's equations,
- Attractive particle dynamics.

Euler system of GD

Context : physical domain $\mathbb{R}^d$, finite mass M and initial energy E_0.

Recall the Div-free tensor

$$A = \begin{pmatrix} \rho & \rho u^T \\ \rho u & \rho u \otimes u + pI_d \end{pmatrix} = \rho \begin{pmatrix} 1 \\ u \end{pmatrix} \otimes \begin{pmatrix} 1 \\ u \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & pI_d \end{pmatrix}.$$

We have

$$\det A = \rho p^d.$$

Cauchy–Schwarz gives

$$\|\rho u(t)\|_1 \leq \sqrt{2ME_0}\,.$$

Whence

Estimate 1

We have

$$\int_0^{+\infty} dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \leq_d M^{\frac{1}{d}} \sqrt{ME_0}\,.$$

The action of the *projective group*, yields an improved dispersion :

Estimate 2 (mono-atomic gas : $\gamma = 1 + \frac{2}{d}$.)

Suppose $p = \frac{2}{d}\,\rho\varepsilon$ ($p = \rho^{1+\frac{2}{d}}$ *for isentropic flow). We have*

$$\int_0^{+\infty} t\,dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p\,dy \leq_d M^{\frac{1}{d}} \sqrt{MI_0}\,.$$

The technique being more flexible when $n = 2$, Besov regularity can be achieved in one space dimension :

Theorem 7 ($d = 1$, F. Golse, 2008.)

Assume a mono-atomic gas ($p = \rho^3$). *Then admissible flows satisfy*

$$\rho, u \in B^{\frac{1}{4},4}_{\infty,\mathrm{loc}}.$$

Other results for $p = \rho^\gamma$ *with* $1 < \gamma < 3$...

Improved estimates

The time-space integrals do not depend upon the choice of the Galilean frame.
The right-hand sides do ... Optimize the choice !

This lets us replace

$$ME_0 \quad \longmapsto \quad \frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\rho_0(y)\rho_0(z)|u_0(z)-u_0(y)|^2\,dz\,dy$$
$$+M\int_{\mathbb{R}^d}\rho_0\varepsilon_0\,dy,$$

and

$$MI_0 \quad \longmapsto \quad \frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\rho_0(y)\rho_0(z)|z-y|^2\,dz\,dy.$$

Comments

- The first estimate assumes neither an equation of state (only $p \geq 0$), nor an entropy condition. It involves only the decay of the mechanical energy $t \mapsto E(t)$.
- The second one does not depend at all upon the initial velocity field !
- Say that the gas is barotropic ($p(\rho) = \rho^\gamma$ for $\gamma > 1$). Then

$$\rho \in \underbrace{L_t^\infty(L_y^1)}_{\text{mass}} \bigcap \underbrace{L_t^\infty(L_y^\gamma)}_{\text{energy}} \bigcap \underbrace{L_{t,y}^{\gamma+\frac{1}{d}}}_{\text{C.I.}}.$$

The internal energy may not concentrate.

- Strichartz-like estimates are new for the Euler system. Previous dispersive estimates, like (J.-Y. Chemin, Mono-atomic, 1990)

$$t^2 \int_{\mathbb{R}^d} p \, dy \leq \frac{2}{d} I_0 ,$$

involve only a space integral.

Further estimates (I)

Denote $\widetilde{p} : \mathbb{R}^{1+d} \to \mathbb{R}_+$ the extension of p by 0 on $t < 0$.

Applying multi-linearized C.I., with again

$$A_1 = \phi(r) g(\frac{x}{r}) \, \frac{x \otimes x}{r^{n+1}}$$

and $A_2 = \cdots = A_n = A$ (the mass-momentum tensor), and noting that

$$A \succ \begin{pmatrix} 0 & 0 \\ 0 & pI_d \end{pmatrix},$$

we obtain an estimate for the Radon-like transform ($\omega \in S^d$)

$$Tp(\tau, z; \omega) := \left[\frac{\omega_0^2}{(E_0\omega_0^2 + M|\omega'|^2)^{\frac{d}{2}+1}} \right]^{\frac{1}{d}} \int_{\mathbb{R}} r^{d-1} \widetilde{p}(\tau + r\omega_0, z + r\omega') \, dr.$$

$\ldots \longrightarrow$

Further estimates (II)

Estimate 3

Admissible flows of finite mass and energy satisfy

$$\sup_{\tau,z} \|Tp(\tau,z;\cdot)\|_{L^{\frac{d}{d-1}}(S^d)} \leq_d E_0^{\frac{1}{2}-\frac{1}{d}}.$$

Combining with $L^{\frac{d}{d-1}}(S^d) \subset L^1(S^d)$, this implies the more readible (but weaker)

$$\sup_{\tau,z} \int_0^{+\infty} \int_{\mathbb{R}^d} \left[\frac{(\tau-t)^2}{(E_0(\tau-t)^2 + M|z-y|^2)^{\frac{d}{2}+1}} \right]^{\frac{1}{d}} p(t,y)\,dy\,dt \leq_d E_0^{\frac{1}{2}-\frac{1}{d}}$$

where the kernel in the singular integral has degree -1.

Further estimates (III)

So far, the velocity was estimated only through the kinetic energy $E_{\rm kin}[t] \le E[t] \le E_0$. Is there a Strichartz-like estimate involving u ?

Apply multi-lin C.I. to shifts of the mass-momentum tensor : $A_j(t,y) = A(t, y+h_j)$. Noting that

$$A \succ \rho \begin{pmatrix} 1 \\ u \end{pmatrix} \otimes \begin{pmatrix} 1 \\ u \end{pmatrix},$$

we obtain

Estimate 4

Admissible flows of finite mass and energy satisfy

$$\sup_{h_0,\dots,h_d} \int_0^{+\infty} \int_{\mathbb{R}^d} \left(\prod_0^d \rho_j \cdot V(u_0,\dots,u_d)^2 \right)^{\frac{1}{d}} dy\, dt \le_d M^{\frac{1}{d}} \sqrt{ME_0}\,,$$

where V is the volume of the d-simplex spanned by $u_0,\dots,u_d$.

- The expression estimated above

$$\left(\prod_0^d \rho_j \cdot V(u_0,\dots,u_d)^2\right)^{\frac{1}{d}}$$

is quadratic in the velocity, like the density of kinetic energy $\rho|u|^2$, ...

- but it is contains something like

$$\rho^{1+\frac{1}{d}}$$

instead of ρ.
The same gain of a factor $\rho^{\frac{1}{d}}$, as in

$$\int_0^T \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p \, dy \, dt \leq_d M^{\frac{1}{d}} \sqrt{ME_0}\,,$$

compared to (perfect gas)

$$\sup_t \int_{\mathbb{R}^d} p \, dy \leq (\gamma - 1) E_0.$$

Kinetic equations (Boltzman)

$f(t,y,\xi)$ the distribution of mass, ξ the velocity of particles.

Essentially the same estimates, but $\rho^{\frac{1}{d}}p$ is replaced by its kinetic counterpart $(\det \Xi)^{\frac{1}{d}}$ where

$$\Xi(t,y) = \int_{\mathbb{R}^d} f(t,y,\xi) \begin{pmatrix} 1 & \xi^T \\ \xi & \xi \otimes \xi \end{pmatrix} d\xi.$$

That is

$$\det \Xi = \frac{d!}{d+1} \int_{\mathbb{R}^d}^{\otimes(1+d)} f(\xi_0)\cdots f(\xi_d)\, V(\xi_0,\dots,\xi_d)^2\, d\xi_0 \cdots d\xi_d.$$

Notice the homogeneity :

$$(\det \Xi)^{\frac{1}{d}} \sim f^{1+\frac{1}{d}} |u|^2.$$

The 1-D estimate

$$\int_0^{+\infty} dt \int_{\mathbb{R}}\int_{\mathbb{R}} f(t,y,\xi)f(t,y,\xi')|\xi'-\xi|^2 d\xi\, d\xi' \leq cM\sqrt{ME_0}$$

was known to J.-M. Bony (1987). Used by C. Cercignani (2005) to prove that DiPerna–Lions' renormalized solutions are distributional.

Open Problem 1

Can one use our Strichartz-like estimate in order to prove that multi-d Boltzmann solutions are distributional ?

Hard spheres dynamics

Large number of spherical particles $B_\alpha(t)$, $\alpha \in [\![1, N]\!]$. Elastic collisions. Total mass $M = Nm$.

Initial data : positions/velocities. Yields conserved quantities :

$$\text{energy} \qquad E_0 = \frac{m}{2} \sum |u_\alpha(0)|^2,$$

$$\text{standard deviation of velocity} \qquad \bar{u}.$$

Theorem 8 (R. K. Alexander 1975.)

Global existence with pairwise collisions only, for almost every initial data.

Ya. Sinai's question :

> *Is the number K of collisions finite ? If so, how does it behave with N ?*

Answers :

- **Yes** (Vaserstein 1979, Illner 1989),
- $\log K = O(N^2 \log N)$ (Burago & al. 1998),
- $\log K = O(N \log N)$ (Burdzy 2022),
- For some configuration, $\log K \sim \frac{N}{2} \log 2$ (Burago & Ivanov 2018) : the collision number may be really (exponentially) large !

The above estimates don't involve Functional Analysis. The upper bounds are huge (useless ?).

B.&I.'s explicit construction is discouraging ...

Weighted estimate, using C.I.

The dynamics is encoded into a positive Div-free tensor A_{hs}, though a singular one : its support is a graph.

The tensor A_{hs} is rank-one a.e : $\det A_{hs} \equiv 0$!

Apply a modified version of C.I., adapted to singular supports : $(\det A)^{\frac{1}{n-1}}$ is a set of Dirac masses at the nodes of the graph.

Estimate 5 (D.S., ARMA 2021.)

We have

$$\sum_{\text{coll.}} \underbrace{|u_{\text{out}} - u_{\text{in}}|}_{\text{weight}} \leq_d N^2 \bar{u}.$$

Way better than N^N ... ! Even in B. & I.'s example $(K \sim 2^{N/2})$, almost every collision is “exponentially small”.

In other words ($q_\alpha = m u_\alpha$ the linear momenta)

$$\text{mean}\left[TV(t \mapsto q_\alpha(t))\right] \leq_d \sqrt{ME_0}\,.$$

Thank you for your attention !