In Search of Euler Equilibria via the MR Equations

MRE

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Topological Fluid Dynamics

19C: Helmholtz, Tait, Maxwell, Kelvin

20C: V. I. Arnold : geometry of infinite.
1965 dimensional groups of volume preserving
diffeomorphisms applied to ideal fluids

1970 Ebin & Marsden used this concept
to obtain sharp existence and uniqueness
theorems for the Euler equations (& NSE)

Shnirelman, Holm, Ratiu, Khesin

1998 Arnold & Khesin
2021 Topological methods in hydrodynamics.

1969 Moffatt
1985 Topological aspects of fluid dynamics

Ideal MHD

$$\partial_t B + u \cdot \nabla B = B \cdot \nabla u$$

$$\partial_t u + u \cdot \nabla u + \nabla p = B \cdot \nabla B$$

$$\nabla \cdot u = 0, \quad \nabla \cdot B = 0$$

Magnetostatic equilibria $\bar{B}$

$$\nabla \bar{p} = \bar{B} \cdot \nabla \bar{B}, \quad \nabla \cdot \bar{B} = 0$$

Euler equilibria $\bar{u}$

$$\bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} = 0, \quad \nabla \cdot \bar{u} = 0$$

Stability/instability <u>not</u> the same.

Many examples of 3D Euler equilibria

eg: Hill's spherical vortex
Hick's vortex with swirl
Kida's elliptic vortex
Gavrilov Euler equilibria, compact support

Chaotic: Beltrami flows

Arnold observed not enough Beltrami fields to cover every conceivable topology that may be used as the initial field

Magnetic relaxation procedure preserves the stream line topology of an initial 3D div free vector B_0 as it evolves under the "frozen" field equation

$$\partial_t B + u \cdot \nabla B = B \cdot \nabla u$$

via a vector field $u(x,t)$ which is related to $B(x,t)$ by a suitable constitutive law where

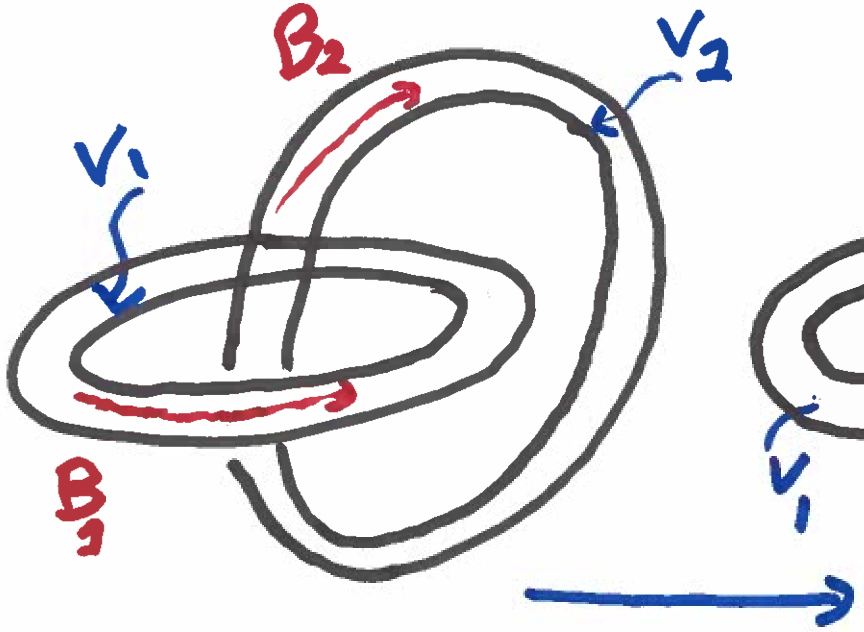
$$\lim_{t \to \infty} u(x,t) \to 0 \text{, formally}$$

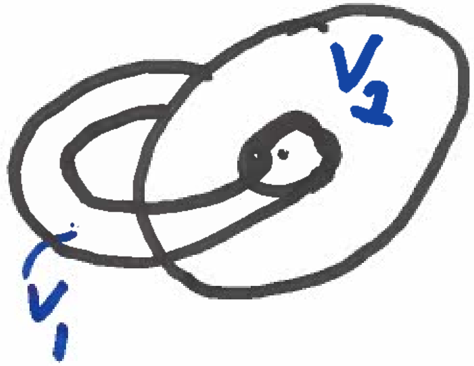
Moffatt (JFM, 2021)

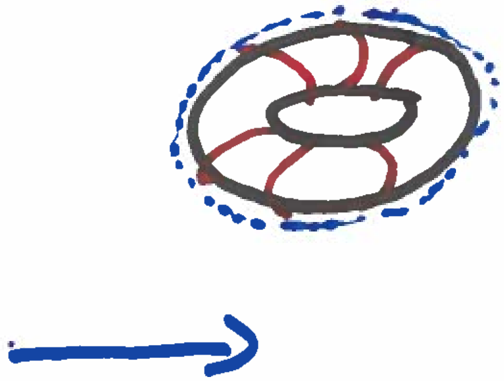
Magnetic relaxation procedure preserves the stream line topology of an initial div free vector field but abandons the constraint that it remains smooth as $t \to \infty$.

Topological accessibility, weaker than topological equivalence because it allows for the appearance of discontinuities in current sheets $J = \nabla \times B$ as $t \to \infty$.

Hopf Link







2 untwisted but linked flux tubes

contracted state when the tubes make contact

fully relaxed axisymmetric state

The discontinuity is a current sheet on the surface of contact.

Active vector PDE: MRE

$$\partial_t B + u\cdot\nabla B = B\cdot\nabla u \qquad (1)$$

$$u = B\cdot\nabla B + \nabla P\,, \quad \nabla\cdot u = 0 \qquad (2)$$

$$B(x,0) = B_0(x) \quad \text{with} \quad \nabla\cdot B_0 = 0$$

note: cubic nonlinearity

$$\nabla\cdot u = 0 \Rightarrow \nabla\cdot(B\cdot\nabla B) + \Delta P = 0$$

or Stokes type regularisation:

$$(-\Delta)^\gamma u = B\cdot\nabla B + \nabla P\,, \quad \nabla\cdot u = 0 \qquad (2a)$$

also preserves topology

Dissipative nature of MRE.

Magnetic energy estimate

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 = \int_{\mathbb{T}^d} B\cdot(B\cdot\nabla u) = -\int_{\mathbb{T}^d} u\cdot(B\cdot\nabla B)$$

$$= -\int u\cdot\left((-\Delta)^\gamma u - \nabla p\right) = -\|u\|^2_{\dot{H}^\gamma}$$

Energy is strictly decreasing for $u \neq 0$

Global lower bound (Arnold's inequality)

$$\|B(\cdot,t)\|_{L^2}^2 \geq |\mathcal{H}(0)| \quad \text{(topologically non trivial)}$$

Magnetic helicity

$$\mathcal{H}(t) = \int A\cdot B\,dx, \quad \nabla\times A = B$$

Brenier (2014) 2 dimensions $\gamma = 0$

Admissible dissipative solutions satisfying an energy inequality

$$\frac{d}{dt}\int |B|^2 dx + \int |u|^2 dx + \int |\mathbb{P}\nabla\cdot(B\otimes B)|^2 dx \le 0$$

Ignored local well posedness.

Proved uniquenes of smooth solutions among all dissipative solutions for any given prescribed smooth initial cond?

B-F-V: Local existence in Sobolev
(2022) spaces for all $\gamma \geq 0$.

Let $B_0 \in H^s(\mathbb{T}^d)$, div free, $s > \frac{d}{2} + 1$

Then $\exists\, T_* \geq (C\|B_0\|_{H^s})^{-2}$ such that MRE has a unique solution $B \in C^0([0, T_*); H^s(\mathbb{T}^d))$ with associated velocity u.

The $\|B(\cdot, t)\|^2_{H^s}$ is bounded and the energy inequality is satisfied for $t \in [0, T_*)$.

Th 2. Global existence for $\gamma > \frac{d}{2}+1$

Let $\gamma, s > \frac{d}{2}+1$, $B_0 \in H^s(\mathbb{T}^d)$, div free

Then $T_* = +\infty$. Moreover

$$\|B(\cdot,t)\|^2_{\dot H^s} \le \|B_0\|^2_{\dot H^s} \exp\left(Ct^{1/2}\|B_0\|_{L^2}\right)$$

$$\times \exp\left(Ct\left(\|B_0\|^2_{L^\infty} + Ct^2\|B_0\|^6_{L^\infty}\right)\right)\exp\left(Ct^{1/2}\|B_0\|_{L^2}\right)$$

for all $t \geq 0$ where $C(\gamma, s, d) > 0$

Proof: via showing the Lipschitz norm of u is integrable in time and the Lipschitz norm of B is square integrable in time.

Convergence as $t \to \infty$ for $\gamma > \frac{d}{2}+1$

Th3 asymptotic behavior for velocity

Under conditions of Th2, the zero mean velocity u associated to $B \in C^0([0,\infty]; H^s(\mathbb{T}^d))$ has the property that

$$\lim_{t \to \infty} \| \nabla u(\cdot,t) \|_{L^\infty} = 0$$

Open: Is the decay fast enough to ensure $\| \nabla u(\cdot,t)\|_{L^\infty} \in L^1(0,\infty)$?

Open: does $B(x,t)$ itself converge to $\bar{B}(x)$ which is an Euler equilibria

2D stability of $B = \hat{e}_1$, $u = 0$

Set $\gamma = 0$:

Perturbation: $b = B - \hat{e}_1$, $v = b\cdot\nabla b + \nabla p$

$$\partial_t b + v\cdot\nabla b - b\cdot\nabla v - \partial_1^2 b = 2\,\mathbb{P}(b\cdot\nabla\partial_1 b)$$

$$v = b\cdot\nabla b + \nabla p$$

$$\nabla\cdot v = 0 \quad , \quad \nabla\cdot b = 0$$

<u>Note</u>: dissipative role of $-\partial_1^2 b$

<u>but</u> this vanishes on functions independent of x_1

Project onto the x_1-independent and x_1-dependent components of $b(x_1, x_2, t)$

Th 4. Stability and Relaxation.

There exists a unique global in time solution (b, v) where $\|b(\cdot,t)\|_{L^2} \le \varepsilon$.

The total velocity $u(\cdot,t) \to 0$ as $t \to \infty$.

The total magnetic field $B = \hat{e}_1 + b(\cdot,t)$ relaxes to a steady state $\overline{B}$ with

$$\|\overline{B} - \hat{e}_1\|_{H^{k+2}} \le 4\varepsilon$$

Both convergences take place with respect to strong topologies

Nonlinear Instability in 3D

Recall 3D Euler exact solutions (Yudovich, 74, 00)

$$u(x,t) = (v(x_H), g(x_H, t))$$

where $v = \nabla_H^\perp \phi(x_H)$, $\Delta_H \phi = F(\phi)$

and $\partial_t g + (v \cdot \nabla_H) g = 0$, $g(x_H, 0) = g_0(x_H)$

ex shear flow $v(x_H) = (V(x_2), 0)$

exact Euler solution

$$u(x,t) = (V(x_2), 0, g_0(x_1 - tV(x_2), x_2))$$

$u(\cdot, t)$ bounded in L^∞ but $\|\nabla \times u\|_{L^\infty}$ grows like t

Analogous exact sol^{n}s for 3D MRE ($\gamma=0$)

$$B = (v(x_H),\ g(x_H,t)),\quad u = (0,\ 0,\ (v\cdot\nabla_H)g)$$

$$\partial_t g = (v\cdot\nabla_H)^2 g \quad \text{rank 1 diffusion eqn}$$

ex shear flow $v(x_H) = (V(x_2), 0)$

$$\partial_t g = V^2(x_2)\,\partial_{11} g$$

choose $g_0(x_1)$: $-\partial_{11} g_0(x_1) = \lambda^2\left(g_0(x_1) - \fint g_0\right)$

$$g(x_1,x_2,t) = \fint g_0 + \exp\left[-\lambda^2 V^2(x_2)\,t\right]\left(g_0(x_1) - \fint g_0\right)$$

note: $\lim_{t\to\infty} u \to 0$, $\lim_{t\to\infty} B \to (v(x_H), 0)$

$\boxed{\text{but}}$ there is infinite time growth in gradients of B

Specific Example:

$$B_0 = \hat{e}_3 + \varepsilon(\sin x_2, 0, \cos x_1)$$

$$u_0 = -\varepsilon^2(0, 0, \sin x_2 \sin x_1)$$

Calculate: $g(x_1, x_2, t) = 1 + \varepsilon \cos x_1 \exp(-\varepsilon^2 \sin^2(x_2) t)$

$$\partial_2 B_3(x,t) = -2t\varepsilon^3 \sin x_2 \cos x_2 \cos x_1 \exp(-\varepsilon^2 \sin^2(x_2) t)$$

Compute:

$$\lim_{t \to \infty} \frac{1}{c_1 \varepsilon^{3/2} t^{1/4}} \|\partial_2 B_3(\cdot, t)\|^2_{L_{x_1, x_2}} = 1$$

$$\lim_{t \to \infty} \frac{1}{c_2 \varepsilon^2 t^{1/2}} \|\partial_2 B(\cdot, t)\|_{L^2_{x_1} L^\infty_{x_2}} = 1$$

Relaxation to $\overline{B}$ in weak topologies, eg L^2

nonlinear instability in stronger topologies

Hyperbolic Flow (exp growth int)

$$v(x_H) = \nabla_H^{\perp}(\sin x_1 \sin x_2)$$

$$\partial_t g = (v \cdot \nabla_H)^2 g,$$ exist and unique (see Ebin & Marsden)

$g_0 \in H^k,\ k \geq 3$

MRE solution satisfies

$$|\nabla_H g_0(0,0)|\, e^t \leq \|\nabla B(\cdot,t)\|_{L^\infty} \leq C \|B_0\|_{H^k} e^{ct}$$

where $c > 0$, constant only on k

i.e. we have a 3D MRE example which exhibits exponential growth in their gradients

Concluding Comments

MRE is a challenging and unusual PDE

An active vector equation with a cubic nonlinearity.

Many open questions: for example

Global existence when $\gamma \in [0, \frac{d}{2}+1]$

Given a global sol^n $B(\cdot,t)$ does $\lim_{t\to\infty} B \to \overline{B} \in L^2(\mathbb{T}^d)$

Our special 2½D solutions show that generically we can not expect magnetic relaxation with respect to strong norms.

What is the asymptotic structure of $\overline{B}$ when the initial field is chaotic?

Thank You

for your kind invitation