Eckhaus instability of the compressible Taylor vortices

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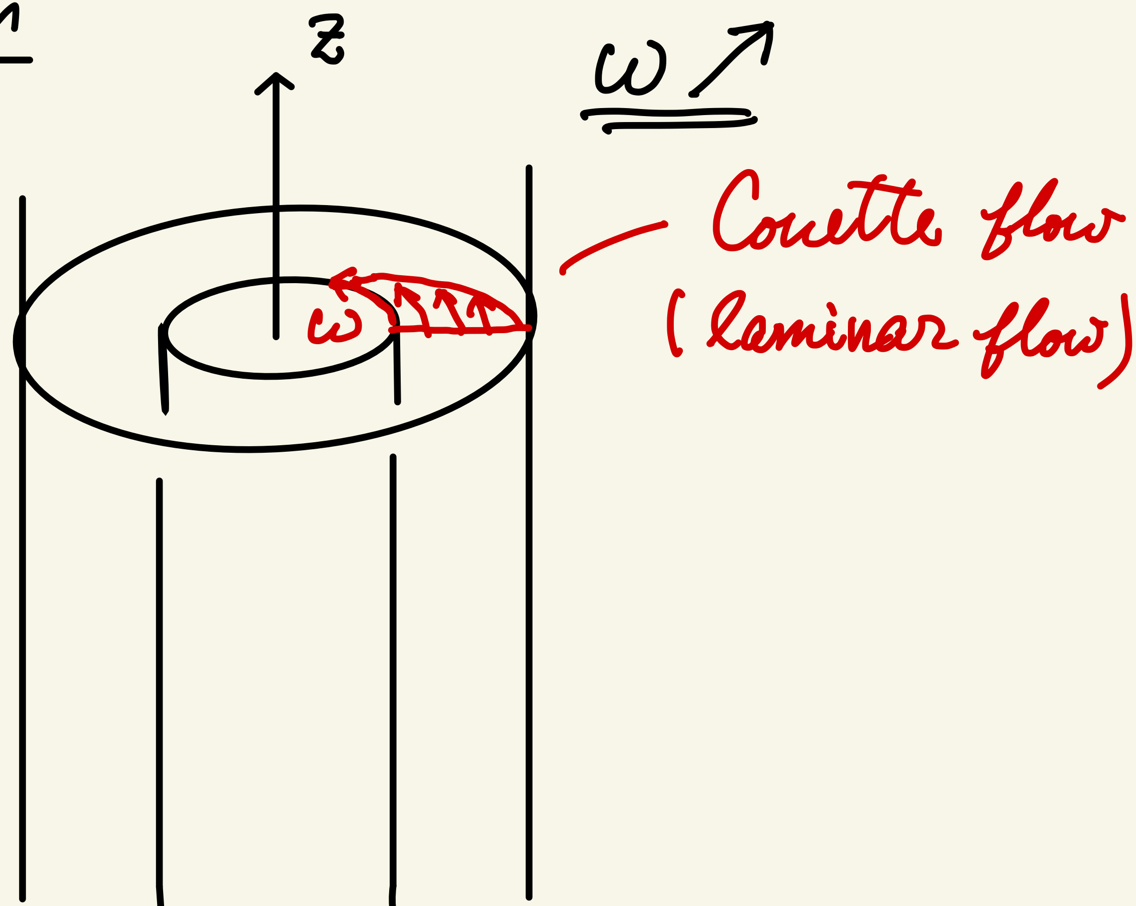
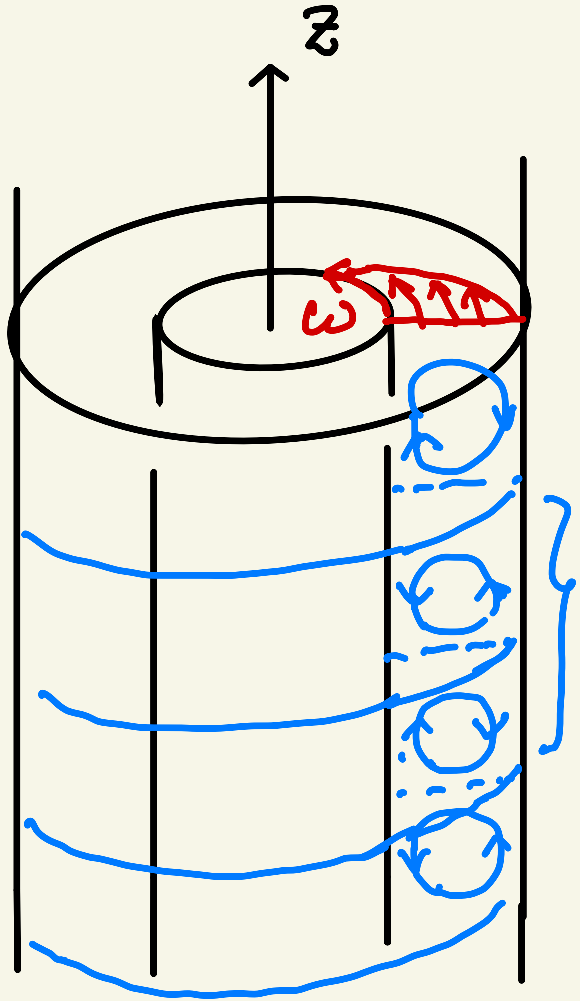
1. Taylor Problem



$\underline{\underline{\omega \nearrow}}$



$\exists\, \omega_c > 0$ s.t.

$\omega > \omega_c$

$\Rightarrow$ Taylor vortex bifurcates

$\frac{2\pi}{\alpha}$-periodic

$\underline{\omega_c = \omega_c(\alpha)}$

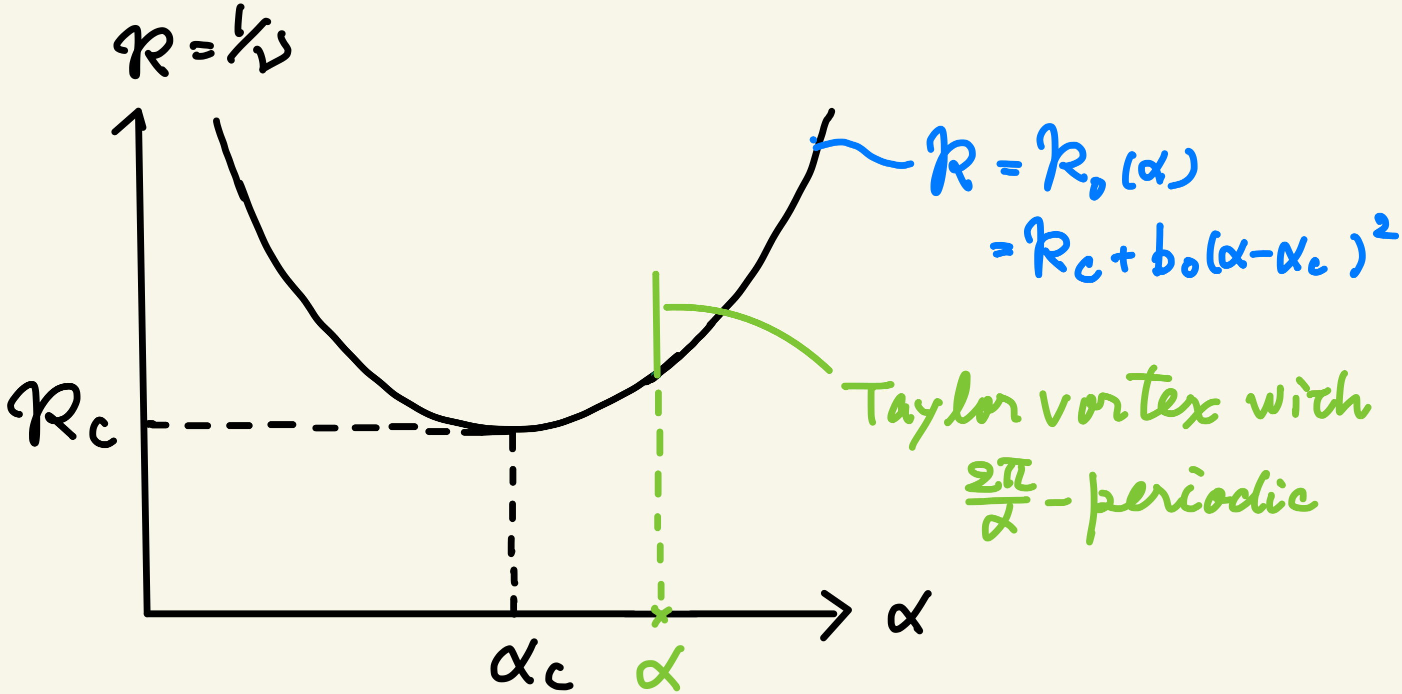
2. Incompressible Problem

Incompressible Navier-Stokes system:

$$\text{(NS)} \qquad \begin{cases} \operatorname{div} \boldsymbol{v} & = & 0, \\ \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \frac{1}{\mathcal{R}} \Delta \boldsymbol{v} + \nabla p & = & \mathbf{0}. \end{cases}$$

- Unknowns: p: pressure, $\boldsymbol{v}$: velocity.
- $\mathcal{R}$: Reynolds number; $\mathcal{R} \propto \omega/\mu$
- semilinear parabolic system.

- Bifurcation of incompressible Taylor vortex:
  - Velte (1966), Iudovich (1966), Kirchgässner-Sorger (1969), ...
  - $\exists$ critical value $\mathcal{R}_0 > 0$ such that
    - $\mathcal{R} < \mathcal{R}_0 \Rightarrow$ Couette flow $\boldsymbol{v}_C$ is stable,
    - $\mathcal{R} > \mathcal{R}_0 \Rightarrow$ Couette flow $\boldsymbol{v}_C$ is unstable and Taylor vortex $\boldsymbol{v}_T$ bifurcates from the Couette flow.
    - $\boldsymbol{v}_T$ is stable for $\mathcal{R} \sim \mathcal{R}_0$.
- Crandall-Rabinowitz theory (1971, 1973) is applicable to show the bifurcation and stability.



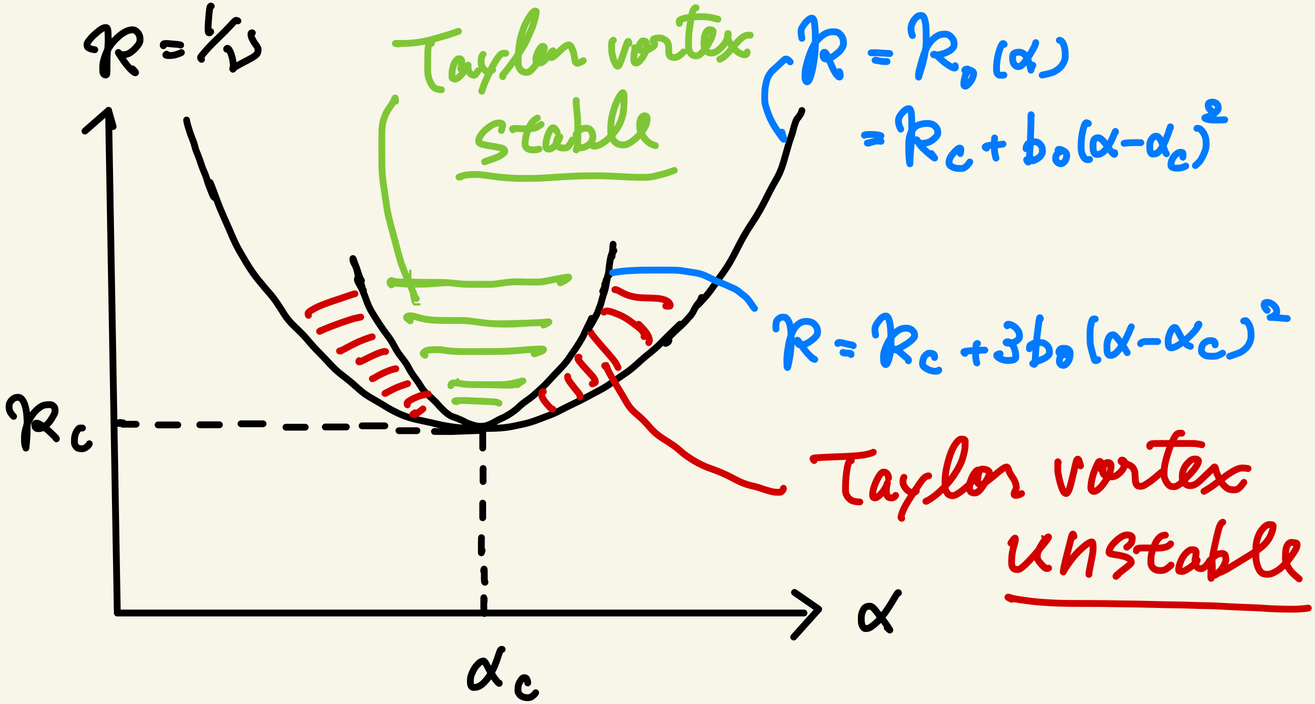
· For $^\forall \alpha \sim \alpha_c$, $\exists$ Taylor vortex with $\frac{2\pi}{\alpha}$-peridic in z

for $R > R_0(\alpha)$, $R \sim R_c$

Stability of incompressible Taylor vortex under localized perturbations

- Stability of incompressible Taylor vortex under localized perturbations (non-periodic, decaying at $z = \pm\infty$): Eckhaus instability occurs.

Eckhaus instability



Stability of incompressible Taylor vortex under localized perturbations

- G. Schneider (1998): Stability of Taylor vortex $\boldsymbol{v}_T$ with wave number α_c

$$\begin{aligned} \boldsymbol{w}(x', z, t) \quad &\sim \quad \partial_z \boldsymbol{v}_T(x', z)\, t^{-\frac{1}{2}} g_d\left(\tfrac{z}{\sqrt{t}}\right) \ \text{ as } t \to \infty, \\ & \qquad g_\kappa(z) = (4\pi d)^{-\frac{1}{2}} e^{-\frac{|z|^2}{4d}} \quad (d > 0\text{: constant}). \end{aligned}$$

3. Compressible Navier-Stokes equations

$$
\text{(CNS)} \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \frac{1}{\mathcal{R}} \Delta \boldsymbol{v} - \frac{1}{3\mathcal{R}} \nabla \operatorname{div} \boldsymbol{v} + \frac{1}{\varepsilon^2} \nabla p(\rho) = \mathbf{0}. \end{cases}
$$

- Unknowns: ρ: density, $\boldsymbol{v}$: velocity.
- $p(\rho)$: pressure, $\left.\dfrac{dp}{d\rho}\right|_{\rho=1} = p'(1) = 1$.
- $\varepsilon > 0$: Mach number (in this talk, $0 < \varepsilon \ll 1$).
- quasilinear hyperbolic-parabolic system.

Compressible Taylor Problem

$$\text{(CNS)} \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \frac{1}{\mathcal{R}} \Delta \boldsymbol{v} - \frac{1}{3\mathcal{R}} \nabla \operatorname{div} \boldsymbol{v} + \frac{1}{\varepsilon^2} \nabla p(\rho) = \mathbf{0}. \end{cases}$$

- Cylindrical coordinates (r, θ, z):

$$\Omega_\alpha = \{(r, \theta, z) : r_1 < r < r_2,\ \theta \in \mathbb{T}_{2\pi},\ z \in \mathbb{T}_{\frac{2\pi}{\alpha}}\} =: D \times \mathbb{T}_{\frac{2\pi}{\alpha}}$$
$$(0 < r_1 < r_2 < \infty,\ \alpha > 0).$$

- Boundary conditions on $r = r_1, r_2$: $\boldsymbol{v} = v^r \boldsymbol{e}_r + v^\theta \boldsymbol{e}_\theta + v^z \boldsymbol{e}_z$

$$v^\theta|_{r=r_1} = 1,\ v^\theta|_{r=r_2} = 0,\ v^r = v^z = 0 \text{ on } r = r_1, r_2.$$

4. Compressible Couette flow

- (CNS) has a stationary solution (Couette flow):

$$u_C^\varepsilon = \begin{pmatrix} \rho_C^\varepsilon \\ \boldsymbol{v}_C \end{pmatrix},$$

$$\boldsymbol{v}_C = v_C^\theta(r)\boldsymbol{e}_\theta, \quad v_C^\theta(r) = c_1 r + c_2 \frac{1}{r},$$

$$\rho_C^\varepsilon = \rho_C^\varepsilon(r) = 1 + O(\varepsilon^2) \quad (\varepsilon \to 0).$$

- If $\mathcal{R} \ll 1$ and $0 < \varepsilon \ll 1$, then $u_C^\varepsilon = \begin{pmatrix} \rho_C^\varepsilon \\ \boldsymbol{v}_C \end{pmatrix}$ is asymptotically stable.
- Question: Stability of u_C^ε when $\mathcal{R} \nearrow$?

Equations for perturbation

Perturbation: $u = {}^\top(\phi, \boldsymbol{w})$; $\rho = \rho_C^\varepsilon + \varepsilon^2\phi$, $\boldsymbol{v} = \boldsymbol{v}_C + \boldsymbol{w}$

$$\begin{cases} \partial_t\phi + \mathrm{div}\,(\boldsymbol{v}_C\phi) + \frac{1}{\varepsilon^2}\mathrm{div}\,(\rho_C^\varepsilon\boldsymbol{w}) = -\mathrm{div}\,(\phi\boldsymbol{w}), \\ \partial_t\boldsymbol{w} - \frac{1}{\mathcal{R}\rho_C^\varepsilon}\Delta\boldsymbol{w} - \frac{1}{3\mathcal{R}\rho_C^\varepsilon}\nabla\mathrm{div}\,\boldsymbol{w} + \nabla\left(\frac{p'(\rho_C^\varepsilon)}{\rho_C^\varepsilon}\phi\right) \\ \qquad +\boldsymbol{v}_C\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_C = -\boldsymbol{g}(\varepsilon, \mathcal{R}; u, \partial_x u, \partial_x^2\boldsymbol{w}). \end{cases} \tag{1}$$

- $\boldsymbol{g} = \boldsymbol{w}\cdot\nabla\boldsymbol{w} + \varepsilon^2\tilde{\boldsymbol{g}}(\varepsilon, \mathcal{R}; u, \partial_x u, \partial_x^2\boldsymbol{w})$: nonlinearity
- $\varepsilon \to 0$: since $\rho_C^\varepsilon = 1 + O(\varepsilon^2)$,

$$\Rightarrow \begin{cases} \mathrm{div}\,\boldsymbol{w} = 0, \\ \partial_t\boldsymbol{w} - \frac{1}{\mathcal{R}}\Delta\boldsymbol{w} + \nabla\phi + \boldsymbol{v}_C\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_C = -\boldsymbol{w}\cdot\nabla\boldsymbol{w}. \end{cases}$$

- Main Difference to Incompressible NS :

  $-\mathrm{div}\,(\phi\boldsymbol{w})$ causes derivative loss (not Fréchet differentiable) !

Bifurcation of compressible Taylor vortex

- Stationary problem:

$$L^{\varepsilon}_{\mathcal{R}} u + N(u)u + G(u) = 0. \tag{2}$$

  Here

$$N(\tilde{u})u = \begin{pmatrix} \operatorname{div}(\phi \tilde{\boldsymbol{w}}) \\ \mathbf{0} \end{pmatrix}, \quad G(u) = \begin{pmatrix} 0 \\ \boldsymbol{g}(\varepsilon, \mathcal{R}; u, \partial_x u, \partial_x^2 \boldsymbol{w}) \end{pmatrix}$$

  for $\tilde{u} = {}^{\top}(\tilde{\phi}, \tilde{\boldsymbol{w}})$ and $u = {}^{\top}(\phi, \boldsymbol{w})$.

- $u = 0$ is a stationary solution of (2) corresponding to the Couette flow u^{ε}_{C}.

Theorem 1. *Fix* $\alpha \sim \alpha_c$.*Then there exist critical Reynolds number* $\mathcal{R}_0^\varepsilon = \mathcal{R}_0^\varepsilon(\alpha)$ *and nontrivial stationary solution branch* $(\mathcal{R}, u) = (\mathcal{R}_\delta^\varepsilon, u_\delta^\varepsilon)$ $(|\delta| \ll 1)$ *with*

$$\mathcal{R}_\delta^\varepsilon = \mathcal{R}_0^\varepsilon + a^\varepsilon \delta^2 + O(\delta^3) > \mathcal{R}_0^\varepsilon$$

and

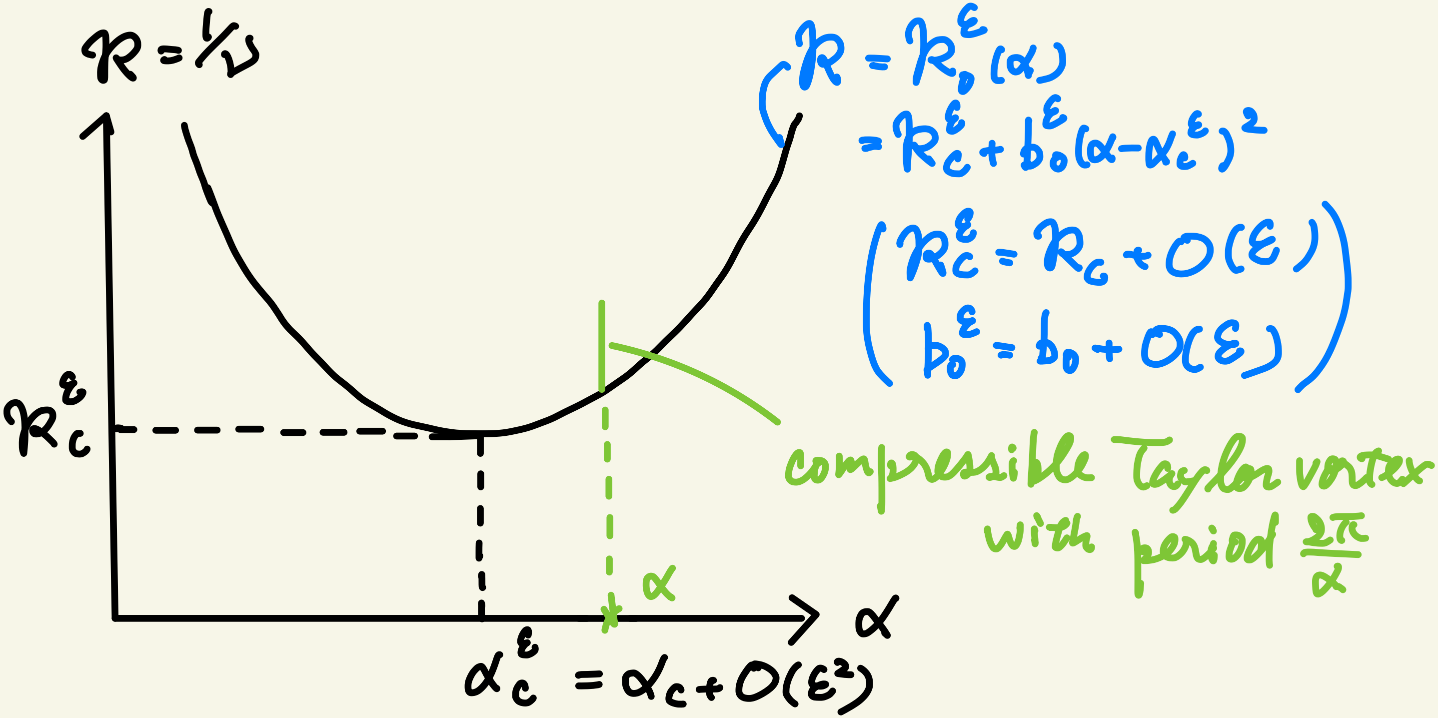
$$u_\delta^\varepsilon = {}^\top(\phi_\delta^\varepsilon, \boldsymbol{w}_\delta^\varepsilon) \in H^4(\Omega_\alpha) \times H^5(\Omega_\alpha)^3, \quad \int_{\Omega_\alpha} \phi_\delta^\varepsilon \, dx = 0$$

Here $\mathcal{R}_0^\varepsilon(\alpha) = \mathcal{R}_0(\alpha) + O(\varepsilon^2)$, $a^\varepsilon > 0$ *and*

$$u_\delta^\varepsilon = \delta(u^{(0),\varepsilon} + \delta U_\delta^\varepsilon),$$

where $L_{\mathcal{R}_0^\varepsilon}^\varepsilon u^{(0),\varepsilon} = 0$, $u^{(0),\varepsilon} \neq 0$; *and* U_δ^ε *is a* C^ℓ *function of* δ *with values in* $H^{3-\ell} \times H^{4-\ell}$ $(0 \le \ell \le 3)$.

Remark. One can also construct the solution $u = u_{\delta,m}^\varepsilon = {}^\top(\phi_{\delta,m}^\varepsilon, \boldsymbol{w}_{\delta,m}^\varepsilon)$ with $\int_{\Omega_\alpha} \phi_{\delta,m}^\varepsilon \, dx = m$ for $0 < |m| \ll 1$.



$$\mathcal{R} = \mathcal{R}_0^\varepsilon(\alpha) = \mathcal{R}_c^\varepsilon + b_0^\varepsilon(\alpha - \alpha_c^\varepsilon)^2$$

$$\begin{pmatrix} \mathcal{R}_c^\varepsilon = \mathcal{R}_c + O(\varepsilon) \\ b_0^\varepsilon = b_0 + O(\varepsilon) \end{pmatrix}$$

- For $^\forall \alpha \sim \alpha_c^\varepsilon$, $\exists$ Taylor vortex with $\frac{2\pi}{\alpha}$-peridic in z

  for $\mathcal{R} > \mathcal{R}_0^\varepsilon(\alpha)$, $\mathcal{R} \sim \mathcal{R}_c^\varepsilon$

Bifurcation results on compressible Navier-Stokes equations

- Nishida-Padula-Teramoto (2013)
  Bifurcation of compressible thermal convection patterns and Incompressible limit ("$\varepsilon \to 0$") of the bifurcating patterns (convergence of the compressible solutions to the Oberbeck-Boussinesq ones):
- Nishida-Y.K. (2019)
  Bifurcation of traveling waves from the plane Poiseuille flow: $\varepsilon \sim 1.5$
- Stability results on bifurcating solutions have been yet missing.

Stability under axisymmetric periodic perturbations

- Linearized operator: $\tilde{L}_\delta^\varepsilon$ on $H^1_{axi}(\Omega_\alpha) \times L^2_{axi}(\Omega_\alpha)^3$,

$$\tilde{L}_\delta^\varepsilon u := L^\varepsilon_{\mathcal{R}_\delta^\varepsilon} u + M(\tilde{u}_\delta^\varepsilon)u + \partial_u G(\tilde{u}_\delta^\varepsilon)u,$$

where

$$u_\delta^\varepsilon = u_C^\varepsilon + \tilde{u}_\delta^\varepsilon, \quad \tilde{u}_\delta^\varepsilon = O(\delta),$$

$$M(\tilde{u}_\delta^\varepsilon)u = N(\tilde{u}_\delta^\varepsilon)u + N(u)\tilde{u}_\delta^\varepsilon = \begin{pmatrix} \tilde{\boldsymbol{w}}_\delta^\varepsilon \cdot \nabla\phi + \cdots \\ \mathbf{0} \end{pmatrix}.$$

Theorem 2. *Fix* $\alpha \sim \alpha_c^\varepsilon$. *Then there are* $\varepsilon_2 > 0$, $\Lambda_2 > 0$ *and* $\delta_2 > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_2$ *and* $|\delta| \leq \delta_2$,

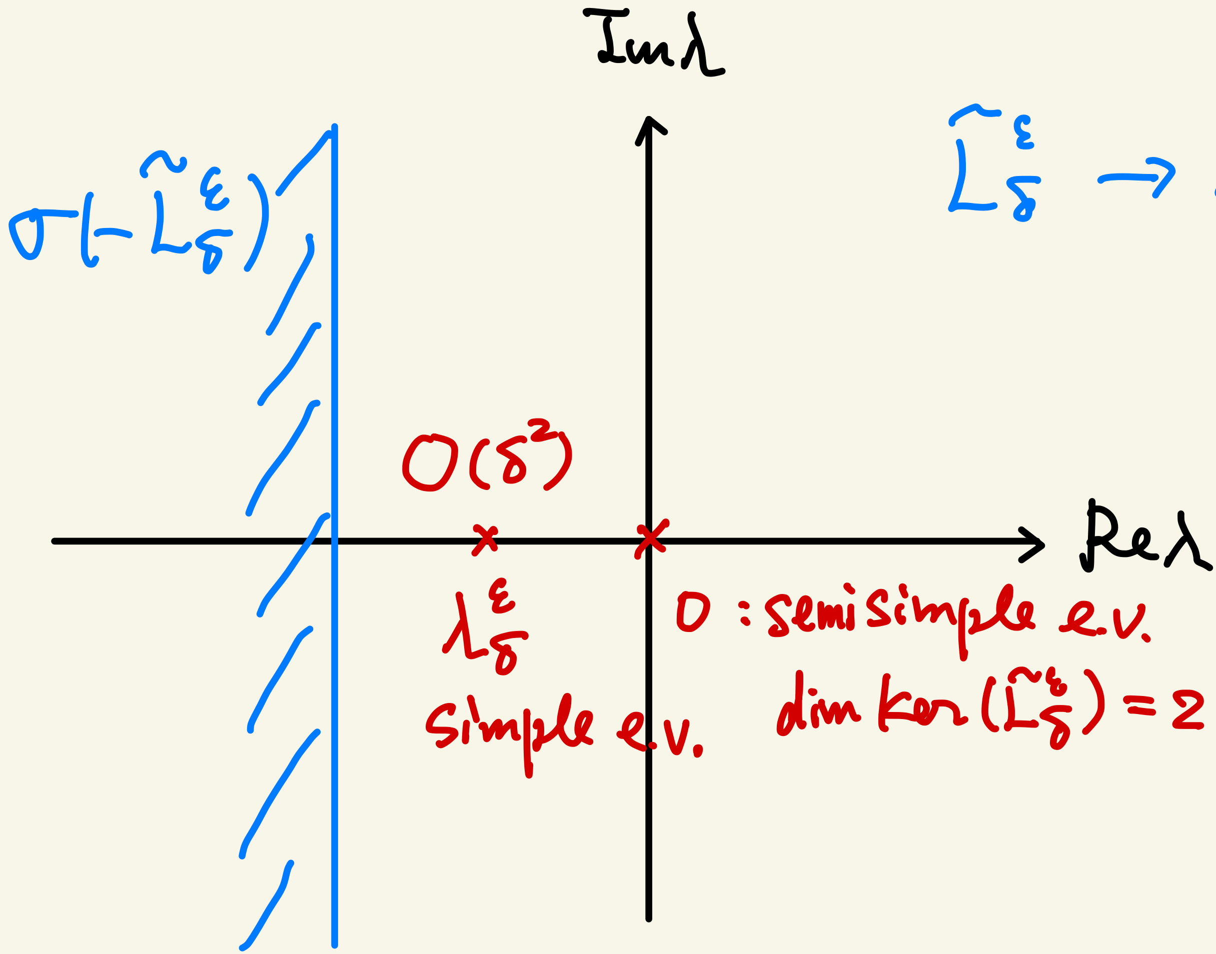
$$\rho(-\tilde{L}_\delta^\varepsilon) \supset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -\Lambda_2\} \setminus \{0, \lambda_\delta^\varepsilon\}.$$

Here 0 *is a semisimple eigenvalue and* $\lambda_\delta^\varepsilon$ *is a simple eigenvalue;* $\lambda_\delta^\varepsilon$ *is a* C^2 *function of* δ*; and*

$$\lambda_\delta^\varepsilon = -\tilde{a}^\varepsilon \delta^2 + O(\delta^3) \ (\tilde{a}^\varepsilon > 0 \text{ constant}).$$

Furthermore, $\operatorname{Ker}(-\tilde{L}_\delta^\varepsilon) = \operatorname{span}\{f_{1,\delta}^\varepsilon, f_{2,\delta}^\varepsilon\}$. *Here*

$$f_{1,\delta}^\varepsilon = (\partial_m \tilde{u}_{\delta,m}^\varepsilon)|_{m=0}, \quad f_{2,\delta}^\varepsilon = \partial_z \tilde{u}_\delta^\varepsilon.$$



Im λ
Re λ
σ(−L̃^ε_δ)
L̃^ε_δ → L^ε_{R_0^ε}
O(δ²)
λ^ε_δ
simple e.v.
0 : semisimple e.v.
dim Ker(L̃^ε_δ) = 2

Resolvent

- Linearized operator $\tilde{L}^{\varepsilon}_{\delta} u = L^{\varepsilon}_{\mathcal{R}^{\varepsilon}_{\delta}} u + M(\tilde{u}^{\varepsilon}_{\delta}) u + \partial_u G(\tilde{u}^{\varepsilon}_{\delta}) u$
- 0 is a semisimple eigenvalue of $-L^{\varepsilon}_{\mathcal{R}^{\varepsilon}_{0}} = -\tilde{L}^{\varepsilon}_{\delta}|_{\delta=0}$.
- Asymptotic expansions of the critical eigenvalues in δ
  $\leftarrow$ smoothness (C^2) of the eigenvalues
  $\leftarrow$ the regularity of the Taylor vortex u^{ε}_{δ} and the resolvent of $-\tilde{L}^{\varepsilon}_{\delta}$ in δ:

$$\mathcal{R}^{\varepsilon}_{\delta}(\lambda) := (\lambda + \tilde{L}^{\varepsilon}_{\delta})^{-1}$$

Proposition 1

(i) *If* $0 < \varepsilon \leq \varepsilon_2$, $|\delta| \leq \delta_2$, *then*

$$\rho(-\tilde{L}_\delta^\varepsilon) \supset \{\mathrm{Re}\,\lambda \geq -\Lambda_2, |\lambda| \geq \tilde{r}_0\}$$

and the resolvent $\mathcal{R}_\delta^\varepsilon(\lambda) = (\lambda + \tilde{L}_\delta^\varepsilon)^{-1}$ *satisfies*

$$\|\mathcal{R}_\delta^\varepsilon(\lambda)F\|_{H^1 \times H^2} \leq C\|F\|_{H^1 \times L^2}$$

uniformly for small ε, δ *and* λ *with* $\mathrm{Re}\,\lambda \geq -\Lambda_2, |\lambda| \geq \tilde{r}_0$.

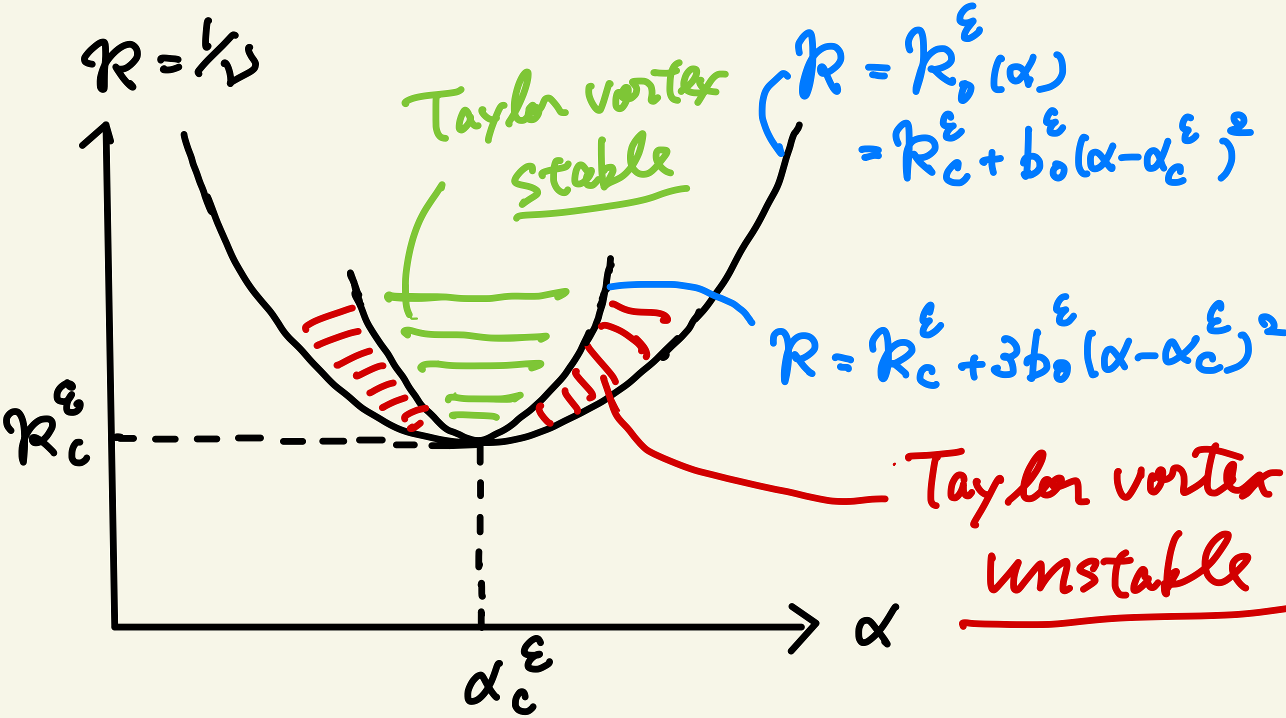
(ii) *Let* $\ell \in \mathbb{Z}$ *with* $1 \leq \ell \leq 2$ *and let* $F \in H^3 \times H^2$*. Then* $\mathcal{R}_\delta^\varepsilon(\lambda)F$ *is a* C^ℓ *function of* δ *with values in* $H^{2-\ell} \times H^{3-\ell}$ *and* $\partial_\delta^\ell \mathcal{R}_\delta^\varepsilon(\lambda)F$ *satisfies*

$$\|\partial_\delta^\ell \mathcal{R}_\delta^\varepsilon(\lambda)F\|_{H^{2-\ell} \times H^{3-\ell}} \leq C\|F\|_{H^3 \times H^2}.$$

Stability under axisymmetric localized perturbations

- Stability of compressible Taylor vortex under localized perturbations (axisymmetric, non-periodic, decaying at $z = \pm\infty$)
- Linearized operator $\tilde{L}_\delta^\varepsilon$ is considered as an operator on $H^1_{axi}(\Omega) \times L^2_{axi}(\Omega)^3$,
- Perturbation: $u = {}^\top(\phi, \boldsymbol{w}) \in H^1_{axi}(\Omega) \times L^2_{axi}(\Omega)^3$

Theorem 3 (Eckhaus instability)



Proposition 2 $\exists\ \tilde{\Lambda}_0 > 0$ *s.t.*

$$\rho(-\tilde{L}_\delta^\varepsilon) \supset \{\mathrm{Re}\,\lambda \geq -\tilde{\Lambda}_0\} \setminus \{\lambda_{1,\xi}, \lambda_{2,\xi}\},$$

where

$$\lambda_{1,\xi} \sim -\frac{c_1 \mathcal{R}_\delta^\varepsilon}{\varepsilon^2}\xi^2, \quad \lambda_{2,\xi} \sim -\frac{c_2 \beta_\delta^\varepsilon}{\delta^2}\xi^2 \quad (|\xi| \ll 1)$$

with constants $c_1, c_2 > 0$ *and*

$$\beta_\delta^\varepsilon := (\mathcal{R} - \mathcal{R}_c^\varepsilon) - 3b_0^\varepsilon(\alpha - \alpha_c^\varepsilon)^2.$$

- $\beta_\delta^\varepsilon < 0 \quad \Leftrightarrow \quad \mathcal{R} - \mathcal{R}_c^\varepsilon < 3b_0^\varepsilon(\alpha - \alpha_c^\varepsilon)^2$ (Eckhaus instability)

Nonlinear stability

$$\partial_t u + \tilde{L}_\delta^\varepsilon u + N(u) + G(u) = 0, \quad u|_{t=0} = u_0. \tag{3}$$

Thm 3. *Let* $\alpha = \alpha_c^\varepsilon$. *If* $u_0 = {}^\top(\phi_0, \boldsymbol{w}_0) \in [H^2_{axi}(\Omega) \times H^2_{axi}(\Omega)^3] \cap [L^1 \times (L^1)^3]$ *with* $\boldsymbol{w}_0 \in H^1_0(\Omega)^3$ *is sufficiently small, then there exists a unique global solution* $u(t) \in C([0,\infty); H^2_{axi}(\Omega) \times H^2_{axi}(\Omega)^3)$ *of* (3) *and* $u(t)$ *satisfies, as* $t \to \infty$,

$$\|\partial_z^\ell u(t)\|_{L^2} = O(t^{-\frac{1}{4}-\frac{\ell}{2}}) \quad (\ell = 0, 1)$$

$$\|u(t) - \sigma_1(z,t) f^\varepsilon_{1,\delta} - \sigma_2(z,t) f^\varepsilon_{2,\delta}\|_{L^2} = O(t^{-\frac{3}{4}+\kappa}) \quad (\kappa > 0)$$

Here $f^\varepsilon_{1,\delta} = (\partial_m \tilde{u}^\varepsilon_{\delta,m}[\alpha_c^\varepsilon])|_{m=0}$, $f^\varepsilon_{2,\delta} = \partial_z \tilde{u}^\varepsilon_\delta[\alpha_c^\varepsilon]$; *and* $\sigma_j = \sigma_j(z,t)$ *satisfy*

$$\begin{cases} \partial_t \sigma_1 - \frac{c_1}{\varepsilon^2 \nu} \partial_z^2 \sigma_1 + d_1 \partial_z(\sigma_1^2) = 0, \\ \partial_t \sigma_2 - c_2 \gamma_\delta^\varepsilon \partial_z^2 \sigma_2 + d_2 \partial_z(\sigma_1^2) = 0, \end{cases}$$

Here $c_j > 0$, $\gamma_\delta > 0$ *and* $d_j \in \mathbb{R}$ $(j = 1, 2)$.

Cf., Johnson-Noble-Rodrigues-Zumbrun (2014)

5. Remarks

(1) Stability of Couette flow under non-axisymmetric periodic perturbations ?

$\Rightarrow$ Stability of Taylor vortex under non-axisymmetric perturbations

(2) Hopf bifurcation ?

Let ω_1:angular velocity of the inner cylinder, ω_2:angular velocity of the outer cylinder

If $\omega_1 < 0 < \omega_2$ and $\omega_2 - \omega_1 \geq \tilde{\omega} > 0$, then Hopf bifurcation from non-axisymmetric modes occurs in the incompressible system.

(1) $\Rightarrow$ Hopf bifurcation in the compressible system could be proved.

- Zero Mach number limit $\varepsilon \to 0$?
- Stability of bifurcating solutions ?

(3) Existence of front solutions ? Incompressible case: Haragus-Schneider (1999)

Thank you for your kind attention !